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A Refined Saddle Point Theorem and Applications

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Abstract

Under adequate conditions on g, we show the density in $L^2((0,\pi),(0,2\pi))$ of the set of functions p for which

$$u_{tt}(x,t) - u_{xx}(x,t) = g(u(x,t)) + p(x,t)$$

has a weak solution subject to

$$\begin{aligned} u(x,t) &= u(x,t+2\pi) \\ u(0,t) &= u(\pi,t) = 0. \end{aligned}$$

To achieve this, we prove a Saddle Point Principle by means of a refined variant of the deformation lemma of Rabinowitz.

Generally, inf-sup techniques allow the characterization of critical values by taking the minimum of the maximae on some particular class of sets. In this version of the Saddle Point Principle, we introduce sufficient conditions for the existence of a saddle structure which is not restricted to finite-dimensional subspaces.

Contents

Chapter 1

Introduction

In this thesis, under adequate conditions on g, we show the density in $L^2((0,\pi),(0,2\pi))$ of the set of functions p for which

$$u_{tt}(x,t) - u_{xx}(x,t) - g(u(x,t)) - p(x,t) = 0 \quad (1.1)$$

has a *weak solution* subject to

$$u(x,t) = u(x,t+2\pi) \text{ and } u(0,t) \quad = u(\pi,t) = 0 \quad (1.2)$$

for all $x \in [0,\pi]$, $t \in \mathbb{R}$.

This extends the results of Willem (1981) and Hoffer (1982), which relate the asymptotic behavior of g to the eigenvalues of $\Box$ subject to Equation 1.2.

To achieve this, we prove an extension of the Saddle Point Principle (Rabinowitz, 1984: Theorem 4.6) by means of a refined variant of the deformation lemma (Rabinowitz, 1984: Lemma A.4). *Sup-inf* techniques such as used here allow the characterization of critical values by taking the minimum of the maximae achieved by some particular class of sets. In this version of the Saddle Point Principle, we place more stringent restrictions on the functional I in order to permit a saddle-structure which is not restricted to a finite-dimensional subspace.

In order to state our Saddle Point Principle, recall that a functional $K : E \to E$ on a real Hilbert space E is compact if it is continuous and if for every bounded sequence $\{w_n\} \subset E$, $\{K(w_n)\}$ has a convergent subsequence.

Theorem 1 (Refined Saddle Point Principle)**.** *Let E be a real Hilbert space, and let $E = U \oplus V$ where U and V are closed subspaces. Let $P : E \to V$ be the projection of E onto V.*

Consider $I \in C^1(E,\mathbb{R})$ such that

(S1) *$P\nabla I = P\mathrm{id} + K_0(u+v)$ for some compact function $K_0 : E \to V$,*

(S2) *There exists a compact function $\phi : V \to U$, an $\alpha \in \mathbb{R}$, and a neighborhood D of 0 in V such that $I(v + \phi(v)) \le \alpha$ for all $v \in \partial D$,*

(S3) *There exists $\beta > \alpha$ such that $I_U \ge \beta$.*

Define

$$\begin{aligned}\Gamma = \{h \in & C(\bar{D}, E) | h \text{ is bounded}, \\ & Ph(v) = d(v)u + K(v) \\ & \qquad \text{for } K \text{ compact}, d(v) > \delta_1 > 0 \in C(\bar{D}, \mathbb{R}), \\ & \text{and } K|_{\partial D} = 0 \text{ and } d|_{\partial D} = 1.\}\end{aligned} \tag{1.3}$$

Then

$$c = \inf_{h \in \Gamma} \sup_{v \in \bar{D}} I(h(v)) \tag{1.4}$$

is either a critical value of I, or else there exist $\{w_n\} \in E$ such that $I(w_n) \to c$ and $||\nabla I(w_n)|| \to 0$.

This saddle point principle can be applied to a suitable functional to allow us to find weak solutions of Equation 1.1 subject to Equation 1.2. We first construct this functional.

Let $\Omega = [0, 2\pi] \times [0, \pi]$.

The eigenvalues of $\Box = \partial_{tt} - \partial_{xx}$ subject to Equation 1.2 are $\lambda_{k,j} = k^2 - j^2$, for $k = 1, 2, \ldots$ and $j = 0, 1, \ldots$. Each $\lambda_{k,j}$ has orthonormal associated eigenfunctions

$$\phi_{k,j} = \frac{\sin kx \cos jt}{\int_\Omega (\sin kx \cos jt)^2 dx dt},$$
$$\psi_{k,j} = \frac{\sin kx \sin jt}{\int_\Omega (\sin kx \cos jt)^2 dx dt}.$$

Let σ be the set of these eigenvalues.

Definition 2. We will let H be the subspace of $L^2(\Omega)$ of elements of the form

$$\sum_{k=1, j=0}^{\infty, \infty} a_{k,j}\phi_{k,j} + b_{k,j}\psi_{k,j}$$

where

$$\sum_{k=1, j=0}^{\infty, \infty} |\lambda_{k,j}|(a_{k,j}^2 + b_{k,j}^2) < \infty.$$

We define the inner product

$$\left\langle \sum_{k=1,j=0}^{\infty,\infty} a_{k,j}\phi_{k,j} + b_{k,j}\psi_{k,j}, \sum_{k=1,j=0}^{\infty,\infty} \alpha_{k,j}\phi_{k,j} + \beta_{k,j}\psi_{k,j} \right\rangle_1$$
$$= \sum_{k=1,j=0}^{\infty,\infty} (1 + |\lambda_{k,j}|)(a_{k,j}\alpha_{k,j} + b_{k,j}\beta_{k,j}).$$

We denote by $||\cdot||_1$ the norm defined by $< \cdot, \cdot >_1$.

Define the subspace U as the closure of

$$\text{span}\{\phi_{k,j}, \psi_{k,j} | k^2 - j^2 \leq 1\} \tag{1.5}$$

and V as the closure of

$$\text{span}\{\phi_{k,j}, \psi_{k,j} | k^2 - j^2 > 1\}. \tag{1.6}$$

For $w_1 = \sum_{j,k} a_{k,j}\phi_{k,j} + b_{k,j}\psi_{k,j} \in H$ and $w_2 = \sum_{k,j} \alpha_{k,j}\phi_{k,j} + \beta_{k,j}\psi_{k,j} \in H$, define

$$B(w_1, w_2) = \sum_{k,j} \lambda_{k,j}(a_{k,j}\alpha_{k,j} + b_{k,j}\beta_{k,j}). \tag{1.7}$$

Note that for $w \in C^1(\Omega)$, $B(w,w) = \int_\Omega (w_t)^2 - (w_x)^2 dxdt$.

Define $J : E \to \mathbb{R}$ by

$$J(w) = \frac{B(w,w)}{2} - \int_\Omega (G(w(x,t)) + pw(x,t))dxdt, \tag{1.8}$$

where $G = \int_0^s g(t)dt$.

Definition 3 (Weak Solution)**.** We say $w \in E$ is a *weak solution* of Equation 1.1 subject to Equation 1.2 if w is a critical point of J.

Note that for all $w, y \in E$,

$$< \nabla J(w), y >= B(w,y) - \int_\Omega (g(w) + p)ydxdt. \tag{1.9}$$

In defining the asymptotic behavior of $G(x)$, we make use of the *Fucik spectrum* (see Chapter 2), and in particular the characterization of $b_1(a)$, the smallest $b \geq a$ such that (a,b) is in the Fucik spectrum, found in Castro and Chang (2010).

Theorem 4. *Let $a_0, a \in (1,3)$ and $b \in (a, b_1(a))$, C and $M \in \mathbb{R}$. If $g(s)$ is Lipschitz continuous, and for all $s \in \mathbb{R}$,*

$$\frac{a_0 s^2}{2} - C < G(s) < \frac{a(s_+)^2}{2} + \frac{b(s_-)^2}{2} + C, \tag{1.10}$$

either Equation 1.1 subject to Equation 1.2 has a weak solution, or there exists some sequence p_n converging in $L^2(\Omega)$ to p such that there exists a weak solution to $\Box u = g(u + p_n$ for all n.

