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Explorations of the Aldous Order on Representations of the Symmetric Group

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Abstract

The Aldous order is an ordering of representations of the symmetric group motivated by the Aldous conjecture, a conjecture about random processes proved in 2009. In general, the Aldous order is very difficult to compute, and the proper relations have yet to be determined even for small cases. However, by restricting the problem down to Young–Jucys–Murphy elements, the problem becomes explicitly combinatorial. This approach has led to many novel insights, whose proofs are simple and elegant. However, there remain many open questions related to the Aldous order, both in general and for the Young–Jucys–Murphy elements.

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Part I

Background Material

Chapter 1

Representation Theory of Finite Groups

The Aldous order is almost entirely dependent on the machinery of representation theory in general, and in particular that of the symmetric groups. While we assume an acquaintance with the basic properties of groups and rings as may be found in an introductory abstract algebra course, this part seeks to provide an introduction to the more complicated machinery necessary to understand and manipulate the Aldous order. This chapter provides an introduction to general representation theory, giving basic definitions of the group algebra, representations, and irreducibility. This chapter seeks to give a brief introduction to the relevant aspects of the topic. A more comprehensive treatment can be found in James and Liebeck (2001).

1.1 The Group Algebra

Before proceeding to representations, it is important to be familiar with the group algebra of a finite group G over some field F. The group algebra is defined as follows.

Definition 1.1 (Group Algebra)**.** For a finite group G and a field F, the group algebra FG is constructed by defining addition and multiplication on finite formal linear combinations $\sum_{i=1}^{k} c_i g_i$, with $c_i \in F$, $g_i \in G$. Addition is given componentwise, with the elements of G serving as a basis, while multiplication is extended from the multiplications of F and G.

Throughout, we will restrict our field to the complex numbers $\mathbb{C}$, denoting the group algebra of a group G over the complex numbers as $\mathbb{C}G$.

Then, note that, as a vector space, $\mathbb{C}G$ is isomorphic to $\mathbb{C}^n$, where $n = |G|$. Representation theory is fundamentally concerned with using linear algebra to investigate $\mathbb{C}G$ in order to discover new insights into the structure of G.

1.2 Modules

In order to define a representation, we must first recall the definition of an R-module.

Definition 1.2 (R-Module)**.** For a ring R, an R-Module M is an abelian group $(M, +)$ along with an operation $R \times M \to M$ such that for all $r, s \in R$, $x, y \in M$,

(i) $r(x + y) = rx + ry$

(ii) $(r + s)x = rx + sx$

(iii) $(rs)x = r(sx)$

(iv) $1_R x = x$ if R has multiplicative identity 1_R.

Modules are a generalization of vector spaces, and as such any vector space V over a field F is also an F-module. Additionally, any abelian ring R will be an R-module over itself, with the module action given by the original ring multiplication. Thus, $\mathbb{C}G$ will always be a $\mathbb{C}G$-module.

As with any algebraic structure, there is the natural definition of submodule, as a subgroup of M which is closed under multiplication of R. This leads to a natural definition of an irreducible module.

Definition 1.3 (Irreducible Module)**.** An R-module M is said to be *irreducible* if it contains no proper nontrivial submodules. That is, the module is irreducible if the only submodules are $\{0\}$ and M.

Depending on the ring, it is often possible to decompose an R-module into the direct sum of several irreducible modules. This turns out to be the case for $\mathbb{C}G$-modules, and is fundamental to much of representation theory.

Additionally, $\mathbb{C}G$-modules arise naturally from group actions on vector spaces.

Lemma 1.1. *If G is a finite group, M is a vector space over $\mathbb{C}$ and there exists some group action $G \times M \to M$ such that G acts linearly on M, then there is a natural* *construction of M as a $\mathbb{C}G$-module. This construction is given by extending the action of G in the following manner:*

$$(\sum_{i=1}^{k} c_i g_i)(m) = \sum_{i=1} c_i(g_i m).$$

In performing this construction, we are able to translate a map (the group action), into an algebraic structure (the $\mathbb{C}G$-module).

1.3 Representations of a Finite Group

Recall that for the complex numbers $\mathbb{C}$, $GL(n, \mathbb{C})$ is the group of invertible linear transformations on $\mathbb{C}^n$ (equivalently, the group of invertible $n \times n$ matrices). Then, we can finally define a representation as follows:

Definition 1.4 (Representation)**.** A *complex representation* of a finite group G is a group homomorphism $\rho : G \to GL(n, \mathbb{C})$ for some n.

General representations can be constructed over any field, but we will restrict our focus to complex representations, which will hereafter be refered to merely as representations.

While this is a straightforward definition, representations are seldom analyzed solely at the level of groups. Instead, representations will frequently refer jointly to the following four algebraic structures:

(i) The group homomorphism $\rho_g : G \to GL(\mathbb{C}, F)$ given in Definition 1.4.

(ii) The ring homomorphism $\rho_r : \mathbb{C}G \to \mathbb{C}^{n\times n}$ given by extending the map ρ_g to an algebra homomorphism.

(iii) The group action $\rho_a : G \times \mathbb{C}^n \to \mathbb{C}^n$ given by $\rho_a(g, v) = \rho_g(g)v$ under usual vector-matrix multiplication.

(iv) the $\mathbb{C}G$-module $\mathbb{C}^n$ constructed by extending the group action ρ_a as in Lemma 1.1.

Throughout representation theory, it is common practice to conflate these four objects. Note that any one can be derived readily and uniquely from the others, mainly by appropriate restriction or extension of the proper maps. As such, we will do without the subscripts on ρ_g, ρ_r, and ρ_a and instead use ρ throughout. In general, which is actually being used does not matter, and when it does, context should make it apparent.

As with most algebraic structures, there is a notion of equivalence between represenions.

Definition 1.5 (Equivalent Representations)**.** For a finite group G and representations $\rho : G \to GL(n, \mathbb{C})$, $\sigma : G \to GL(m, \mathbb{C})$, ρ and σ are *equivalent* if $n = m$ and there exists some $T \in GL(n, \mathbb{C})$ such that for all $g \in G$

$$\sigma g = T(\rho g)T^{-1}.$$

In other words, two representations are equivalent if σg and ρg are similar under the same matrix T for all $g \in G$. As expected, such a T induces a $\mathbb{C}G$-module isomorphism between the two $\mathbb{C}G$-modules given by ρ and σ. Thus, two representations are equivalent if their modules are isomorphic. Additionally, as implied by the name, equivalence between representations is an equivalence relation.

Analogous to how vector spaces or groups may be summed together, we can define the direct sum of a representation.

Definition 1.6 (Direct Sum of a Representation)**.** If ρ and σ are representations of G of degree n and m, then $\rho \oplus \sigma$ is a representation of G of degree $n + m$ given by

$$(\rho \oplus \sigma)(g) = \rho(g) \oplus \sigma(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \sigma(g) \end{pmatrix}.$$

Next, in analogy to irreducible modules, an irreducible representation is defined as follows.

