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Find, Process, and Share: An Optimal Control in the Vidale-Wolfe Marketing Model

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Abstract: The Vidale-Wolfe marketing model is a first-order, linear, non-homogeneous ordinary differential equation (ODE) where the forcing term is proportional to advertising expenditure. With an initial response in sales as the initial condition, the solution of the initial value problem is straightforward for a first undergraduate ODE course. The model serves as an excellent example of many relevant topics for those students whose interests lie in economics, finance, or marketing. Its inclusion in the curriculum is particularly rewarding at an institution without a physics program. The model is not new, but it was novel to us when a group of students chose it for an exploratory project that we designed in order to help students acquire the ability to interpret and communicate mathematical results. In addition to describing the project in this work, we discuss the Vidale-Wolfe model and show how it can lead one to use Green's theorem in a real situation.

1 Introduction

As mathematicians and educators in higher education, we are steeped in a find, process, and share mentality. Indeed, we spend hours scouring the literature for research related to our own. We process what we have found and synthesize new directions. After much work, we share our results with the community through papers and presentations. If education is our focus, we perform similar steps: finding and learning about new pedagogical developments; processing to determine what will work within our classroom and teaching mode or style; and, ultimately, sharing our work with students, fellow educators, and the larger academic community through classes, discussion at workshops, and by other means. In this article, we are excited to report on our finding of the Vidale-Wolfe marketing model. Our process has shed light on some great mathematics that can be gleaned from a careful analysis of the model. We share our story with the hope that other educators might find the model appropriate for processing and sharing with their own students.

We begin in Section 2 with details about an ordinary differential equations (ODEs) reading and writing course project. Section 3 presents the Vidale-Wolfe marketing model along with suggestions for how it can be used in early-level undergraduate mathematics courses. Section 4 makes a case for how Green's theorem, appearing within the analysis of an optimal control problem related to the Vidale-Wolfe model, might be incorporated into a first ODE course for those students who are highly motivated. This work concludes with some data and anecdotal commentary in Section 5.

2 FINDING THE MODEL

As undergraduate students, we recall completing assignments like:

> **Assignment 1**: Read the paragraph and respond.
>
> "It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to Heaven, we were all going direct the other way - in short, the period was so far like the present period, that some of its noisiest authorities insisted on its being received, for good or for evil, in the superlative degree of comparison only." - Charles Dickens: A Tale of Two Cities. Accessed on January 18, 2018 from https://archive.org/stream/adventuresofoliv00dickiala#page/n401/mode/2up.

Surely current university students in English courses perform similar exercises. Another type of assignment that we completed is:

> **Assignment 2**: View the painting. Describe it and how it makes you feel.

Figure 1: Vincent van Gogh: The Starry Night. Accessed on January 18, 2018, from https://commons.wikimedia.org/wiki/File:Van_Gogh_-_Starry_Night_-_Google_Art_Project.jpg#file.

On the other hand, we venture to say that most of our students have probably not undertaken an exercise such as:

Assignment 3: Read the mathematics article excerpt. Explain.

> A mathematical model of sales response to advertising, based on these parameters, is represented by:
>
> $$dS/dt = r\,A(t)\,(M-S)/M - \lambda S, \qquad (1)$$
>
> where S is the rate of sales at time t, and $A(t)$ is the rate of advertising expenditure. This equation has the following interpretation: the increase in the rate of sales, dS/dt, is proportional to the intensity of the advertising effort, A, reaching the fraction of potential customers, $(M-S)/M$, less the number of customers that are being lost, λS.

Figure 2: Excerpt from [4, p. 377].

Perhaps in addition to asking our students for an explanation in Assignment 3, we should also ask them to consider the tasks of Assignment 2. We could ask questions like: How does viewing the mathematics in the excerpt make you feel? What are its aesthetic qualities? We have introduced a reading and writing project into our ODE course that prompts the students to perform tasks akin to Assignment 3. Not surprisingly, writing about mathematics seems new to the students. From previous coursework, they are familiar with *obtaining* solutions of problems. With our project, we desire just *descriptions* of solutions and problems.