Chapter 2

Preliminaries

Let Ω be an open subset of $\mathbb{R}^m$.

Let E be a real Hilbert space and $E^\star$ its dual. If the functional $I : E \to \mathbb{R}$ is continuous, and there is a continuous linear operator $A(u) : E \to \mathbb{R}$ such that for all $\epsilon > 0$, there is a neighborhood $\delta_r(0)$ of 0 such that for all $h \in \delta_r(0)$,

$$|I(u+h) - I(u) - < A(u), h > | \leq \epsilon ||h||_E,$$

then we say I is differentiable at u. If $A : E \to E^\star$ is continuous, we say $I \in C^1(E, \mathbb{R})$. By the Riezs representation theorem, there exists a unique element $v \in E$ such that $< v, y >=< A(u), y >$ for all $y \in E$. We denote v as $\nabla I(u)$ and call it the gradient of I at u.

2.1 L^p spaces

Let $L^p(\Omega)$ be

$$\{u : \Omega \to R | u \text{ is measurable and } \int_\Omega |u|^p d\mu < \infty\},$$

and then let

$$||u||_{L^p} = (\int_\Omega |u|^p d\mu)^{\frac{1}{p}}.$$

2.2 Sobolev Space

In order to define Sobolev spaces, we first define the notion of weak derivative (or distributional derivative), which extends the standard notion of derivative. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a n-tuple of nonnegative integers, and let

$$D^\alpha = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

If $\phi \in C^\infty(\Omega)$, we say that ϕ has compact support in Ω if the closure of $\{x|\phi(x) \neq 0\}$ is a compact subset of Ω. If, for any C^∞ function ϕ with compact support in Ω,

$$\int_\Omega v\phi d\mu = (-1)^{\alpha_1+\alpha_2+\cdots+\alpha_n} \int_\Omega u D^\alpha \phi d\mu,$$

we say that v is a weak α-derivative of u, or the α-derivative of u in the sense of distributions. We write $D^\alpha u = v$.

Let k be a positive integer. If for all α with $|\alpha| \leq k$, $D^\alpha u \in L^p(\Omega)$, we say u belongs to the Sobolev space $W^{k,p}(\Omega)$ with norm

$$||u||_{W^{k,p}} = (\sum_{|\alpha| \leq k} ||D^\alpha u||_{L^p}^p)^{1/p}.$$

Note that $|| \cdot ||_{w^{1,2}} = || \cdot ||_1$.

2.3 Fucik Spectrum

Let Ω be a measurable subset of $\mathbb{R}^n$. Let $u_+ = \max(0, u)$ and $u_- = \max(0, -u)$.

Let $\mathcal{L}$ be a linear operator defined in a dense subset of $L^2(\Omega)$. The Fucik spectrum of $\mathcal{L}$ is defined as

$$\{(a, b)|\mathcal{L}(u) = au_+ - bu_- \text{ has nontrivial solutions on } \Omega\}.$$

We will make use of the variational characterization of the Fucik spectrum of $\Box = \partial_{tt} - \partial_{xx}$ found in Castro and Chang (2010).

Let B be as in Equation 1.7, and define

$$g_{a,b}(s) = as_+ - bs_-.$$

Just as $J : E \to \mathbb{R}$ corresponds to Equation 1.1 subject to Equation 1.2, we define

$$J_{a,b}(w) = \frac{1}{2}\left[B(w, w) - \int_0^{2\pi} \int_0^\pi w(g_{a,b}(w))dxdt\right]$$

corresponding to $\Box u = g_{a,b}(u)$.

Lemma 5. *Let E be a separable real Hilbert space with closed subspaces X, Y such that $E = X \oplus Y$. Let $J : E \to \mathbb{R}$ be a functional of class C^1. If there exists $m > 0$ such that*

$$< \nabla J(x_1 + y) - \nabla J(x_2 + y), x_1 - x_2 > \leq -m||x_1 - x_2||^2$$

for all $x_1, x_2 \in X$, $y \in Y$, then there exists a function $r : Y \to X$ such that $J(y + r(y)) = \max\{J(y + x) | x \in X\}$.

This result is taken from Castro and Chang (2010), Theorem 1.

Let U and V be as defined in Equations 1.5 and 1.6.

Corollary 6. *For $a \in (1, 3)$ and $b \geq a$, there exists a function $r_{a,b} : V \to U$ such that $J_{a,b}(v + r_{a,b}(v)) = \max\{J_{a,b}(v + u) | u \in U\}$.*

For convenience, write

$$\tilde{J}_{a,b}(v) = J_{a,b}(v + r_{a,b}(v)).$$

Proof. This proof follows Castro and Chang (2010).

For all $u_1, u_2 \in U, v \in V$,

$$\begin{aligned} &< \nabla J_{a,b}(v + u_1) - \nabla J_{a,b}(v + u_2), u_1 - u_2 >_1 \\ &\quad = B(u_1 - u_2, u_1 - u_2) - \int_0^{2\pi} \int_0^{\pi} (u_1 - u_2)(a(v + u_1)_+ \\ &\quad\quad - a(v + u_2)_+ - b(v + u_1)_- + b(v + u_2)_-)dxdt \\ &\quad = B(u_1 - u_2, u_1 - u_2) \\ &\quad\quad - \int_0^{2\pi} \int_0^{\pi} (u_1 - u_2)(g(v + u_1) - g(v + u_2))dxdt \\ &\quad \leq B(u_1 - u_2, u_1 - u_2) - a||v + u_1||_0^2 - a||v + u_2||_0^2 \\ &\quad \leq B(u_1 - u_2, u_1 - u_2) - a||u_1 - u_2||_0^2 \\ &\quad \leq -m||u_1 - u_2||_1^2, \end{aligned} \tag{2.1}$$

where $m = \inf\{(a - \lambda_{k,j})/(1 + |\lambda_{k,j}|) > 0 | \lambda_{k,j} \leq 1\}$. Either $\lambda_{k,j} \in (0, a)$ and $(a - \lambda_{k,j})/(1 + |\lambda_{k,j}|)$ is positive (and there exist only finitely many such $\lambda_{k,j}$), or $\lambda_{k,j} < 0$ and $(a - \lambda_{k,j})/(1 + |\lambda_{k,j}|)$ approaches 1 as $\lambda_{k,j} \to -\infty$. In either case, $m > 0$.

Then by Lemma 5, there exists a function $r_{a,b} : V \to U$ as desired. □

Note that if v_n converges to $\bar{v}$ in $L^2(\Omega)$, then $r_{a,b}(v_n)$ converges to some $\bar{z}$ in $L^2(\Omega)$. Then

$$0 = B(\bar{z}, u) - \int_\Omega g_{a,b}(\bar{z} + \bar{v})udxdt \tag{2.2}$$

for any $u \in U$. See (16) of Castro and Chang (2010).

Lemma 7. *For $a \in \mathbb{R}^+ \setminus \sigma$, define*

$$b_1(a) = \sup\{b \geq a | J_{a,\beta}(v + r_{a,\beta}(v)) > 0 \text{ for all } \beta \in (a,b), v \in V \setminus \{0\}.$$

Then

- $(a, b_1(a))$ *is in the Fucik spectrum whenever* $b_1(a) < \infty$,
- *If* $b \in (a, b_1(a))$, (a,b) *is not in the Fucik spectrum, and*
- *For* $b > a$, (a,b) *is in the Fucik spectrum if and only if the restriction of* $\tilde{J}_{a,b}(v)$ *to* $\{v \in V | ||v||_1 = 1\}$ *has a critical point on* $\{v \in V | ||v||_1 = 1, \tilde{J}_{a,b} = 0\}$.