Definition 1.7 (Irreducible Representation)**.** A representation ρ of σ is irreducible if the corresponding $\mathbb{C}G$ module is irreducible.

This then gives us the machinery to state the main results from general representation theory which will later be useful:

Theorem 1.1 (Irreducible Representations)**.** *For any finite group G, any (complex) representation of G over is the direct sum of irreducible representations. Moreover, this decomposition is unique up to isomorphism. Additionally, up to isomorphism, the number of distinct irreducible representations of G is equal to the number of conjugacy classes of G.*

Because of this theorem, in order to gain an understanding of the representation theory of a specific group, it is usually sufficient to describe the finite number of distinct irreducible representations. Because of the special symmetries inherent in the symmetric group, the irreducible representations of S_n possess remarkable structure, which is detailed in Chapter 2.

Chapter 2

Representation Theory of the Symmetric Groups

The symmetric groups S_n are one of the most important groups in mathematics, appearing in a wide variety of contexts and with a large array of applications. Understanding the representations of S_n can lead to new insights in the structure of the group. Additionally, the representation theory of the symmetric groups yields much beautiful mathematics and is fascinating in its own respect. In this chapter we seek to give an overview of those aspects of the representation theory of the symmetric groups which will be used in Parts II and III. For those interested in a more in depth treatment, Chapters 1–3 of Ceccherini-Silberstein et al. (2010) cover these topics extensively.

2.1 Partitions

Much of the representation theory of symmetric groups is combinatorial in nature, with algebraic structures frequently corresponding to combinatorial objects. The most basic of these objects are partitions and set partitions.

Definition 2.1 (Partition)**.** A partition σ of an integer n, frequently written $\sigma \dashv n$, is a multiset of positive integers $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_k]$ such that $\sum_{i=1}^{k} \sigma_i = n$. Alternately, one may view a partition as a sequence of integers which is either nonincreasing or nondecreasing. We will assume throughout that the sequence is nonincreasing.

Definition 2.2 (Set Partition)**.** A set partition of size n is a disjoint collection $\{S_i\}$ of nonempty subsets of $[n] = \{1, 2, \ldots, n\}$ such that $\cup S_i = [n]$. Each set partition corresponds to an equivalence relation on $[n]$, and the number of set partitions $[n]$ is given by the nth Bell number, B_n. The partition of a set partition is given by the sequence $[|S_1|, |S_2|, \ldots, |S_k|]$, in decreasing order.

Definition 2.3 (Ordered Set Partition)**.** An ordered set partition consists of a set partition with an order on the sets $\{S_i\}$, subject to the constraint that $|S_i| \geq |S_{i+1}|$. Two ordered set partitions are distinct if the sets are of in a different order, even if the collection of sets is the same

Example 2.1. The set partitions of [3] are given by

$$\begin{gathered}\{(123)\}\\ \{(12),(3)\}\\ \{(13),(2)\}\\ \{(23),(1)\}\\ \{(1),(2),(3)\}\end{gathered}$$

while the partitions are $[3]$, $[2, 1]$, and $[1, 1, 1]$. The ordered set partitions of $[3]$ are the same as the unordered set partitions. However, for $n > 3$, this is not the case. For example $\{(12),(34)\}$ and $\{(34),(12)\}$ are distinct ordered set partitions, but have the same unorderd set partition.

Now, recall that each conjugacy class of S_n consists of all elements with the same cycle structure. Then, there is a natural correspondence between the conjugacy classes of S_n and partitions of n.

Lemma 2.1. *There exists a bijection between the conjugacy classes of S_n and the partitions of n given as follows: Each element of $\pi \in S_n$ has a corresponding set partition, given by the cycles of π. Then, each conjugacy class of S_n consists of all elements of S_n whose set partitions have a given partition.*

Therefore, because of Theorem 1.1, this implies that the number of irreducible representations of S_n is equal to the number of partitions. In fact, there is a natural identification of the irreducible representations of S_n with the partitions of n. A full explanation of this approach to the symmetric groups' representation theory can be found in Ceccherini-Silberstein et al. (2010). For our purposes, it will suffice to describe several of the combinatorial and algebraic objects related to the irreducible representations which will be useful in analyzing the Aldous Order. Depending on the context, the irreducible representation will be denoted by either ρ or, in some cases, S^ρ.

2.2 Young Diagrams and Standard Young Tableaux

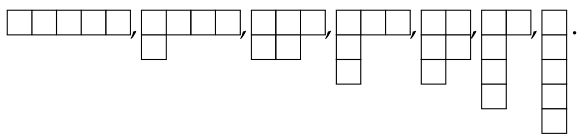
There is a standard diagramatic way of displaying a partition, known as a Young diagram.

Definition 2.4 (Young Diagram)**.** For a partition $\sigma \dashv n$ where $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_k]$ and $\sigma_i \geq \sigma_{i+1}$, the corresponding *Young Diagram* is formed using squares by placing σ_i squares in the ith column, with each row left justified.

For example the partitions of 5 are given by

$$[5], [4,1], [3,2], [3,1,1], [2,2,1], [2,1,1,1], [1,1,1,1,1],$$

and the corresponding Young diagrams are



Initially, the Young diagrams may seem to offer no advantages over merely listing the elements of the partition. However, the geometric properties of the Young diagram prove to be crucial to cleanly stating many of the most powerful statements about the symmetric groups' representations. Without the diagrams, the statements would be complicated and opaque. For instance, the transpose of a partition, an involution on the set of partitions, is most easily defined as follows, using the Young diagrams.

Definition 2.5 (Transpose of a Partition or Young Diagram)**.** The *transpose* of a Young diagram is attained by reflecting the diagram over the standard diagonal; that is, the $y = -x$ line.

The *transpose* of a partition is the partition corresponding to the transpose of its Young diagram. Alternately, the kth largest element of the transpose is the number of elements of the original partition of size k or greater.

The transpose of a partition $\rho \dashv n$ is denoted by ρ'.

The transpose could also be defined arithmetically, where $(\rho')_k$ is the number of elements of ρ of length at least k.

2.3 Standard Young Tableaux

The standard Young tableaux of a Young diagram are defined as follows and are fundamental to the representation corresponding to that diagram.

Definition 2.6 (Standard Young Tableaux (SYT))**.** Given a partition $\sigma \dashv n$ and its associated Young diagram, a *standard Young Tableau (SYT)* is constructed by placing each numbers 1 through n inclusive in one of the boxes of the diagram such that the numbers are increasing from left to right and top to bottom.