Similar to how many of us perform scholarly work, the project follows the form of **F**ind, **P**rocess, **S**hare. To elaborate: **F**ind - Students browse online journal articles and print journals, ultimately selecting a topic that involves an accessible ODE model of a "real-life" phenomenon. We often discuss with the students the appropriateness, in terms of accessibility, of any relevant articles that they obtain. **P**rocess - Students *read* and *understand* the mathematical model in the articles that they found. The model generally takes the form of an initial value problem (IVP). The students also read and seek to understand the "real-life" interpretation of the parameters and variables of their chosen model. **S**hare - Students *describe* and *show* the mathematical model and "real-life" interpretation of parameters and variables. This is accomplished through a brief synopsis paper and a class presentation.

Our home institution, Niagara University (NU), is a small, private, liberal arts university. While our science programs are quite strong, we lack a physics major, and we do not offer any engineering courses. As such, the typical audience in our ODE course consists primarily of mathematics majors and science majors (biochemistry, chemistry, biology), while some business school students (majoring in economics or finance) sometimes take the course, too. Due to the varied backgrounds and interests of the students in the course, we attempt to provide numerous examples of applications of differential equations within many diverse fields. By implementing our project, the students are given the opportunity to select a topic for their group project that appeals to them. Appreciating the utility of

ODEs is a wonderful by-product of the project. However, the main goal of the project is to enhance students' mathematical communication ability. This includes improving their reading comprehension and their ability to explain mathematics through both writing and speaking.

The project is designed to parallel the content development in the course. Deadlines for the different parts of the project are carefully chosen throughout the semester so that students in the course will have had a chance to learn any relevant mathematics prior to submitting their project work. The assessment of the group project consists of grading a group topic proposal for completeness, grading the short paper, and evaluating the presentation. Based on preliminary analysis of course and exam scores, the project is associated with improved individual performance in the course, including an increase in content comprehension. The project has seen some major and minor successes, but is constantly undergoing refinement for improvement in future semesters. One major success of the project is that it led a team of students to pursue further study on their topic, the Vidale-Wolfe marketing model. By locating Vidale and Wolfe [4], these two students "discovered" the Vidale-Wolfe marketing model, and they created an excellent course project. As a consequence, we too found the Vidale-Wolfe model.

3 PROCESSING THE VIDALE-WOLFE MODEL

As we discussed the Vidale-Wolfe marketing model with our students, it became apparent that it contained much interesting mathematics. In this section, we delve further into this model to show how it can serve as a fresh example of various mathematical techniques in a first ODE course.

3.1 The Model

In an effort to understand advertising expenditure and sales, Vidale and Wolfe developed a differential equation model capable of measuring the effectiveness of advertising campaigns for the companies and products that they considered in Vidale and Wolfe [4]. Other questions that they sought to answer, and that relied on a quantitative understanding of the effectiveness of an advertising campaign, concerned allocation of resources amongst multiple products and determining an appropriate size for an advertising budget. Their work was extended by other researchers (see Sethi [2] and the references therein). Of particular note, the work in Sethi [2] provides optimal controls for the Vidale-Wolfe marketing model by developing various optimal advertising strategies. Section 4 of this work will discuss some of those details.

The original model proposed by Vidale and Wolfe in 1957 is a first-order, linear, nonhomogeneous ordinary differential equation with an initial condition. The IVP is

$$\frac{dS}{dt} = rA(t)\left(\frac{M-S}{M}\right) - \lambda S, \qquad S(0) = S_0$$

where the dependent variable, $S = S(t)$, is the rate of sales at time t. The initial rate of sales is a constant, $S(0) = S_0$. The inhomogeneity, i.e., the forcing term, is $rA(t)$, where $A(t)$ is the rate of advertising expenditure and r is a response constant. The other parameters are M, the saturation level, and λ, a sales decay constant. While we refer the interested reader to Vidale and Wolfe [4] for a complete description of these quantities, our students are expected to give further explanation about the variables and parameters in their project, including units. For example, they might explain how M, in units of dollars per month, is determined for a particular product within the article and what real-life quantity it represents. Groups might present information like in Figure 2, where Vidale and Wolfe point out that the increase in the rate of change of sales, i.e., $\frac{dS}{dt}$, is proportional to the advertising expenditure and the fraction of potential customers $\frac{M-S}{M}$. The increase is lessened by a loss of customers, reflected in the term λS for $\lambda > 0$. In other words, sales decrease at an exponential rate of λ in the absence of advertising.