This is Theorem 2, Castro and Chang (2010).

2.4 Schauder Fixed Point Theorem

The Saddle Point Principle, as formulated in Rabinowitz (1984), applies on a Hilbert space $E = U \oplus V$, where U and V are closed subspaces and V has finite dimension. A vital step in this result is to show that all the deformations of a neighborhood $D \subset V$ which fix ∂D intersect the subspace U. Provided V has finite dimension, this may be established through the Brouwer fixed-point theorem:

Theorem 8 (Brower Fixed-Point Theorem)**.** *Let B be a compact, convex, and nonempty subset of $\mathbb{R}^n$. For any continuous function $f : B \to B$, there exists some $x \in \mathbb{R}^n$ such that f has a fixed point $f(x) = x$.*

This is a well-known result, see, for example, Zeidler (1985).

In this work V need not be finite dimensional. Here we make use of an extension of Theorem 8.

Theorem 9 (Schauder Fixed Point Theorem)**.** *Let B be a closed, bounded, convex, and nonempty subset of a Hilbert space E. For any compact function $K : B \to B$, there exists some $x \in E$ such that $K(x) = x$.*

This can likewise be found in Zeidler (1985).

The following corollary follows from Theorem 9.

Corollary 10 (Leray-Schauder principle)**.** *Let E be a Hilbert space. If $K : E \to E$ is compact, and if there is some $r > 0$ such that for all $x \in E$ with $||x|| = r$, $K(x) \neq \lambda x$ for all $\lambda > 1$, then K has a fixed point $x = K(x)$, with $||x|| \leq r$.*

2.5 Palais-Smale Condition

Let E be a real Hilbert space. A functional $I \in C^1(E, \mathbb{R})$ is said to satisfy a Palais-Smale condition at c if, for every $\{u_n\} \subset E$ for which $I(u_n) \to c$ and $\nabla I(u_n) \to 0$, $\{u_n\}$ has a convergent subsequence.

This condition acts as a substitute for compactness, by allowing us to find convergent subsequences in certain cases.

Chapter 3

Proof of the Deformation Lemma

Let $E, I, \nabla I$ be as defined in Chapter 2.

Lemma 11. *Let $\tilde{E} = \{x \in E | \nabla I(x) \neq 0\}$. There exists a locally Lipschitz continuous pseudogradient vector field $V : \tilde{E} \to E$ satisfying for all $u \in \tilde{E}$*

(V1) $||V(u)|| \leq 2||\nabla I(u)||$ *, and*

(V2) $< \nabla I(u), V(u) > \geq \frac{1}{2}||\nabla I(x)||^2$.

A proof can be found in, for example, Rabinowitz (1984: Lemma A.2).

Before we prove Theorem 1, we prove a variation of the Deformation lemma, as in Rabinowitz (1984).

Lemma 12 (Deformation Lemma)**.** *Let E be a real Hilbert space and let $I \in C^1(E, \mathbb{R})$ satisfy the Palais-Smale Condition at $c \in \mathbb{R}$.*

Then for all $\bar{\epsilon} > 0$, there exists an $\epsilon \in (0, \bar{\epsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

(I1) $\eta(0, u) = u$ *for all* $u \in E$.

(I2) $\eta(t, u) = u$ *for all* $t \in [0,1]$ *if* $I(u) \notin [c - \bar{\epsilon}, c + \bar{\epsilon}]$.

(I3) $||\eta(t, u) - u|| \leq \frac{1}{4}$ *for all* $t \in [0,1]$, $u \in E$.

(I4) $I(\eta(t, u)) \leq I(u)$ *for all* $t \in [0,1]$ *and* $u \in E$.

(I5) If $K_c := \{u \in E | I(u) = c$ *and* $\nabla I(u) = 0\} = \emptyset$, $\eta(1, A_{c+\epsilon}) \subset A_{c-\epsilon}$ *where* $A_\zeta := \{x \in E | I(x) < \zeta\}$.

Proof: This proof is adapted from Rabinowitz (1984).

We claim there exist constants $b, \hat{\epsilon} > 0$ such that

$$||V(u)|| \geq b \text{ for all } u \in A_{c+\hat{\epsilon}} \setminus A_{c-\hat{\epsilon}}. \tag{3.1}$$

If not, there are sequences $b_n \to 0$, $\hat{\epsilon}_n \to 0$, and $u_n \in A_{c+\hat{\epsilon_n}} \setminus A_{c-\hat{\epsilon_n}}$ such that $||V(u_n)|| < b_n$. Because I satisfies Palais-Smale at c, a subsequence of u_n converges to some $u \in K_c$, which is empty. Thus, by contradiction, there are constants $b, \hat{\epsilon}$ as in Equation 3.1.

Then as Equation 3.1 still holds with $\hat{\epsilon}$ reduced, assume

$$0 < \hat{\epsilon} \leq \min(\bar{\epsilon}, \frac{b^2}{8}, \frac{1}{16}). \tag{3.2}$$

Take $\epsilon \in (0, \hat{\epsilon})$.

Define

$$A = \{u \in E; I(u) \leq c - \hat{\epsilon}\} \cup \{u \in E; I(u) \geq c + \hat{\epsilon}\}$$

and

$$B = \{u \in E; c - \epsilon \leq I(u) \leq c + \epsilon\}.$$

Therefore $A \cap B = \emptyset$. Let

$$g(x) = \frac{\text{dist}(x, A)}{\text{dist}(x, A) + \text{dist}(x, B)}.$$

Then $g = 0$ on A, $g = 1$ on B, and $0 \leq g \leq 1$. Define $h(s) = 1$ if $s \in [0, 1]$ and $h(s) = \frac{1}{s}$ otherwise.

Let $V(x) : E \to E$ be a locally Lipschitz continuous pseudogradient vector field; one must exist by Lemma 11.

Set

$$W(x) = -\frac{g(x)h(||V(x)||)V(x)}{4}. \tag{3.3}$$

Note that $W(x)$ is locally Lipschitz continuous, by construction.

Then define $\eta : [0, 1] \times E$ as

$$\frac{d\eta}{dt} = W(\eta), \text{ and } \eta(0, u) = u. \tag{3.4}$$

The Picard-Lindelöf theorem implies that for each $u \in E$, Equation 3.4 has a unique solution defined for t in a maximal interval $(t^-(u), t^+(u))$. We claim $t^{\pm} = \pm\infty$. Otherwise take $t_n \to t^+(u)$ with $t_n < t^+(u)$. Integrating Equation 3.4 shows

$$||\eta(t_{n+1}, u) - \eta(t_n, u)|| \leq \frac{|t_{n+1} - t_n|}{4}, \tag{3.5}$$

since $W(\cdot) \leq \frac{1}{4}$. Then $\eta(t_n, u)$ is a Cauchy sequence and hence converges to some $\bar{u}$ as $t_n \to t^+(u)$. Then the solution to Equation 3.4 with $\bar{u}$ as initial data furnishes a continuation of $\eta(t, u)$ to values of $t > t^+$, contradicting the maximality of $t^+(u)$. Similarly, $t^- = -\infty$.

The continuous dependence of Equation 3.4 on u implies $\eta \in C([0, 1] \times E, E)$ and the initial condition implies that (I1) holds. Also, as $\hat{\epsilon} < \bar{\epsilon}$,

$$\{u | I(u) < c - \bar{\epsilon}\} \cup \{u | I(u) > c + \bar{\epsilon}\} \subset A,$$

on which $g(x) = 0$. Thus (I2) holds.

Using $W(\cdot) \leq \frac{1}{4}$,

$$\begin{aligned} ||\eta(t, u) - u|| &= ||\eta(t, u) - \eta(0, u)|| \\ &= || \int_0^t \frac{d\eta}{ds} ds || \\ &\leq \frac{t}{4} \leq \frac{1}{4} \end{aligned}$$

for all $t \in [0, 1]$. Thus (I3) holds.