For example, the valid SYT of the partition $\rho = [3,2]$ are given by

$$\ytableaushort{123,45}\,,\ \ytableaushort{124,35}\,,\ \ytableaushort{125,34}\,,\ \ytableaushort{134,25}\,,\ \ytableaushort{135,24}\,.$$

The definition of SYT may initially appear arbitrary. To motivate the definition, note that each tableau can be associated with a unique way to build the associated Young diagram from smaller diagrams. For instance, the tableau $\ytableaushort{124,35}$ gives the path

$$\ydiagram{1} \to \ydiagram{2} \to \ydiagram{2,1} \to \ydiagram{3,1} \to \ydiagram{3,2}\,.$$

Alternately, this can be viewed as a path of SYT:

$$\ytableaushort{1} \to \ytableaushort{12} \to \ytableaushort{12,3} \to \ytableaushort{124,3} \to \ytableaushort{124,35}\,.$$

As such, it is common to construct the SYT recursively.

As we will see, the position of the particular numbers have a particular importance with the algebraic meaning of the SYT. Of particular importance is the *content* of a SYT.

Definition 2.7 (Content of a SYT)**.** Consider a partition $\sigma \dashv n$. Then, for some SYT T of σ, the *content* of T, written $C(T)$, is the vector $C(T) = (a_1, a_2, \ldots, a_n)$, where $a_k = c_k - r_k$. Here, c_k, r_k denote the row and column of the box in T containing k. Rows are counted top to bottom, columns left to right.

Often, when the SYT is understood, we will refer to a_k as the "content of k". Alternately, for Young diagrams, the content of a box will refer to the column minus the row for the box in question.

Example 2.2. the SYT $\ytableaushort{124,35}$ has content $C(T) = (0, 1, -1, 2, 0)$, and the content of 3 is -1. In a similar vein the upper-rightmost box of $[3,2]$ will have content 2.

2.4 The Irreducible Representations of S_n

As has been mentioned in Section 2.1, the irreducible representations of S_n can be placed into a correspondence with the partitions of S_n. The following theorem further explicates this correspondence.

Theorem 2.1 (Partitions and Irreducible Representations of S_n)**.** *For every partition $\sigma \dashv n$, there exists an irreducible representation of S_n, which has dimension equal to the number of unique SYTs of σ. Moreover, the representations can be realized by viewing S_n as acting on linear combinations of the SYTs of σ, with the action defined based on properties of the SYT. More details can be found in Section 3.4 of Ceccherini-Silberstein et al. (2010).*

(Note: Throughout, σ will be used to indicate the partition, the Young diagram, *and* the representation. Which meaning is intended should be clear by context.)

The particulars of realizing the representation ρ prove unimportant for this thesis, but for explicit computation it is useful to note that it is possible to realize the representation within the rationals. The action is more complicated than simply permuting the elements of the SYT. For those interested, consult Ceccherini-Silberstein et al. (2010).

Instead of explicitly computing the representations, much of our focus has been on interactions with the following elements of the group algebra $\mathbb{C}S_n$:

Definition 2.8 (Young–Jucys–Murphy Elements (YJM))**.** The *Young–Jucys–Murphy (YJM) elements* are given by

$$\chi_k = \sum_{i<k} (i\ k).$$

Thus,

$$\chi_2 = (12), \chi_3 = (13) + (23).$$

That is, χ_k is the sum of the transpositions of k with every positive integer less than k.

Our interest in the YJM stems primarily from the following theorem:

Theorem 2.2 (Young–Jucys–Murphy Elements Acting on Standard Young Tableaux)**.** *Consider a partition $\sigma \dashv n$ and its corresponding irreducible representation of S_n. Then, this representation may be realized with $T_\sigma = \{T_1, T_2, \ldots, T_m\}$, the SYT of σ, as a basis. Moreover, if the content of T_k is given by $C(T_k) =$*

$(a_1, a_2, \ldots, a_n)$, then T_k is an eigenvector of $\sigma(\chi_j)$ for any j, with eigenvalue a_j. That is,

$$\sigma(\chi_j)T_k = a_j T_k.$$

Thus, T_σ forms an eigenbasis for all of the YJM elements.

This theorem lies at the heart of the Okounkov-Vershik approach, which is detailed in Chapter 3 of Ceccherini-Silberstein et al. (2010).

2.5 The Trivial and Alternating Representations

In general, we will ignore the specific details of the various representations. However, it will prove useful to be acquainted with the two one-dimensional representations of S_n, the trivial and alternating representations.

Definition 2.9 (The Trivial and Alternating Representations)**.** There exist two 1-dimensional representations of S_n, the *trivial* and *alternating* representations. The trivial representation is given by the map $g \mapsto 1$ for all S_n (every group has a trivial representation). The trivial representation corresponds to the partition $[n]$, and thus has a Young diagram consisting of a single row.

The alternating representation is given by the map $g \to sgn(g)$, the sign of g. The sign of g is 1 if g is an even permutation, and -1 if g is odd. The alternating representation corresponds to the partition $[1^n]$, and thus has a Young diagram consisting of a single column.

2.6 Lexicographic and Dominance Orders

As we will see in Part II, the Aldous order defines a novel order on the irreducible representations of S_n, and thus on the partitions on n. However, there are already several well-established partial orders on partitions of which we should be aware, namely the lexicographic and dominance orders. The lexicographic order is straightforward and should be familiar to anyone who has experience with partial orders.

Definition 2.10 (Lexicographic Order)**.** Let $\sigma, \rho \dashv$ be two partitions, with $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_s]$, $\rho = [\rho_1, \rho_2, \ldots, \rho_r]$. Then, assuming that $\sigma \neq \rho$, then there is some first i such that $\sigma_i \neq \rho_i$. If $\sigma_i < \rho_i$, then $\sigma < \rho$ in the dominance order, while if $\sigma_i > \rho_i$, then $\sigma > \rho$. This lexicographic is the same as the one constructed constructed on general sequences of integer sequences, then restricted to partitions of n.

As can be seen by the definition, the lexicographic order is a total order. The next order, the dominance order, is a suborder of the lexicographic and is fundamental to the representations of S_n.

Definition 2.11 (Dominance Order)**.** Let $\rho = [\rho_1, \ldots, \rho_r]$ and $\sigma = [\sigma_1, \ldots, \sigma_s]$ be two partitions of n, written in nonincreasing order. Then, ρ precedes σ in the dominance order (written $\rho \trianglelefteq \sigma$) if $s \leq r$ and

$$\sum_{i=1}^{m} \rho_i \leq \sum_{i=1}^{m} \sigma_i,$$

for all $m = 1, 2, \ldots, s$.

The dominance order plays a crucial role in many of the theorems of the representation theory of symmetric groups, such as characterizing the Young Modules (see Ceccherini-Silberstein et al. (2010)). For our purposes, it will prove useful to have several equivalent definitions for the dominance order. First, we need the idea of a *box-up move*.

Definition 2.12 (Box-up Move)**.** Given a Young diagram for some partition $\sigma \dashv n$, a box-up move on the diagram consists of moving a box from the end of one row and placing it on the end of another, higher row.