3.2 The Features

In the remainder of this work, we consider a scaled version of the model, following Sethi [2]. In Sethi's notation, the scaled problem is

$$\frac{dx}{dt} = \rho u(t)(1-x) - kx, \quad x(0) = x_0 \tag{3.1}$$

where $x = S/M$ is market share with initial constant value $x(0) = x_0$. The response constant is $\rho = r/M$, advertising is $u = A$, and $k = \lambda$. In Vidale and Wolfe [4], two different advertising strategies are considered: constant advertising over a finite time interval and pulse advertising. Pulse advertising is a very large magnitude campaign of brief duration. Including a discussion of how the model reacts to these different advertising inputs is instructive for a first ODE course.

Suppose the constant advertising strategy is given by

$$u(t) = \begin{cases} \bar{u} & \text{if} \quad 0 \le t \le \tau \\ 0 & \text{if} \quad t > \tau \end{cases} \tag{3.2}$$

where $\bar{u} > 0$ is constant and $\tau > 0$ is the constant length of the advertising campaign. Using standard methods from a first ODE course (perhaps the method of integrating factors), one can compute a solution to (3.1). A representative trajectory is depicted in Figure 3(a) for illustrative parameter values $t \in [0, 12]$, $\tau = 6$, $\bar{u} = 1$, $\rho = 0.25$, $k = 0.1$, and $x_0 = 0.2$. The solution is quite similar to that encountered in break-point models that students in a first ODE course learn about in the context of ODEs for the velocity in air resistance/parachute models or ODEs in compartmental mixing models (see, e.g., Nagle et al. [1], Exercise 7, p. 122). We provide a simple example. Suppose that, for 10 seconds, a salt solution of concentration $\frac{1}{2}$ kg/L is pumped into a large tank containing 1L of pure water at a rate of 1 L/sec. At the same time, assuming the tank is kept well-mixed, solution is pumped out of the tank at a rate of 1 L/sec. After these initial 10 seconds, pure water is pumped into the tank at a rate of $\frac{1}{2}$ L/sec and the solution is pumped out at a rate of $\frac{1}{2}$ L/sec. If $y(t)$ is the amount of salt in the tank at time t, this scenario can be modeled by

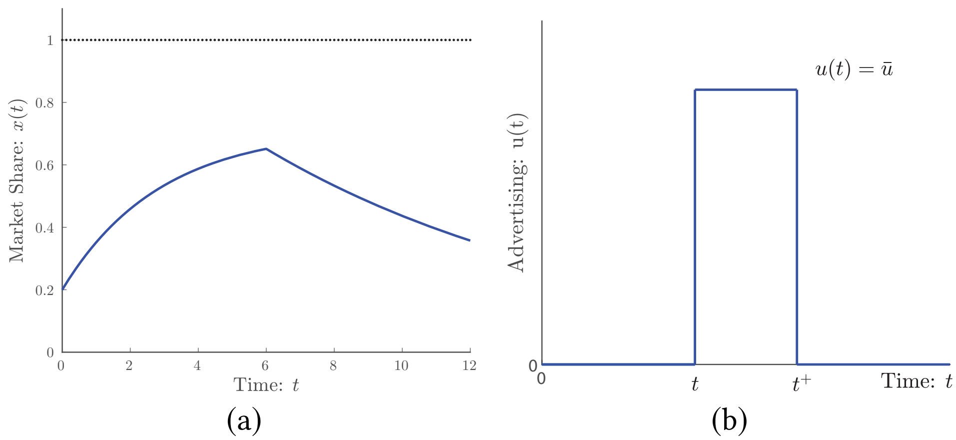

Figure 3: (a) Solution, $x(t)$, in constant advertising case. (b) Advertising pulse.

two ODEs similar to (3.1) with $u(t)$ given by (3.2).

First 10 seconds	Next 10 seconds
$\frac{dy}{dt} = \frac{1}{2} - y$	$\frac{dy}{dt} = -\frac{1}{2}y$
This is (3.1) with $\rho\bar{u} = \frac{1}{2}, k = \frac{1}{2}$.	This is (3.1) with $u = 0, k = \frac{1}{2}$.