To verify (I4), first note that if $W(u) = 0$, $\eta(t, u) = u$ is the solution of Equation 3.4 and by uniqueness (I4) is trivially satisfied. Otherwise, if $W(u) \neq 0$, $V(u) \neq 0$ and because $\eta(t, u)$ exists and is unique, $V(\eta(t, u))$ is well defined. Then applying (V2),

$$\begin{aligned} \frac{dI(\eta(t, u))}{dt} &= -\nabla I(\eta(t, u)) \cdot \frac{d\eta(t, u)}{dt} \\ &= - < \nabla I(\eta(t, u)), \frac{g(\eta(t, u))h(||V(\eta(t, u))||)V(\eta(t, u))}{4} > \\ &\leq -\frac{g(\eta(t, u))h(||V(\eta(t, u))||)||\nabla I(\eta(t, u))||^2}{8} \\ &\leq 0, \end{aligned} \tag{3.6}$$

which proves (I4).

Finally, to prove (I5), observe that if $u \in A_{c-\epsilon}$ then $I(\eta(t, u)) < c - \epsilon$ by (I4). Thus we need only consider the case of $u \in Y \equiv A_{c+\epsilon} \setminus A_{c-\epsilon}$.

Let $u \in Y$. Since $g = 0$ on $A_{c-\hat{\epsilon}}$, the orbit $\eta(t,u)$ cannot enter $A_{c-\hat{\epsilon}}$; that is, $I(\eta(t,u)) \geq c - \hat{\epsilon}$ for all $t \geq 0$. Then

$$I(\eta(0,u)) - I(\eta(t,u)) \leq \epsilon + \hat{\epsilon} \leq 2\hat{\epsilon} \tag{3.7}$$

for all $t \geq 0$.

Suppose by way of contradiction that $\eta(t,u)$ does not enter $A_{c-\epsilon}$ for any $t \in [0,1]$. Then $g(\eta(t,u)) = 1$ for all $t \in [0,1]$.

If for $t \in [0,1]$, $||V(\eta(t,u))|| \leq 1$, then $h(||V(\eta(t,u))||) = 1$ and by Equations 3.1 and 3.6,

$$\frac{dI(\eta(t,u))}{dt} \leq -\frac{b^2}{4}. \tag{3.8}$$

On the other hand, if for $t \in [0,1]$ $||V(\eta(t,u))|| > 1$, then $h(||V(\eta(t,u))||) = ||V(\eta(t,u))||^{-1}$, so Equation 3.6 implies

$$\begin{aligned}\frac{dI(\eta(t,u))}{dt} &= -\frac{||V(\eta(t,u))||}{8}\\ &\leq -\frac{1}{8}.\end{aligned}$$

Thus for all $t \in (0,1)$, $u \in A_{c-\epsilon} \setminus A_{c+\epsilon}$,

$$\frac{dI(\eta(t,u))}{dt} \leq -\min(\frac{b^2}{4}, \frac{1}{8}), \tag{3.9}$$

and integrating this, together with Equation 3.7, gives

$$\min(\frac{b^2}{4}, \frac{1}{8}) \leq I(\eta(0,u)) - I(\eta(t,u)) \leq 2\hat{\epsilon}, \tag{3.10}$$

which contradicts Equation 3.2.

Thus $\eta(1,u) \in A_{c-\epsilon}$. Thus 6. is proved and the proof is complete. □

In preparation for our proof of the saddle point principle, we prove two lemmas about compact functions.

Lemma 13. *Let E be a real Hilbert space. Let $K : [0,1] \times E \to E$ be compact. Then $K_1(v) = \int_0^1 K(t,v)dt$ is compact.*

Proof. Consider a sequence $\{v_n\}$ that is bounded in E.

The compactness of K implies that for $j = 1, 2, \ldots$, there exists a finite dimensional subspace Z_j with projection $P_j : V \to Z_j$ such that for all t, v_n,

$$K(t,v_n) = P_j(K(t,v_n)) + y_n$$

where $||y_n|| < \frac{1}{2^j}$ and $Z_n \subset Z_{n+1}$.

Let $\{v_{n(1,k)}\}$ be a subsequence of $\{v_n\}$ such that $\int_0^1 P_1(K(t, v_{n_k}))dt$ converges, and call this limit A_1.

Inductively define $\{v_{n(j+1,k)}\}$ as a subsequence of $\{v_{n(j,k)}\}$ such that $\int_0^1 P_{j+1}(K(t, h(v_{n(j+1),k})))dt$ converges, and call this limit A_{j+1}.

For any A_j, projecting into Z_{j-1} shows

$$A_j = A_{j-1} + C_{j-1}$$

for some $||C_{j-1}|| < \frac{1}{2^{j-1}}$.

Fix $\epsilon > 0$. Then consider

$$\begin{aligned} &||\int_0^1 P_k(K(t, v_{n(k,k)})) - P_{k+1}(K(t, v_{n(k+1,k+1)}))dt|| \\ &= ||\int_0^1 P_k(K(t, v_{n(k,k)})) - P_{k+1}(K(t, v_{n(k+1,k+1)}))dt \\ &\quad + \int_0^1 P_k(K(t, v_{n(k,k+1)})) - P_k(K(t, v_{n(k,k+1)}))dt||. \end{aligned}$$

Choose N_1 such that $||K(t, v_n) - P_{N_1}(K(t, v_n))|| \leq \frac{1}{2^{N_1}} \leq \frac{\epsilon}{4}$. Then for $k > N_1$,

$$\begin{aligned} &||\int_0^1 P_k(K(t, v_{n(k,k)})) - P_k(K(t, v_{n(k,k+1)}))|| \\ &< ||\int_0^1 P_{N_1}(K(t, v_{n(N_1,k)})) - P_{N_1}(K(t, v_{n(N_1,k+1)}))|| + \frac{\epsilon}{4}. \end{aligned} \tag{3.11}$$

And because $v_{N_1,k}$ converges in Z_{N_1}, there is N_2 such that for $k > N_2$,

$$||\int_0^1 P_{N_1}(K(t, v_{n(N_1,k)})) - P_{N_1}(K(t, v_{n(N_1,k+1)}))|| \leq \frac{\epsilon}{4}. \tag{3.12}$$

Then for $k > \max\{N_1, N_2\}$,

$$||\int_0^1 P_k(K(t, v_{n(k,k)})) - P_k(K(t, v_{n(k,k+1)}))|| \leq \frac{\epsilon}{2}. \tag{3.13}$$

Additionally, choose N_2 sufficiently large such that for all $k > N_2$,

$$\begin{aligned} &||\int_0^1 P_k(K(t, v_{n(k,k+1)})) - P_{k+1}(K(t, v_{n(k+1,k+1)}))dt|| \\ &\leq ||\int_0^1 C_k dt|| < \frac{1}{2^k} < \frac{\epsilon}{2}. \end{aligned} \tag{3.14}$$

Then taken together, Equation 3.13 and Equation 3.14 imply

$$|| \int_0^1 P_k(K(t, v_{n(k,k)})) - P_{k+1}(K(t, v_{n(k+1,k+1)}))dt|| < \epsilon,$$

and

$$\{\int_0^1 P_k(K(t, v_{n(k,k)})dt\}$$

is thus a Cauchy sequence. Thus

$$\int_0^1 K(t, v_n)dt$$

is compact. □

Lemma 14. *Let E be a real Hilbert space. Let $M : E \to \mathbb{R}$ be continuous and map bounded sets onto bounded sets, and let $K : E \to E$ be compact. Then $M(v)K(v)$ is compact.*

Proof. Consider a bounded sequence $\{v_n\} \subset E$.