For example, for the partition $[3, 2, 1]$ there are three possible box-up moves, producing the three Young diagrams shown.

$$\ydiagram{3,2,1} \rightarrow \ydiagram{4,2}, \ydiagram{4,1,1}, \ydiagram{3,3}.$$

Then, the next diagram gives the different definitions of the dominance order which will be useful.

Theorem 2.3 (Dominance Order Characterizations)**.** *Let $\sigma = [\sigma_1, \ldots, \sigma_s]$ and $\sigma = [\rho_1, \ldots, \rho_r]$ be partitions of n. Then, the following are equivalent:*

(i) $\sigma \trianglelefteq \rho$.

(ii) For all $k \leq s$,

$$\sum_{i=k}^{s} \sigma_i \geq \sum_{i=k}^{r} \rho_i.$$

(iii) The Young diagram of ρ can be attained from σ by some sequence of box-up moves.

(iv) $\sigma' \trianglerighteq \rho'$.

The proofs of this are rather straightforward, and can be found in Section 3.6 of Ceccherini-Silberstein et al. (2010).

2.7 Young Modules

In addition to its corresponding irreducible representation, each partition corresponds naturally to a *Young module*.

Definition 2.13 (Young Module of a Partition)**.** For a partition $\rho \dashv n$, there exists a $\mathbb{C}S_n$ module M^ρ known as the Young Module. The underlying abelian structure is a vector space, with each basis vector corresponding to an ordered set partition with partition ρ. Then, the action of $\pi \in S_n$ on these basis elements is given by simply permuting the elements of the set partition according to π.

Example 2.3. Some straightforward examples include

(i) If $\rho = [n-k, k]$, M^ρ corresponds to the k-subsets of $\{1, \ldots, n\}$.

(ii) If $\rho = [1^n]$, then M^ρ has dimension $n!$, with each basis element corresponding to an element of S_n. This is thus isomorphic to the *regular representation*, corresponding to the action of S_n on $\mathbb{C}S_n$.

(iii) If $\rho = [2^n]$, then each basis element of M^ρ corresponds to pairings of $2n$ people.

In discussing Young modules, it becomes apparent how readily we conflate the various ways of thinking about representations. It would be acceptable to discuss the "Young representations", but because the basis vectors have such a concrete combinatorial interpretation as ordered set partitions, it is customary to focus on the module aspect.

2.8 Decomposition of Young Modules

Like any representation, the Young Modules can be decomposed into the irreducible modules of S_n. Here we will denote the module corresponding to ρ as S^ρ, to help limit confusion. Then, the decomposition of Young

Modules follows the corresponding theorem, Theorem 3.6.11 in Ceccherini-Silberstein et al. (2010).

Theorem 2.4. *Denote by $K(\mu, \lambda)$ the multiplicity of S^μ in M^λ, where S^μ is the irreducible S_n representation corresponding to μ. Then,*

$$K(\mu, \lambda) \begin{cases} = 0 \text{ if } \lambda \ntrianglelefteq \mu \\ = 1 \text{ if } \lambda = \mu \\ \geq 1 \text{ if } \lambda \triangleleft \mu \end{cases}$$

so that

$$M^\lambda \simeq \oplus_{\lambda \trianglelefteq \sigma} K(\mu, \lambda) S^\mu.$$

Thus, the module M^λ consists of the direct sums of the representations of every shape which dominates λ, possibly with multiplicity. This result will prove important to motivating the definition of the Aldous order.

This concludes the necessary background for understanding the work that has been done on the Aldous order, which occupies Parts II and III.

Part II

General Aldous Order

Chapter 3

Random σ-Interchange Processes

The general Aldous order was first defined in Alon and Kozma (2011) as a novel partial order defined on the partitions of S_n, and thus the associated S_n irreducible representations. The Aldous order was based on the Aldous conjecture, and this chapter describes a class of continuous-time Markov processes from which the Aldous conjecture naturally arises.

3.1 The Interchange Process

If n distinct, colored balls are placed on the n vertices of a possibly weighted graph G, then swapped along the vertices according to some fixed rule, then the result will be some deterministic process, moving the balls through a variety of positions. If, however, the swapping occurs according to some stochastic rule, then the result is a random process. In particular, if the next swap along edge $\{i, j\}$ occurs with exponential rate a_{ij}, the weight of the edge, then the result is called the *interchange process*.

By using different weighting schemes, a variety of processes can be realized. However, the interchange process can also be viewed as a special case of a *generalized interchange process* a class that also includes exclusion processes and simple random walks.

Definition 3.1 (σ-Interchange Process)**.** For a partition $\sigma \dashv n$, the σ-interchange process follows the same setup as the standard interchange process, except that instead of the n balls having n distinct colors, there are σ_1 balls of the first color, σ_2 balls of the second, and so forth. Balls of the same color are regarded as being indistinguishable. Thus, any particular arrangement of balls corresponds to an ordered set partition, with $S_i = \{$ vertices of color $i\}$. The σ-Interchange processes are all continuous-time Markov processes, with the state space given the ordered set partitions of shape σ.

In addition to the usual interchange process, there are several natural examples of σ-interchange processes.

Example 3.1 (Trivial Process)**.** This corresponds to $\sigma = [n]$ and is a trivial example, as the state of the system never changes.

Example 3.2 (Simple Random Walk)**.** The (continuous) simple random walk on a graph corresponds to just following one ball along the interchange process. As such, it corresponds to the σ-interchange process where $\sigma = [n-1, 1]$.

Example 3.3 (Paired Random Walk)**.** Suppose that n men and n women are seated across a table from each other and paired off based on who is sitting across from whom. Then, if the men and women each undergo interchange processes with graphs G_1 and G_2, the pairings also undergo some random process. This process can be modeled as a σ-interchange process with $\sigma = [2^n]$ and $G = G_1 \cup G_2$.

All of the σ-interchange processes can be investigated using the machinery developed in Chapter 2. In particular, any σ-interchange process is closely related to the Young Module M^σ. In general, when discussing a σ-interchange process on n elements, we will think of the corresponding graph G as being a complete weighted graph on n elements, but with some of the weights possibly being zero. This ties the σ-interchange processes directly to S_n.

3.2 Intensity Matrices and Young Modules

Recall from Section 2.7 that the Young module M^σ corresponds to a vector space with basis given by the ordered set partitions with partition structure σ. Thus the σ-interchange process can be viewed as taking values on the basis of M^σ. Then, the probabilities of the process taking on some future value can be viewed as a function $p(t)$, where $p_i(t)$ is the probability of being in the ith state at time t. Then, p is a function $p : \mathbb{R}^+ \to M^\sigma$.