For students whose interests lean more toward business, the Vidale-Wolfe model provides a more relevant application of such a break-point situation.

Another topic often studied in a first ODE course is pulse forcing. In past courses we have introduced the idea of a pulse by suggesting that a mass-spring system be set into motion by a quick hammer strike to the mass. This scenario from the realm of physics might be easy for some students to visualize, but we posit that marketing students would be more likely to appreciate an application where the pulse occurs within a sales response model. Figure 3(b) shows a square wave representing an advertising pulse of magnitude $\bar{u}$ and duration $t^+ - t$, where t^+ is a time "just after" t. An immediate and interesting application of pulse forcing in the Vidale-Wolfe model is to determine the magnitude of an advertising pulse that is required in order to increase market share by some predetermined amount. A result to this end can be found in Lemma 3.1 from Sethi [3], Lemma 3.1, p. 5. We reproduce the lemma here since its consideration leads to additional analysis that we find important for our students to see.

Lemma 3.1 (Sethi, 1972). *Let $P(a, b, t)$ denote the magnitude of an advertising pulse necessary to increase market share $x(t) = a$ at time t to $x(t^+) = b$ at time t^+ (just after t); then*

$$P(a, b, t) = 1/\rho \ln [(1 - a)/(1 - b)] = P(a, b) \tag{3.3}$$

for all $0 \le a \le b \le 1$ and for all t. Furthermore, if $a \le c \le b$, then $P(a, b, t) = P(a, c, t) + P(c, b, t)$.

Proof. The idea of the proof is to solve the ODE in (3.1) via definite integration and then take appropriate limits to represent the behavior of the intense advertising campaign pulse for an extremely short time interval.

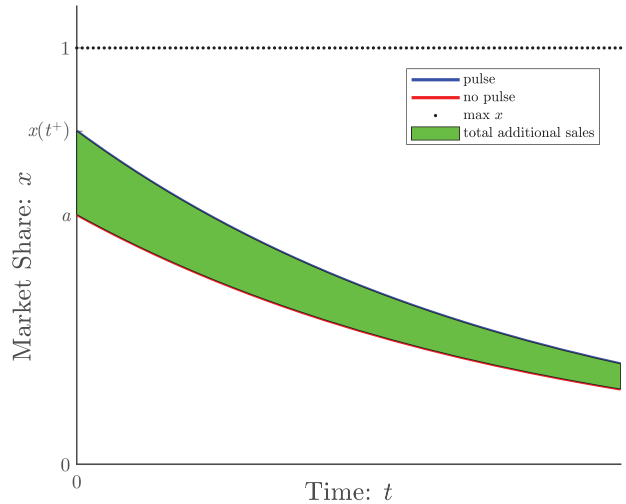

Figure 4: Total additional market share due to advertising pulse.

Temporarily fixing u and integrating (3.1) from t to $t + \delta t$ leads to

$$x(t + \delta t) = \mathrm{e}^{-(k+\rho u)\delta t} a + \mathrm{e}^{-(k+\rho u)(t+\delta t)} \rho u \int_t^{t+\delta t} \mathrm{e}^{(k+\rho u)s} \, ds$$

Evaluating the integral and letting $u \longrightarrow \infty$, $\delta t \longrightarrow 0$, and $u\delta t \longrightarrow P$ gives

$$x(t^+) = \mathrm{e}^{-\rho P}(a - 1) + 1$$

Since $x(t^+) = b$, the first part of the lemma follows readily.

The additive property in the lemma is a simple exercise using algebraic properties of logarithms. □

Some additional quantities of interest can be computed with the aid of Lemma 3.1. In particular, Vidale and Wolfe show that the immediate market share increase due to the advertising pulse is

$$x(t^+) - a = (1 - a)(1 - \mathrm{e}^{-\rho P})$$

Moreover, the total additional market share generated by the pulse campaign is

$$\int_0^\infty (x(t^+) - a)\mathrm{e}^{-kt} \, dt = \frac{(1-a)}{k}(1 - \mathrm{e}^{-\rho P}) \approx \frac{\rho P}{k}(1 - a) \tag{3.4}$$