$M(v_n)$ and $K(v_n)$ are both bounded, thanks to the compactness of K, so choose $\mathcal{K} > ||K(v_n)||$ and $\mathcal{M} > |M(v_n)|$.

There is some subsequence v_{n_k} such that $M(v_{n_k})$ converges to some $\bar{M}$ and $K(v_{n_k})$ converges to some $\bar{K}$.

Fix $\epsilon > 0$ and take R such that for all $k > R$,

$$||\Delta(K)|| = ||\bar{K} - K(v_{n_k})|| < \frac{\epsilon}{6\max\{\mathcal{M}, \mathcal{K}\}}$$

and

$$||\Delta(M)|| = |\bar{M} - M(v_{n_k})| < \frac{\epsilon}{6\max\{\mathcal{M}, \mathcal{K}\}}.$$

Then for all $k > R$,

$$\begin{aligned}||\bar{M}\bar{K} - M(v_{n_k})K(v_{n_k})|| &= ||(K(v_{n_k}) + \Delta(K))(M(v_{n_k}) + \Delta(M)) - M(v_{n_k})K(v_{n_k})|| \\ &= ||M(v_{n_k})\Delta(K) + \Delta(M)K(v_{n_k}) + \Delta(M)\Delta(K)|| \\ &< \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{3} < \epsilon.\end{aligned}$$

Thus $M(v)K(v)$ is compact. □

Chapter 4

Proof of the Saddle Point Principle

Proof. Assume $I \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale Condition at $c \in \mathbb{R}$.

Choose $h \in \Gamma$. Then for $v \in \bar{D}$, $Ph(v) = d(v)v + K(v)$, where $d(v) > \delta > 0$, $K|_{\partial D} = 0$, $Ph|\partial D = id$, where K is compact. Note that $\frac{K(v)}{d(v)}$ is likewise compact, by Lemma 14.

Define

$$\bar{K} = \begin{cases} -\frac{K(v)}{d(v)} & v \in \bar{D} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

$\bar{K}$ is continuous and compact. Furthermore, taking $r = \max\{||x|||x \in \partial D\}$, $K(x) = 0$ when $||x|| = r$. Then Corollary 10 implies that $\bar{K}$ has a fixed point z with $||z|| \leq r$. In fact, $z \in \bar{D}$; if not, $z = 0$, which is in $\bar{D}$. So $Ph(z) = 0$ and $h(z) = (id - P)(h(z)) \in U$. Thus by hypothesis (S2),

$$\sup_{v \in \bar{D}} I(h(v)) \geq I(h(z)) \geq \beta. \tag{4.2}$$

By Equation 1.4, this implies that $c \geq \beta$.

Let $A_s = \{u \in E | I(u) \leq s\}$ and $K_c = \{u \in E | I(u) = c \text{ and } V(u) = 0\}$.

Suppose by way of contradiction that c is not a critical value of I, so that $K_c = \emptyset$.

We invoke Lemma 12 with $\hat{\epsilon} = \frac{1}{2}(\beta - \alpha)$ to obtain $\epsilon > 0$ and $\eta : [0,1] \times E \to E$ satisfying properties (I1) through (I5).

Take $h \in \Gamma$ such that

$$\max_{v \in \bar{D}} I(h(v)) \leq c + \epsilon; \tag{4.3}$$

note that $\eta(1, h(v)) \leq c - \epsilon$, for all $v \in \bar{D}$. But if $\eta(1, h) \in \Gamma$, then there exists some $v_0 \in \bar{D}$ such that $I(h(v_0)) > c$, by Equation 1.4.

Thus it suffices to show that $\eta(1, h) \in \Gamma$ to establish a contradiction.

There are four conditions necessary and sufficient for $\eta(1, h) \in \Gamma$.

First, for $v \in \bar{D}$, $\eta(1, h(v)) \in C^1(\bar{D}, E)$, by Lemma 12.

Second, for $\{v_n\} \in \bar{D}$, $\{h(v_n)\}$ is bounded, and so too is $\{\eta(1, h(v_n))\}$ by (I3).

Third, taking $\eta(t, v)$ from Lemma 12,

$$\eta(t, h) = h - \int_0^t M(\eta(s, h))V(\eta(s, h))ds \tag{4.4}$$

where

$$M(x) = \frac{g(x)h(V(||x||))}{4}. \tag{4.5}$$

Recall that $P_V \nabla I = Pid + K_0$ for some compact K_0.

Let us construct a pseudogradient V for ∇I. Fix $x \in \tilde{E}$. Choose $r \in (0, 1)$ such that for all u satisfying $||u - x|| < r$,

$$\begin{aligned} ||Pu + K_0(x)|| &\leq ||Pu + K_0(u)|| + ||K_0(u) - K_0(x)|| \\ &< 2||Pu + K_0(u)|| + ||K_0(u) - K_0(x)|| \\ &< 2||\nabla I(u)||, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} < Pu + K_0(u), Pu + K_0(x) > &= ||Pu + K_0(u)||^2 + < Pu + K_0(u), K_0(x) - K_0(u) > \\ &> ||Pu + K_0(u)||^2 - \frac{1}{2}||Pu + K_0(u)||^2 \\ &= \frac{1}{2}||\nabla I(u)||^2. \end{aligned} \tag{4.7}$$

Note that we make use of $||\nabla I(u)|| > 0$.

Note also that any convex combination of vectors satisfying (V1) and (V2) also satisfies both conditions.

We can choose neighborhoods in E on which (V1) and (V2) are satisfied. Call this covering $\{N_u\}$.

Consider a locally finite refinement $\{M_j\}$. Let $\rho_j(x)$ be the distance from x to the complement of M_j. As x is in only finitely many of M_j, $\sum_k \rho_k(x)$ is finite.

Let $\beta_j = \frac{\rho_j(x)}{\sum_k \rho_k(x)}$, and define

$$V = \sum_j (\beta_j(u))(Pu + K_0(x_j)). \tag{4.8}$$

This still satisfies (V1) and (V2). Furthermore,

$$\sum_j (\beta_j(u))(Pu) = Pu.$$

As for the remaining, it is the finite sum of a bounded, continuous, scalar-valued function with a compact function, and by Lemma 14, is itself compact. Thus $V(u)$ is locally Lipschitz continuous, satisfies (V1) and (V2), and

$$V(u) = Pu + K_1(u) \tag{4.9}$$

for some compact function K_1.

Then applying the definition of $\eta(s, w)$ from Equation 3.4 with this choice of pseudogradient,

$$\begin{aligned} P(\eta(1, h(v))) &= P[h(v) - \int_0^1 M(\eta(s_1, h(v)))V(\eta(s_1, h(v)))ds_1] \\ &= P(h(v)) - \int_0^1 M(\eta(s_1, h(v)))PV(\eta(s_1, h(v)))ds_1. \end{aligned}$$

This can be expanded inductively.