Within this framework, there exists a very important matrix Q, the intensity matrix. Q is a linear transformation of M^σ, and thus has dimensions equal to the number of ordered set partitions with shape σ. For ordered set partitions A and B, the coordinates $\{Q_{AB}\}$ are given as follows:

when $A \neq B$, if there is some transposition (ij) such that $(ij)A = B$, then $Q_{AB} = -a_{ij}$. Otherwise, if no such (ij) exists $Q_{AB} = 0$. Finally,

$$Q_{AA} = -\sum_{B \neq A} Q_{AB}.$$

If all the a_{ij} are either 0 or 1, then Q will correspond to the Laplacian matrix of the induced graph on the basis elements. The intensity matrix relates to the random walk according to the following theorem.

Theorem 3.1. *Let Q be the intensity matrix of a continuous-time Markov process such as the σ-interchange processes. Next, let $p(t) \in M^\sigma$ be the probability distribution at time t. Then, p is a solution to the first-order differential equation*

$$\frac{\partial p}{\partial t} = -Qp.$$

Moreover, if Q is symmetric, then it has real eigenvalues $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$, with corresponding eigenvectors $v_0, v_1, \ldots, v_n$. Then, $\lambda_0 = 0$, and $p(t)$ is of the form

$$p(t) = c_0 v_0 + c_1 e^{-\lambda_1 t} v_1 + \cdots + c_n e^{-\lambda_n t}.$$

The basic decomposition results follow from introductory differential equations and the fact that Q is Hermitian. That $\lambda_0 = 0$ is a consequence of Q being positive semidefinite.

As defined, Q is clearly some square matrix acting on M^σ. In fact it can be constructed using the following theorem.

Theorem 3.2. *If Q is the intensity matrix of a σ-interchange process with weights $\{A_{ij}\}$, then let*

$$\epsilon_A = \sum_{i<j} a_{ij}(Id - (ij)).$$

Then, Q is given by

$$Q = f_\sigma(\epsilon_A),$$

where f_σ is the representation of S_n corresponding to M^σ. (Recall that for any $g \in S_n$, $f_\sigma(g)$ is a permutation matrix corresponding to the way in which g permutes the basis vectors of M^σ).

Thus, representations lie at the heart of understanding these σ-interchange processes.

3.3 Decomposition of M^σ

Recall from Theorem 2.4 that the Young module representations can be decomposed into direct sums of irreducible representations. In particular, there exists a single copy of the trivial representation. Let the corresponding submodule be denoted M^n, as for the trivial partion $[n]$, the irreducible representation is the same as that of the Young module. Then, Let M_0^σ be the submodule such that

$$M^\sigma \simeq M^n \oplus M_0^\sigma.$$

The trivial representation is one dimensional, and thus it must be spanned by a constant vector in order to be invariant under every permutation. Then, it is easy to show that

$$M_0^\sigma = \{v = (v_1, v_2, \ldots, v_m) \in M^\sigma | \sum v_i = 0\}.$$

In fact, we know that

$$M_0^\sigma = \bigoplus_{\sigma \trianglelefteq \rho \neq [n]} K(\rho, \sigma)\rho,$$

where the ρ in the sum is the irreducible representation. As we shall see, this decomposition is of crucial importance in the analysis of the σ-interchange processes.

3.4 The Uniform Distribution

For any continuous-time Markov process such as the σ-interchange processes, the uniform distribution is given by $\pi = (1/n, 1/n, \ldots, 1/n)$, where n is the number of possible states. This corresponds to each state being equally likely. If there is a path from each state to each other with nonzero probability (the process is *transitive*), the process will converge to the uniform distribution. In general, a symmetric Markov process will converge to a constant probability on each connected component.

Since the long-term behavior is well known, it is customary to examine the manner in which the Markov process converges. For connected graphs, this is done by looking at the vector $p(t) - \pi$. Note that since the components of both π and $p(t)$ sum to 1, $p(t) - \pi \in M_0^\sigma$. Now, recall from Theorem 3.1, that

$$p(t) = c_0 v_0 + c_1 e^{-\lambda_1 t} v_1 + \cdots + c_n e^{-\lambda_n t}.$$

We now know that $c_0 v_0 = \pi$. Thus,

$$p(t) - \pi = c_1 e^{-\lambda_1 t} v_1 + \cdots + c_n e^{-\lambda_n t}.$$

Thus, in general, the convergence will be determined by λ_1. This value, the second smallest eigenvalue, is known as the *spectral gap*.

Next, recall the decomposition $M^\sigma \simeq M^n \oplus M_0^\sigma$. Let $g_\sigma : \mathbb{C}G \mapsto M^n \oplus M_0^\sigma$ be the decomposition's representation, which is equivalent to f_σ, the permutation representation on the ordered set partitions of $\{1, 2, \ldots, n\}$. Now, let g_n and g_0 be the restrictions of g_σ to M^n and M_0^σ respectively. Then, g_n is the trivial map, $g \mapsto 1$. Thus,

$$g_n(\epsilon_A) = \sum a_{ij} g_n(Id - (ij)) = 0.$$

Now, since $g_\sigma(\epsilon_A)$ and $f_\sigma(\epsilon_A)$ have the same eigenvalues, it follows that $g_0(\epsilon_A)$ has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Thus, the spectral gap of a σ-interchange process is given by the minimal eigenvalue of ϵ_A under the representation $M_0^\sigma \simeq M^\sigma / M^n$.

Chapter 4

Definitions and Superorders of the Aldous Order

This chapter gives the original definition of the Aldous order, as motivated by the Aldous conjecture. It also includes an overview of previous work, several new, equivalent definitions of the order, and a discussion of superorders of the Aldous order.

4.1 The Aldous Conjecture

In 1992, David Aldous made the following conjecture:

Conjecture 4.1 (Aldous Conjecture (1992)). *For* any *graph G, the spectral gap of the $([1]^n])$-interchange process on G is the same as that of the simple random walk.*

Using the machinery developed in Chapter 3, this can be restated in a number of ways.

Theorem 4.1 (Aldous Conjecture Equivalent Statements). *For* any *graph G, the following are equivalent:*

(i) The spectral gap of the $([1]^n)$-interchange process on G is the same as that of the simple random walk.

(ii) The spectral gap of every nontrivial σ-interchange process equals that of the simple random walk ($\sigma = [n-1, 1]$).

(iii) If $\epsilon_A = \sum a_{ij}(Id - (ij))$, and $\sigma = [n-1, 1]$, ρ nontrivial, then the minimal eigenvalue of $\sigma(\epsilon_A)$ is less than or equal to that of $\rho(\epsilon_A)$.