if sales are small compared to saturation level (see Vidale and Wolfe [4], p. 378). In (3.4), we have a nice use of an improper integral. Visually, our students appreciate the computation as the area of the "infinite" shaded region in Figure 4, where the curves are $x(t^+)\mathrm{e}^{-kt}$ and $a\mathrm{e}^{-kt}$. From an instructional point of view, we are certainly getting excited about the wonderful mathematics that has arisen in our analysis of the Vidale-Wolfe model. Indeed, we have even used a Taylor polynomial approximation to an exponential function in (3.4), i.e., $\mathrm{e}^{-\rho P} \approx 1 - \rho P$. Perhaps one could use this example instead of small amplitude estimations in pendulum analysis (see, e.g., Nagle et al. [1], p. 226). Thus, the Vidale-Wolfe model provides yet another benefit for our students who lack background in physics.

4 OPTIMAL CONTROL AND GREEN'S THEOREM

For those particularly motivated students in our course, we can suggest that the main use of Lemma 3.1 is in specifying an optimal control for the following optimization problem. One wishes to maximize the present value of the profit stream up to horizon T, i.e, over the time interval $[0, T]$. In other words, one attempts to solve

$$\text{Problem (P):} \qquad \max_{0 \le u(t) \le Q} \left\{ J = \int_0^T (\pi x - u) e^{-it}\, dt \right\}$$

subject to the state equation (3.1) and the fixed endpoint constraint $x(T) = x_T$. Here, π is maximum sales revenue potential, Q is maximum allowable rate of advertising expenditure, and i is discount rate. This is a problem in the calculus of variations. Stated another way, we seek an advertising function $u(t)$ that makes the functional J as large as possible, while $x(t)$ still satisfies the Vidale-Wolfe IVP and the further restriction that the total market share at time T is fixed at a predetermined level, i.e., $x(T) = x_T$.

In Winkel [5], another author points out how an optimal control problem related to the number of worker and queen bees in a colony can find a place in an ODE course with an emphasis on modeling. Our problem is similar. Indeed, Sethi first constructs an optimal strategy using a combination of pulsing, singular control, and bang-bang control in the case when $Q = \infty$, i.e., when the advertising strategy need not be bounded. The details of the solution follow from Green's theorem and are accessible, with some work, to our best students. We present some of the details in the Appendix. Through the analysis, one is led to a switching diagram like Figure 5, where x^s is a constant market share depending on k, i, π, and ρ that arises in the analysis of Problem (P). The formula for x^s is given in the Appendix. The corresponding control is u^s, obtained by solving for u in (3.1) after setting $\frac{dx}{dt} = 0$ and $x = x^s$. Three portions of the tx-plane are identified and labeled as I, II, and III.

$$\begin{aligned}
I &= \{(t,x) \mid x_s < x < 1, 0 < t < T\} \cup \{(t,x) \mid T - \tfrac{1}{k}\ln(x_T/x^s) \le t < T, x = \tfrac{x_s^2}{x_T} e^{-k(t-T)}\} \\
II &= \{(t,x) \mid x = x^s, 0 < t < T - \tfrac{1}{k}\ln(x_T/x^s)\} \\
III &= \{(t,x) \mid 0 < t < T - \tfrac{1}{k}\ln(x_T/x^s), 0 < x < x^s\}
\end{aligned}$$

While the details are best left to the interested reader, we point out the following idea of the switching diagram and optimal advertising policy. If $(t, x) \in III$, apply pulses until reaching x^s. Then apply u^s to maintain x^s until advertising is halted at time $T - \frac{1}{k}\ln(x_T/x^s)$ in order to meet the target x_T. When $(t, x) \in I$, advertising is halted, i.e., $u = 0$. The optimality of this advertising strategy is established in Sethi [3], p. 6-7.