Applying Equation 4.9 to expand $P(V(\cdot))$,

$$\begin{aligned} &\sum_{k=1}^{n-1} (-1)^k \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{k-1}} M(\eta(s_k, h(v)))[P(h(v)) \\ &\qquad + K_1(\eta(s_k, h(v)))]ds_k \dots ds_1 \\ &\quad + (-1)^n \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{n-1}} M(\eta(s_n, h(v)))PV(\eta(s_n, h(v)))ds_n \dots ds_1 \\ &= \sum_{k=1}^{n-1} (-1)^k \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{k-1}} M(\eta(s_k, h(v)))(P(h(v)) \\ &\qquad + K_1(\eta(s_k, h(v)))ds_k \dots ds_1 \\ &\quad + (-1)^n \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{n-1}} M(\eta(s_n, h(v)))(P(\eta(s_n, h(v))) \\ &\qquad + K_1(\eta(s_n, h(v))))ds_n \dots ds_1. \end{aligned} \tag{4.10}$$

Applying Equation 3.4 to expand $\eta(s, h(v))$, this becomes

$$
\begin{aligned}
&\sum_{k=1}^{n-1}(-1)^k \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{k-1}} M(\eta(s_k, h(v)))(P(h(v)) \\
&\quad + K_1(\eta(s_k, h(v)))ds_k \dots ds_1 \\
&+(-1)^n \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{n-1}} M(\eta(s_n, h(v)))(P(h(v)) \\
&\quad + K_1(\eta(s_n, h(v))))ds_n \dots ds_1 \\
&-(-1)^n \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{n-1}} M(\eta(s_n, h(v))) \\
&\quad \int_0^{s_n} M(\eta(s_{n+1}, h(v)))P(V(\eta(s_{n+1}, h(v))ds_{n+1} \dots ds_1 \\
&= \sum_{k=1}^{n}(-1)^k \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{k-1}} M(\eta(s_k, h(v)))(P(h(v)) \\
&\quad + K_1(\eta(s_k, h(v)))ds_k \dots ds_1 \\
&+(-1)^{n+1} \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_n} M(\eta(s_{n+1}, h(v))) \\
&\quad PV(\eta(s_{n+1}, h(v)))ds_{n+1} \dots ds_1.
\end{aligned}
\tag{4.11}
$$

Since

$$
\begin{aligned}
P(\eta(1, h(v))) = P(h(v)) &+ \sum_{k=1}^{0}(-1)^k \int_0^1 M(\eta(s_1, h(v))) \cdots \\
&\int_0^{s_{k-2}} M(\eta(s_{k-1}, h(v)))(P(h(v)) + K_1(\eta(s_{k-1}, h(v)))ds_{k-1} \dots ds_1 \\
&+ (-1)^1 \int_0^1 M(\eta(s_1, h(v)))PV(\eta(s_1, h(v)))ds_1,
\end{aligned}
$$

we have

$$
\begin{aligned}
P(\eta(1, h(v))) = &P(h(v)) + \sum_{k=1}^{N}(-1)^k \int_0^1 M(\eta(s_1, h(v))) \cdots \\
&\int_0^{s_{k-1}} M(\eta(s_k, h(v)))(P(h(v)) + K_1(\eta(s_k, h(v))))ds_k \dots ds_1 \\
&+ (-1)^{N+1} \int_0^1 M(\eta(s_1, h(v))) \cdots \\
&\int_0^{s_N} M(\eta(s_{N+1}, h(v)))V(\eta(s_{N+1}, h(v)))ds_{N+1} \dots ds_1,
\end{aligned}
$$

for all $N \in \mathbb{Z}^+$. This last term decays to 0 at least exponentially as $N \to \infty$, so

$$P(\eta(1,h(v))) = P(h(v)) + \sum_{k=1}^{\infty}(-1)^k \int_0^1 M(\eta(s_1,h(v))) \cdots \int_0^{s_{k-1}} M(\eta(s_k,h(v)))(P(h(v)) + K_1(\eta(s_k,h(v))))ds_k \dots ds_1,$$

which may be rewritten as

$$\begin{aligned} P(\eta(1,h(v))) = P(h(v))(1 + \sum_{k=1}^{\infty}(-1)^k \int_0^1 M(\eta(s_1,h(v))) \cdots \\ \int_0^{s_{k-1}} M(\eta(s_k,h(v)))ds_k \dots ds_1) \\ + \sum_{k=1}^{\infty}(-1)^k \int_0^1 M(\eta(s_1,h(v))) \cdots \\ \int_0^{s_{k-1}} M(\eta(s_k,h(v)))K_1(\eta(s_k,h(v)))ds_k \dots ds_1. \end{aligned} \tag{4.12}$$

Let

$$\begin{aligned} B(v) = P(h(v))(1 + \sum_{k=1}^{\infty}(-1)^k \int_0^1 M(\eta(s_1,h(v))) \cdots \\ \int_0^{s_{k-1}} M(\eta(s_k,h(v)))ds_k \dots ds_1) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} C(v) = \sum_{k=1}^{\infty}(-1)^k \int_0^1 M(\eta(s_1,h(v))) \cdots \\ \int_0^{s_{k-1}} M(\eta(s_1,h(v)))K_1(\eta(s_k,h(v)))ds_k \dots ds_1. \end{aligned} \tag{4.14}$$

Now define $\hat{d} \in C(\bar{D},\mathbb{R})$ as

$$\begin{aligned} \hat{d}(h(v)) &= 1 + \sum_{k=1}^{\infty}(-1)^k \int_0^1 M(\eta(s_1,h(v))) \\ &\cdots \int_0^{s_{k-1}} M(\eta(s_k,h(v)))ds_k \dots ds_1) \\ &\geq 1 - \sum_{k=1}^{\infty}(\frac{1}{4})^k \\ &= \frac{2}{3}, \end{aligned} \tag{4.15}$$

and note that

$$B(v) = P(h(v))\hat{d}(h(v)) = d(v)\hat{d}(v)v + \hat{d}(v)K(v). \tag{4.16}$$

Now consider $C(v)$. Note that $\bar{D}$ is bounded. Let v_n be a subset of $\bar{D}$. Then also $h(v_n)$ is bounded, by the definition of Γ. Then $\eta(t, h(v_n))$ is bounded, by Property (I3). By repeated application of Lemmas 14 and 13,

$$\int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{k-1}} M(\eta(s_k, h(v)))K_1(\eta(s_k, h(v)))ds_k \dots ds_1$$

is compact.

Fix $\epsilon > 0$, and consider $||C(v_n) - C(v_m)||$. Choose N_3 such that

$$\begin{aligned} || \sum_{k=N_3}^{\infty} \Bigg(\int_0^1 & M(\eta(s_1, h(v_n))) \\ & \cdots \int_0^{s_{k-1}} M(\eta(s_k, h(v_n)))K_1(\eta(s_k, h(v_n)))ds_k \dots ds_1 \\ & - \int_0^1 M(\eta(s_1, h(v_m))) \cdots \\ & \int_0^{s_{k-1}} M(\eta(s_k, h(v_m)))K_1(\eta(s_k, h(v_m)))ds_k \dots ds_1 \Bigg) || < \frac{\epsilon}{2}. \end{aligned}$$

regardless of v_n, v_m. Then choose a subsequence of $\{v_n\}$ such that for all $k \leq N_3$,

$$\begin{aligned} || \int_0^1 & M(\eta(s_1, h(v))) \cdots \int_0^{s_{k-1}} M(\eta(s_k, h(v)))K_1(\eta(s_k, h(v)))ds_k \dots ds_1 - \\ & \int_0^1 M(\eta(s_1, h(v))) \cdots \int_0^{s_{k-1}} M(\eta(s_k, h(v)))K_1(\eta(s_k, h(v)))ds_k \dots ds_1 || < \frac{\epsilon}{2N_3}. \end{aligned}$$

Then $||C(v_n) - C(v_m)|| \leq \epsilon$, and thus $C(v)$ is compact.

Then

$$P(\eta(1, h)(v)) = d(v)\hat{d}(v)v + \hat{d}(v)K_1(v) + C(v), \tag{4.17}$$

where $d(v)\hat{d}(v) \geq \frac{2\delta}{3} \geq 0$ and $\hat{d}(v)K_1(v) + C(v)$, is compact.

Finally, for $v \in \partial D$, $I(h(v + \phi(v))) \leq \alpha < \alpha + \hat{\epsilon} \leq \beta - \hat{\epsilon} \leq c - \hat{\epsilon}$, and so by Lemma 12, $\eta(1, h(v + \phi(v))) = h(v + \phi(v))$. Thus $P(\eta(1, h(v + \phi(v)))) = P(h(v + \phi(v))) = v + K(v + \phi(v)) = v$ on ∂D.

Thus $\eta(1, h) \in \Gamma$.