Proof. Recall that the spectral gap of a σ-interchange process is given by the smallest eigenvalue of ϵ_A in the module

$$\bigoplus_{\sigma \trianglelefteq \rho \neq [n]} K(\rho, \sigma)\rho;$$

that is, the module is a direct sum containing every nontrivial representation (with possible multiplicity) which dominates σ. When $\sigma = [n-1, 1]$, then the desired module is simply σ. Moreover, for *any* nontrivial σ, the desired module contains a copy of the $[n-1, 1]$ representation. Thus, the spectral gap is always equal to that of the simple random walk if and only if the minimal eigenvalue lies in the $[n-1, 1]$ representation. This establishes the equivalence between (ii) and (iii). For (i) and (ii), clearly (ii) implies (i). For the other direction, the interchange process corresponds to $\sigma = [1^n]$. Since $[1^n]$ is the minimal element of the dominance order, the corresponding module contains *every* nontrivial module. Thus, any possible counterexample would provide a counterexample in the interchange process case. $\square$

4.2 The Aldous Order Definition

Thus, the Aldous Conjecture boils down to a relation between minimal eigenvalues of ϵ_A over different representations. While the Aldous Conjecture was proved in Caputo et al. (2010), the final formulation suggests a natural extension. Alon and Kozma followed through on this in defining the *Aldous Order* in Alon and Kozma (2011).

Definition 4.1 (Aldous Order, Initial Definition)**.** Let $\rho, \sigma \vdash n$ be partitions of n, $A = \{a_{i,j}\}_{1 \leq i < j \leq n}$, be a set of weights and define ϵ_A as before. Finally, have $\lambda_1(\epsilon_A; \rho)$ denote the smallest eigenvalue of $\rho(\epsilon_A)$. Then, ρ precedes σ in the Aldous order, written $\rho \preceq \sigma$ if

$$\lambda_1(\epsilon_A; \rho) \geq \lambda_1(\epsilon_A; \sigma).$$

For any weights A.

Therefore, the entirety of the Aldous Conjecture is captured in the following theorem,

Theorem 4.2 (Aldous Conjecture, Final Form)**.** *The partition $[n-1, 1]$ is the second greatest element of the Aldous order on partitions of n, after the trivial partition $[n]$.*

4.3 Previous Work

Alon and Kozma's paper is, at the only moment, the only published research on the Aldous order, and thus their findings represent the sum total of knowledge on the subject. The following consists of the main results that they have reported on the Aldous order.

Theorem 4.3 (Previous Theorems)**.** *The following are known results about the Aldous order.*

(i) The greatest and second greatest elements are $[n]$ *and* $[n-1, 1]$*.*

(ii) The least element is the alternating partition, $[1^n]$*.*

(iii) The Aldous order is a suborder of the lexicographic order on partitions.

(iv) The "hook shapes", $[n-k, 1^k]$*, are totally ordered. That is,*

$$[1^n] \prec [1^{n-1}, 1] \prec \cdots [1^2, n-2] \prec [1, n-1].$$

(v) If $n \geq 4k^2 + 4k$*,* ρ *has* $\leq k$ *squares outside of the left column,* σ *has* $\leq k$ *squares outside the first row, then* $\rho \prec \sigma$*, the main result of Alon and Kozma (2011).*

With the exception of (v) (and the Aldous Conjecture), the rest are very straightforward to prove. That the trivial and alternating representations are the greatest and least partitions has a simple algebraic proof. The eigenvalues for the hook shapes are well known and thus the computation of (iv) is straightforward. We provide a simpler proof of (iii) in Chapter 5.

In general, then, the Aldous order is not well known or easy to compute. Even for $n = 6$, much is not currently known.

In addition to the positive results listed, Alon and Kozma provided several negative results which proved inspirational, such as the incomparability of $[2, 1^{n-2}]$ and $[2, 2, 1^{n-4}]$.

4.4 Alternate Formulations

Given the motivation by interchange processes, the formulation of Alon and Kozma is understandable. However, it will prove useful to analyze relationships between a wide variety of eigenvalues. Our first theorem is about relations between λ_1 and $\lambda_{\max}$ for various matrices. It is given by

Theorem 4.4 (Eigenvalue Equalities). *For any nonnegative weights $A = \{a_{i,j}\}$, let ϵ_A be as before, and define*

$$Wt(A) = \sum_{i<j} a_{i,j}.$$

Additionally, let

$$\beta_A = \sum_{i<j} a_{i,j}(ij) = Wt(A)Id - \epsilon_A.$$

Then, for any representation $\rho \dashv n$ we have the following equalities.

(i) $\lambda_{\max}(\beta_A;\rho) + \lambda_1(\epsilon_A;\rho) = \lambda_1(\beta_A;\rho) + \lambda_{\max}(\epsilon_A;\rho) = Wt(A)$,

(ii) $\lambda_{\max}(\beta_A;\rho) + \lambda_1(\beta_A;\rho') = 0$,

(iii) $\lambda_{\max}(\epsilon_A;\rho) + \lambda_1(\epsilon_A;\rho') = \lambda_{\max}(\epsilon_A;\rho') + \lambda_1(\epsilon_A;\rho) = 2Wt(A)$,

(iv) $\lambda_{\max}(\epsilon_A;\rho) = Wt(A) + \lambda_{\max}(\beta_A;\rho')$.

Proof. The first equality follows from the fact that β_A and ϵ_A have the same eigenvectors, as they differ by a negative sign and $Wt(A)id$. Additionally, any eigenvector v of ϵ_A with eigenvalue λ_k will have an eigenvalue of $Wt(A) - \lambda_k$ as an eigenvector of β_A. Thus, if $(\lambda_1, \lambda_2, \ldots, \lambda_{\max})$ are the eigenvalues of $\epsilon_A$in increasing order, then those of β_A are

$$(Wt(A) - \lambda_{\max}, \ldots, Wt(A) - \lambda_1),$$

giving the desired equality.

The second equality is a consequence of the fact that ρ' is the result of tensoring ρ with the alternating representation. Thus, since β_A is the sum of transpositions (which are all odd), $\rho(\beta_A) = -\rho'(\beta_A)$, and the equality follows.

The third equality is a direct consequence of the first two, while the third follows from the first and third, as

$$\begin{aligned}\lambda_{\max}(\epsilon_A;\rho) &= 2Wt(A) - \lambda_1(\epsilon;\rho'),\\ &= Wt(A) + (Wt(A) - \lambda_1(\epsilon;\rho')),\\ &= Wt(A) + \lambda_{\max}(\beta_A;\rho').\end{aligned}$$

□

These equalities allow us to define a norm on the set of weights A corresponding to a given representation ρ.

Theorem 4.5 (ρ-norm on Weights). *For any set of nonnegative weights $A = \{a_{i,j}\}$, and representation $\rho \vdash n$, let*

$$||A||_\rho = Wt(A) + \lambda_{\max}(\beta_A;\rho).$$

Then, $||A||_\rho$ is a norm over the set of weights; that is,

$$||kA||_\rho = k||A||_\rho, ||A+B|| \leq ||A|| + ||B||.$$

Proof. Recall that we have that $||A||_\rho = \lambda_{\max}(\epsilon_A;\rho')$ from the fourth of the eigenvalue equalities. Then the map

$$A \mapsto \rho'(\epsilon_A)$$

is linear and since $\rho'(\epsilon_A)$ is semidefinite, the map

$$\rho'(\epsilon_A) \mapsto \lambda_{\max}(\rho'(\epsilon_A)) = \lambda_{\max}(\epsilon_A;\rho')$$

is the usual operator norm on the matrix $\rho'(\epsilon_A)$ given by

$$||M|| = \max_{||v||=1} ||Mv|| = \max_{\lambda_k} |\lambda_k|.$$

By linearity, the norm properties will be satisfied. □

The previous theorems allow us to give several equivalent formulations of the Aldous order.