5 SHARING THE MODEL AND CONCLUSIONS

We begin with some concluding thoughts about the reading and writing ODE project. Students found many refreshingly different models with broad appeal, an important point considering that NU does not have a physics major. Commentary from course evaluations supports our observations about the appeal and utility of the project. A student wrote: [I]

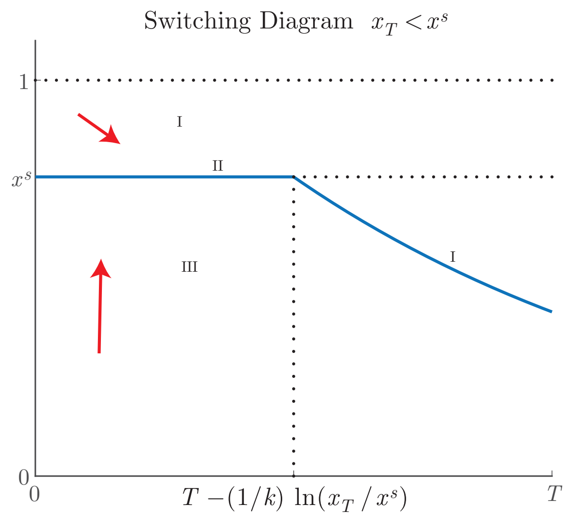

Figure 5: Diagram adapted from Sethi [2], Fig. 1. Arrows indicate solution trajectories under the optimal control policy.

"…Learned how to apply math in things that interest me." Another contribution made note that the "…Real life examples helped to stimulate interest and see applicability." In an effort to quantify the benefits of the project, we have begun to perform statistical analysis on certain final exam questions that test the ability of our students to create initial value problems from a given problem description in words. Preliminary results to this end suggest that the project is associated with improved performance. A larger sample size, in addition to a more robust experimental design, is necessary before statistically significant course improvement can be claimed.

We also provide some concluding thoughts about the Vidale-Wolfe marketing model and its many potential uses in a first ODE course. First and foremost, the model is simple, accessible, and intriguing to students whose majors are not directly STEM related. As such, including the model in a first ODE course is particularly salient at a university without physics students because it provides an alternative approach for content that is usually described within the context of a physical scenario. The analysis touches on many ODE and calculus techniques, including break-points; impulse forcing; improper integrals and area under a curve (representing total additional market share due to pulse advertising); and Taylor polynomials for approximation. For those motivated students that pursue the associated control problem, there is an opportunity to expand their understanding of optimization problems. Calculus of variations is a classical subject, but still an area of active research today. These students will consider a real-life use of Green's theorem by working through the derivation of the optimal control.

As scholar-educators, many of us tinker with the particulars of our courses every time we teach them. This is also the case with our implementation of the ODE project. However, a common theme of the implementations has emerged: **F**ind, **P**rocess, **S**hare. Our ODE project led to our "discovery" of the Vidale-Wolfe marketing model, and subsequent processing of the model has led us to share our story through this work.

Appendix A SOLUTION OF PROBLEM (P)

We present some details, following Sethi [2], concerning the solution of Problem (P) in the case when $Q = \infty$. For ease of exposition, we recall the scaled Vidale-Wolfe ODE (3.1):

$$\frac{dx}{dt} = \rho u(t)(1-x) - kx$$

Solving for $u\,dt$ gives

$$u\,dt = \frac{1}{\rho(1-x)}(dx + kx\,dt)$$

Let Γ be any curve in the tx-plane, and substitute $u\,dt$ into J to obtain

$$J_\Gamma := \int_\Gamma \left(\pi - \frac{1}{\rho(1-x)}k\right) x\mathrm{e}^{-it}dt - \frac{1}{\rho(1-x)}\mathrm{e}^{-it}dx$$

If Γ is simple, piecewise-smooth, and closed, bounding a region R, Green's theorem gives

$$J_\Gamma = \frac{1}{\rho}\int\int_R \left(\frac{i}{1-x} - \pi\rho + \frac{k}{(1-x)^2}\right)\mathrm{e}^{-it}dt\,dx$$

For control, we determine where the integrand of J_Γ is positive and negative. Equating the integrand to zero gives a quadratic equation in $1/(1-x)$. Solving the quadratic for x and requiring $0 \le x \le 1$ leads to

$$x^s = \max\left\{1 - \frac{2k}{\sqrt{i^2 + 4k\pi\rho} - i}, 0\right\}$$

The corresponding control is

$$u^s = \frac{kx^s}{\rho(1-x^s)} = \begin{cases} 0 & \text{if} \quad x^s = 0 \\ \frac{1}{2\rho}(\sqrt{i^2 + 4k\pi\rho} - i - 2k) & \text{if} \quad x^s \neq 0 \end{cases}$$

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