Recall that h was chosen with $\sup_{v \in \bar{D}} h(v) \leq c + \epsilon$, so the contradiction is established. Provided I satisfies (ps) at c, then c is a critical value of I.

Suppose I does not satisfy (PS) at c. Then there exists some $\{w_n\} \in E$ such that $I(w_n) \to c$ and $I'(w_n) \to 0$.

This completes the proof.

$\square$

Chapter 5

Application

We now apply the preceding theorem to show that the semilinear wave equation given in Equation 1.1 subject to Equation 1.2 has weak solution for p in a dense subset of $L^2(\Omega)$, $\Omega = (0, 2\pi) \times (0, \pi)$.

Lemma 15. *Let H, U, and V be defined as in 2. The operator $K : H \to V$ defined by*

$$< K(u), v >= \int_\Omega (g(u) + p)v dx dt$$

is compact.

Proof. Consider a bounded sequence $\{w_n\} \subset L^2$ with

$$g(w_n) = \sum_{k,j} {}_n a_{k,j} \phi_{k,j} + {}_n b_{k,j} \psi_{k,j}.$$

Let

$$v = \sum_{k,j} \alpha_{k,j} \phi_{k,j} + \beta_{k,j} \psi_{k,j}$$

be in V, and thus $\alpha_{k,j} = \beta_{k,j} = 0$ for all $k^2 - j^2 \leq 1$.

Let $\{w_k\} \subset H$. Then for all $n, m \in \mathbb{Z}^+$,

$$\begin{aligned} || \int_0^{2\pi} \int_0^{\pi} (g(w_n) + p) v dx dt - \int_0^{2\pi} \int_0^{\pi} (g(w_m) + p) v dx dt || \\ = || \int_0^{2\pi} \int_0^{\pi} (g(w_n) - g(w_m)) v dx dt || \\ = ||(\sum_{k^2 - j^2 > 0} ({}_n a_{k,j} - {}_m a_{k,j}) \alpha_{k,j} + ({}_n b_{k,j} - {}_m b_{k,j}) \beta_{k,j} || \\ = ||(\sum_{k^2 - j^2 > 0} ({}_n a_{k,j} - {}_m a_{k,j}) \alpha_{k,j} + \sum_{k^2 - j^2 > 0} ({}_n b_{k,j} - {}_m b_{k,j}) \beta_{k,j} || \end{aligned}$$

$$
\begin{aligned}
&\leq ||(\sum_{k^2-j^2>0} \frac{{}_na_{k,j} - {}_ma_{k,j}}{\sqrt{k^2-j^2}} \alpha_{k,j}\sqrt{k^2-j^2} \\
&\quad + \sum_{k^2-j^2>0} \frac{{}_nb_{k,j} - {}_mb_{k,j}}{\sqrt{k^2-j^2}} \beta_{k,j}\sqrt{k^2-j^2}|| \\
&\leq (||(\sum_{k^2-j^2>0} \frac{({}_na_{k,j} - {}_ma_{k,j})^2}{k^2-j^2})^{\frac{1}{2}} (\sum_{k^2-j^2>0} \alpha_{k,j}^2 (k^2-j^2))^{\frac{1}{2}} \\
&\quad + (\sum_{k^2-j^2>0} \frac{({}_nb_{k,j} - {}_mb_{k,j})^2}{k^2-j^2})^{\frac{1}{2}} (\sum_{k^2-j^2>0} \beta_{k,j}^2 (k^2-j^2))^{\frac{1}{2}})||.
\end{aligned}
$$

As H is a Hilbert space and $\{g(w_n)\}$ bounded, $g(w_n) + p$ has a weakly convergent subsequence. Thus individually, the components ${}_na_{k,j}, {}_nb_{k,j}$ converge strongly. Thus for any finite N, there exist $n, m \in \mathbb{Z}^+$ such that

$$
\begin{aligned}
&||(\sum_{N>k^2-j^2>0} \frac{({}_na_{k,j} - {}_ma_{k,j})^2}{k^2-j^2})^{\frac{1}{2}} (\sum_{k^2-j^2>0} \alpha_{k,j}^2 (k^2-j^2))^{\frac{1}{2}} \\
&\quad + (\sum_{N>k^2-j^2>0} \frac{({}_nb_{k,j} - {}_mb_{k,j})^2}{k^2-j^2})^{\frac{1}{2}} (\sum_{k^2-j^2>0} \beta_{k,j}^2 (k^2-j^2))^{\frac{1}{2}})|| \\
&\leq \frac{\epsilon}{2}
\end{aligned}
$$

Choose N such that $||v||\frac{||g(w_n)-g(w_m)||}{\sqrt{N}} < \frac{\epsilon}{2}$, for all $n, m \in \mathbb{Z}^+$. Then

$$
\begin{aligned}
&||(\sum_{k^2-j^2\geq N} \frac{({}_na_{k,j} - {}_ma_{k,j})^2}{k^2-j^2})^{\frac{1}{2}} (\sum_{k^2-j^2\geq N} \alpha_{k,j}^2 (k^2-j^2))^{\frac{1}{2}} \\
&\quad + (\sum_{k^2-j^2\geq N} \frac{({}_nb_{k,j} - {}_mb_{k,j})^2}{k^2-j^2})^{\frac{1}{2}} (\sum_{k^2-j^2\geq N} \beta_{k,j}^2 (k^2-j^2))^{\frac{1}{2}})|| \\
&\leq (||(\sum_{k^2-j^2\geq N} \frac{({}_na_{k,j} - {}_ma_{k,j})^2}{N})^{\frac{1}{2}} (\sum_{k^2-j^2\geq N} \alpha_{k,j}^2 (k^2-j^2))^{\frac{1}{2}} \\
&\quad + (\sum_{k^2-j^2\geq N} \frac{({}_nb_{k,j} - {}_mb_{k,j})^2}{N})^{\frac{1}{2}} (\sum_{k^2-j^2\geq N} \beta_{k,j}^2 (k^2-j^2))^{\frac{1}{2}})|| \\
&\leq ||v||\frac{||g(w_n) - g(w_m)||}{\sqrt{N}} < \frac{\epsilon}{2}.
\end{aligned}
$$

So in total,

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^{\pi}(g(w_n)+p-g(w_m)-p)vdxdt\\
&\leq ((\sum_{k^2-j^2>0}\frac{({}_n a_{k,j}-{}_m a_{k,j})^2}{k^2-j^2})^{\frac{1}{2}}(\sum_{k^2-j^2>0}\alpha_{k,j}^2(k^2-j^2))^{\frac{1}{2}}\\
&\quad+(\sum_{k^2-j^2>0}\frac{({}_n b_{k,j}-{}_m b_{k,j})^2}{k^2-j^2})^{\frac{1}{2}}(\sum_{k^2-j^2>0}\beta_{k,j}^2(k^2-j^2))^{\frac{1}{2}})\\
&\leq ((\sum_{N>k^2-j^2>0}\frac{({}_n a_{k,j}-{}_m a_{k,j})^2}{k^2-j^2})^{\frac{1}{2}}(\sum_{k^2-j^2>0}\alpha_{k,j}^2(k^2-j^2))^{\frac{1}{2}}\\
&\quad+\sum_{N>k^2-j^2>0}\frac{({}_n b_{k,j}-{}_m b_{k,j})^2}{k^2-j^2})^{\frac{1}{2}}(\sum_{k^2-j^2>0}\beta_{k,j}^2(k^2-j^2))^{\frac{1}{2}})\\
&\quad+(\sum_{k^2-j^2\geq N}\frac{({}_n a_{k,j}-{}_m a_{k,j})^2}{k^2-j^2})^{\frac{1}{2}}(\sum_{k^2-j^2>0}\alpha_{k,j}^2(k^2-j^2))^{\frac{1}{2}}\\
&\quad+(\sum_{k^2-j^2\geq N}\frac{({}_n b_{k,j}-{}_m b_{k,j})^2}{k^2-j^2})^{\frac{1}{2}}(\sum_{k^2-j^2>0}\beta_{k,j}^2(k^2-j^2))^{\frac{1}{2}})\\
&\leq \epsilon.
\end{aligned}
$$

Thus $\int_0^{2\pi}\int_0^{\pi}(g(w_n)+p)vdxdt$ is a Cauchy sequence, and the operator K is compact. □

Thus $\nabla I(w) = Pid + K$, where K is compact.