Theorem 4.6 (Alternate Formulations). *For the same nonnegative weights $A = \{a_{i,j}\}$ as before, the following are equivalent:*

(i) $\rho \preceq \sigma$,

(ii) $\lambda_1(\epsilon_A;\rho) \geq \lambda_1(\epsilon_A;\sigma) \forall A$,

(iii) $\lambda_{\max}(\beta_A;\rho) \leq \lambda_{\max}(\beta_A;\sigma) \forall A$,

(iv) $||A||_\rho \leq ||A||_\sigma \forall A$.

The equivalence follows from the definition of the ρ-norm and $\rho \preceq \sigma$, as well as the aforementioned equalities. While equivalent, these formulations, particularly (iii), will prove useful in simplifying some proofs.

4.5 Properties of the ρ-Norm

Recall the ρ-norm defined by

$$||A||_\rho = Wt(A) + \lambda_{\max}(\beta_A;\rho).$$

This defines a total order on weights for each ρ. Additionally, there is a natural partial order on weights, given by the product order. That is, $A \leq B$ if $a_{ij} \leq b_{ij}$ for all i, j. We can then relate the two through the following theorem.

Theorem 4.7. *For nonnegative weights A, B and partition $\rho \vdash n$, if $A \leq B$ then we have that*

$$||A||_\rho \leq ||B||_\rho.$$

Equivalently, for any nonnegative weights C, then

$$||A||_\rho, ||C||_\rho \leq ||A+C||_\rho.$$

Proof. The equivalence of the two should be obvious, by taking $B = A + C$. We will prove the A, C case. Then, suppressing the ρ, we wish to show that

$$\lambda_{\max}(\beta_A) + Wt(A) \leq \lambda_{\max}(\beta_A + \beta_C) + Wt(A+C),$$

or alternately that

$$-Wt(C) \leq \lambda_{\max}(\beta_A + \beta_C) - \lambda_{\max}(\beta_A).$$

Note that this is a weaker inequality than

$$||\lambda_{\max}(\beta_A + \beta_C)| - |\lambda_{\max}(\beta_A)|| \leq Wt(C);$$

that is, the addition of β_A to β_C can't change $\lambda_{\max}$ by more than $Wt(C)$ in either direction, instead of it just not being able to decrease it by more than $Wt(C)$. Now, note that for symmetric matrices, $|\lambda_{\max}|$ is the usual operator norm. Thus, we have that the left-hand side is less than

$$|\lambda_{\max}(\beta_C)|,$$

by the norm inequality $p(u-v) \geq |p(u) - p(v)|$ for any norm p. However, recall that for any transposition (ij), the eigenvalues of $\rho((ij))$ will be either 1 or -1, as $\rho((ij))^2 = \rho((ij)^2) = \rho(id) = I$. Thus, if $||M||$ is the usual matrix norm, then $||\rho((ij))|| = 1$. Then, we have that

$$|\lambda_{\max}(\beta_C)| \leq ||\beta_C|| \leq \sum ||c_{ij}(ij)|| = \sum c_{ij} = Wt(C).$$

Thus, we have the desired result, so if $A \leq B$ in the product order, then $||A||_\rho \leq ||B||_\rho$. $\square$

As a consequence, for representations, ρ, σ, if $X < Y$ and $||Y||_\rho \leq ||X||_\sigma$, then for any $A \in [X, Y]$, we have $||A||_\rho \leq ||A||_\sigma$, which may help in proving comparability results if X and Y have easily computed maximum eigenvalues.

4.6 Superorders of the Aldous Order

Recall that a partial order R is a suborder of another partial order Q if R has the same relations as Q, except that some pairs which are comparable in Q may be incomparable in R. We can also say that Q is a superorder of R. For the Aldous order there is a natural construction of new orders, all of which will be superorders of the Aldous order.

Definition 4.2 (The S-Aldous Order)**.** Let X be the set of all possible weights for our graphs. Then, the Aldous order is given by

$$\rho \preceq \sigma \Leftrightarrow \lambda_{\max}(\beta_A; \rho) \leq \lambda_{\max}(\beta_A; \sigma) \forall A \in X.$$

Now, if $S \subset X$, then a new order, written $\preceq_S$ can be defined by replacing the X in the for all with S, so that

$$\rho \preceq_S \sigma \Leftrightarrow \lambda_{\max}(\beta_A; \rho) \leq \lambda_{\max}(\beta_A; \sigma) \forall A \in S.$$

Note that the suborder/superorder relation on partial orders is in itself a partial order. In this section, let $<<$ denote that relation. Then, the following theorem should be obvious.

Theorem 4.8. *The map from subsets of X to superorders of the Aldous order, given by $S \mapsto \preceq_S$, inverts orderings. Thus, if $A \subseteq B$, then $\preceq_B << \preceq_A$.*

Thus, every S-Aldous order is a superorder of the Aldous order, as $S \subset X$, and $\preceq$ is $\preceq_X$.

Examining the S-Aldous orders can be fruitful for a number of reasons. First of all, it may make sense to want to examine the relative eigenvalues of only certain types of graphs, such as multipartite graphs or trees. These classes may have additional structure and their orders could be interesting in their own right.

Secondly, since they are superorders, any incomparability result in an S-Aldous order must hold for the general Aldous order. Thus, an investigation of S-orders may yield information about the general order while focusing an investigation.

Finally, results on S-Aldous orders can aid in computing the general order. For some strict subsets $S \subset X$, the S-Aldous order and general Aldous order are the same. While it is an open question how small S can be made, the following restrictions are straightforward.

Theorem 4.9 (Restrictions on Weights)**.** *The following restrictions can be made on what weights are considered, while still maintaining the full order:*

(i) The weights can be restricted to any set which is dense in $\mathbb{R}^{\binom{n}{2}}$. In particular, the weights can be restricted to having rational coefficients.

(ii) The order given by a set S and that given by all real scalings of S are the same. In particular, the constraints $a_{ij} \leq 1$ or $\sum a_{ij} = 1$ can be made.

(iii) The coefficients can be constrained to the nonnegative integers.

The first follows from continuity in the definition of the Aldous order. The second comes from the fact that if a weight set A provides a strict inequality in one direction or the other, then so will cA for any positive c. The final condition comes from restricting to rationals, then scaling each weight set so that the resulting weights are integers.

As will be seen in Part III, much can be done with the S-Aldous order given by linear combinations of Young–Jucys–Murphy elements.