Lemma 16. *For $u \in U$, $J(u)$ has an upper bound.*

Proof. Let $\phi_{k,j}$ and $\theta_{k,j}$ be as in Equation 1. Consider $u = \sum_{k,j} a_{k,j}\phi_{k,j} + b_{k,j}\theta_{k,j} \in U$.

Introduce

$$||u||_*^2 := \sum_{k,j}(a_{k,j}^2 + b_{k,j}^2)(|\lambda_{k,j}|).$$

Then

$$\begin{aligned}
J(u) &= \frac{B(u,u)}{2} - \int_0^{2\pi}\int_0^{\pi}(G(u)pu)dxdt \\
&= \sum_{k,j}\frac{\lambda_{k,j}(a_{k,j}^2+b_{k,j}^2)}{2} - \int_0^{2\pi}\int_0^{\pi}(G(u)+pu)dxdt \\
&\leq \sum_{k,j}\frac{\lambda_{k,j}(a_{k,j}^2+b_{k,j}^2)}{2} - \int_0^{2\pi}\int_0^{\pi}(\frac{a_0u^2}{2} - C + pu)dxdt \\
&\leq \sum_{k,j}\frac{(\lambda_{k,j}-a_0)(a_{k,j}^2+b_{k,j}^2)}{2} + 2C\pi^2 + ||p||_{L^2}||-u||_{L^2}.
\end{aligned}$$

Let $u = u_1 + u_2$ where u_1 is in the closure of the span of eigenfunctions with $\lambda_{k,j} \leq 0$, and u_2 is in the closure of the span of eigenfunctions with $0 < \lambda_{k,j} \leq 1$. Then

$$\begin{aligned}
J(u) &\leq \frac{1}{2}(-||u_1||_*^2 + ||u_2||_*^2 - a_0||u||_{L^2}^2) + 2C\pi^2 + ||p||_{L^2}||u||_{1,2} \\
&\leq \frac{1}{2}(-||u_1||_*^2 - (a_0-1)||u||_{L^2}^2) + 2C\pi^2 + ||p||_{L^2}||u||_{1,2} \\
&\leq \frac{1}{2}(-\min(a-1,1)||u||_{1,2}^2) + 2C\pi^2 + ||p||_{L^2}||u||_{1,2},
\end{aligned}$$

and thus $J(u)$ is bounded above on U. □

Lemma 17. *For all $a \in (1,3)$ and $b \in (a, b_1(a))$, there exists $r_{a,b} : V \to U$ which maximizes the functional $J_{a,b}(v + r_{a,b}(v))$ on V.*

$J(v + r_{a,b}(v)) \to \infty$ as $||v||_{1,2} \to \infty$.

Proof. The existence of $r_{a,b}$ is established in Corollary 6.

For all $v \in V$,

$$\begin{aligned}
J(v + r_{a,b}(v)) &= \frac{1}{2}B(r_{a,b}(v)+v, r_{a,b}(v)+v) \\
&\quad - \int_0^{2\pi}\int_0^{\pi}\frac{a(r_{a,b}(v)+v)_+^2 + b(r_{a,b}(v)+v)_-^2}{2} \\
&\quad + G(v + r_{a,b}(v)) + (v + r_{a,b}(v))p \\
&\quad - \frac{a(r_{a,b}(v)+v)_+^2 + b(r_{a,b}(v)+v)_-^2}{2}dxdt \\
&\geq J_{a,b}(v + r_{a,b}(v)) - 2\pi^2C - ||v + r_{a,b}(v)||_2||p||_2.
\end{aligned}$$

Note that $r_{a,b}(\lambda v) = \lambda r_{a,b}(v)$, and $J_{a,b}(\lambda v + r_{a,b}(\lambda v)) = \lambda^2 J_{a,b}(v + r_{a,b}(v))$. Thus

$$\begin{aligned} J(v + r_{a,b}(v)) \geq ||v||^2 J_{a,b}(\frac{v}{||v||} + r_{a,b}(\frac{v}{||v||})) - 2\pi^2 C \\ - ||v||||p|| - ||v||||r_{a,b}(\frac{v}{||v||})||||p||. \end{aligned}$$

Because $J_{a,b}(\frac{v}{||v||} + r_{a,b}(\frac{v}{||v||})) \geq 0$, $J(v + r_{a,b}(v)) \to \infty$ as $||v|| \to \infty$. □

Proof of Theorem 4

By Lemma 17, there is some constant β such that $-J(u) > \beta$ on U. Choose $\alpha < \beta$. We can choose a neighborhood D of 0 in U such that $-J(v + r_{a,b}(v)) \leq \alpha$ on ∂D.

Theorem 1 applied to $-J(u+v)$ shows that either $-J$ has a critical value c, or there exists some sequence $\{w_n\} \in E$ with $-J(w_n) \to c$, $\nabla - J(w_n) \to 0$.

Consider $p \in L^2$.

If $-J$ has a critical value c, Equation 1.1 subject to Equation 1.2 has a weak solution, and we are done.

Otherwise, choose w_n such that $J_p(w_n) \to -c$ and $\nabla J(w_n) \to 0$.

$$\begin{aligned} < \nabla J(w_n), v > &= B(w_n, v) - \int_0^{2\pi} \int_0^{\pi} v(g(w_n) + p) \\ &=< y_n, v >_1 + < z_n, v >_{L^2} \end{aligned} \tag{5.1}$$

for some y_n, z_n which converge to 0 in H and L^2, respectively, z_n being an element of the null space of □ subject to Equation 1.2, and therefore only converging in $L^2(\Omega)$. Note that $z_n \in U$.

Let $y_n = \sum_{k,j} a_{k,j}\phi_{k,j} + b_{k,j}\psi_{k,j}$, let $\hat{a}_{k,j} = \text{sign}(\lambda k, j)a_{k,j}$, and $\hat{b}_{k,j} = \text{sign}(\lambda k, j)b_{k,j}$, and define

$$\hat{y}_n = \sum_{k,j} \hat{a}_{k,j}\phi_{k,j} + \hat{b}_{k,j}\psi_{k,j}$$

so that $B(\hat{y}, v) =< y_n, v >_1$.

Then rearranging Equation 5.1,

$$\begin{aligned} 0 &= B(w_n - \hat{y}_n, v) - \int_0^{2\pi} \int_0^{\pi} v(g(w_n) + p + z_n) \\ &= B(w_n - \hat{y}_n, v) - \int_0^{2\pi} \int_0^{\pi} v(g(w_n) - g(w_n - \hat{y}_n) + g(w_n - \hat{y}_n) + p + z_n) \end{aligned}$$

Since $g(w_n)$ is Lipschitz continuous, there exists K such that $|g(w_n) - g(w_n + \hat{y}_n)| \leq K||\hat{y}_n|| \to 0$ in L^2. So a solution (namely, $w_n - \hat{y}_n$) exists for

$$\partial_{tt}u - \partial_{xx}u = g(u) + p + z_n + (g(w_n - \hat{y}_n) - g(w_n)), \tag{5.2}$$

and

$$p + z_n + (g(w_n + \hat{y}_n) - g(w_n))$$

converges to p in L^2.

Thus the set of p for which Equation 1.1 subject to Equation 1.2 has a weak solution is dense in L^2.

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