Part III

The YJM Aldous Order

Chapter 5

Basic Properties of the YJM Aldous Order

Recall that the standard Young tableaux of an irreducible representation form an eigenbasis for each Young–Jucys–Murphy (YJM) element χ_k. As such, if $\beta_A = \sum_k c_k \chi_k$, then the eigenvectors of $\rho(\beta_A)$ will be given by the SYT of ρ. More specifically, the eigenvalue of any SYT T with contents $C(T) = (a_1, a_2, \ldots, a_n)$ will be a linear function of the c_k, given by $\sum_{k=1}^{n} a_k c_k$. Thus, when the weights correspond to a YJM sum, the problem can be readily solved. This chapter describes the machinery used with the YJM sums, proving some results along the way.

5.1 Computing the YJM Order

Throughout, let $Y = \{A | \beta_A = \sum_k c_k \chi_k, c_k \geq 0\}$. In other words, Y is the set of YJM sums. Alon and Kozma touch on this family briefly, referring to them as *quasicomplete graphs*. Then, this class of graphs correspond to a superorder $\preceq_Y$, as described in Section 4.6.

Then, it is feasible to compute $\preceq_Y$ explicitly, formulating it as a linear program. To easily describe the computations, it helps to define a *dominant pair*.

Definition 5.1 (Dominant Pair)**.** Given two partitions $\sigma, \rho \dashv n$, let (A, v) be a pair consisting of a weight A and a vector v such that v is an eigenvector of $\sigma(\beta_A)$ with eigenvalue λ_v. Then, (A, v) is a *dominant pair* of σ over ρ if

$$\lambda_v > \lambda_{\max}(\beta_A; \rho).$$

Note that $\sigma \preceq \rho$ if and only if σ has no dominant pairs over ρ. Thus, this transforms the definition of $\preceq$ from a for all statement, to a statement about the existence or nonexistence of a counterexample. Additionally, by restricting the weights to be in some desired set S, we get an equivalent formulation for the S-Aldous order.

The YJM sums Y are unique because under any representation their eigenvectors are the same, the SYT. Thus, we can restrict the search for dominant pairs to each of the SYT. This can be done in the linear program which follows. The general idea is to search for weights in Y which form a dominant pair. This can be done by considering each SYT separately and searching for a workable set of weights. This is opposed to simultaneously searching through the weights for the SYT which both has maximal eigenvalue and is greater than the maximal eigenvalue of the other partition.

Definition 5.2 (Dominant Pair Linear Program)**.** Consider partitions $\sigma, \rho \dashv n$. Let σ_T be a SYT of σ, with content $C(\sigma_T) = (a_1, a_2, \ldots, a_n)$. Now, suppose that ρ has r distinct SYT. Denote each of these $\rho_1, \rho_2, \ldots, \rho_r$. Then, an optimization problem can be constructed in nonnegative variables $v = (c_1, c_2, \ldots, c_n)$ and d, subject to the constraints

$$C(T) \cdot v \geq C(\rho_k) + d,$$

for $k = 1, \ldots, r$. Additionally, there is the constraint

$$\sum_i c_i \leq 1.$$

Then, the objective function which is to be maximized is d.

This is a linear program and σ will have a dominant pair over ρ which uses σ_T only if the maximal value of d is nonzero.

Thus, this LP (linear program) can be solved using standard simplex methods. Moreover, the problem is bounded and feasible, as $c_i, d = 0$ provides a feasible solution. This then gives an algorithmic method for determining the order $\preceq_Y$ according to the following theorem:

Theorem 5.1 (LP Definition of $\preceq_Y$)**.** *For partitions $\sigma, \rho \dashv n$, $\sigma \preceq_Y \rho$ if and only if the dominant pair LP for every SYT of σ has maximum solution $d = 0$.*

This follows from the realization that feasible solutions of the LP when $d > 0$ correspond exactly to dominant pairs, and that for the YJM case, the eigenvector of a dominant pair will be a SYT.

This computational method gives exact results for as high as can be computed. Even naive implementations in Mathematica can yield solutions for $n \leq 13$. This proved a valuable source of conjectures and counterexamples.

5.2 Front and Back Sums

Within the set of weights Y, there proved to be two particularly useful classes of weights.

Definition 5.3 (Front and Back Sums)**.** The front sums $\{F_k\}$ are the sum of the first k YJM elements; that is,

$$F_k = \sum_{i=1}^{k} \chi_i.$$

Similarly, the back sums $\{B_k\}$ are the sum of the final k YJM elements; that is,

$$B_k = \sum_{i=n-k+1}^{n} \chi_i.$$

The front and back sums have several delightful properties, which we shall explore in depth. Most trivially, note that $F_n = B_n$ corresponds to the complete graph K_n. In the group algebra this corresponds to the class sum of all the transpositions, and thus in each representation corresponds to some multiple of the identity matrix. In general, $\lambda_{\max}(K_n; \rho)$ is the sum of the contents of ρ. This naturally gives the following theorem:

Theorem 5.2. *Let $\sigma, \rho \dashv n$ be two partitions. Then, if $\sigma \lhd \rho$,*

$$\lambda_{\max}(K_n; \sigma) < \lambda_{\max}(K_n; \rho).$$

This theorem is a simple consequence of realizing that any box-up move increases the content of that box while keeping the rest fixed. Thus, a sequence of box-up moves increases the sum of the contents, giving the result.

This allows us to easily describe SYTs which will have maximal eigenvalues for the front and back sums.

Theorem 5.3. *For any partition $\sigma \dashv n$, the maximal eigenvalues of F_k and B_k are attained by the following SYT.*

- *A SYT with maximal eigenvalue for the front sums F_k is given by the* front maximal *SYT, which is formed by filling in the numbers* $1, \dots, n$ *from left to right, beginning in the first row and working down.*
- *Similarly, the maximal SYT for the back sums B_k (the* back maximal *SYT) is formed by filling in 1 through n from top to bottom, beginning in the first column and working towards the right.*

Example 5.1. If $\rho = [3, 3, 2]$, then the front maximal SYT is

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} \end{array} \,.$$

while the back maximal SYT is

$$\begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 2 & 5 & 8 \\ \hline 3 & 6 \\ \cline{1-2} \end{array} \,.$$

As a proof of Theorem 5.3, consider the following argument. First, for the front sums, the eigenvalue under F_k for any SYT is determined by the placement of the first k numbers within the tableau. This will give correspond to some partition of k. The eigenvalue will then be the content sum of this k-partition. Moreover, the possible k-partitions are exactly those which fit inside the original partition σ.

Thus, it remains to determine which of those k-partitions will have the maximal content sum. Note that the k-partitions can still be ordered by the dominance ordering. This gives an ordering of the k-partitions contained in σ. As noted before, this implies that the maximal k-partition is also maximal in the dominance ordering (under the restriction that all partitions considered lie in σ). Thus, it is the shape with no admissible box-up moves inside σ. This shape is given by some number of full rows of σ and exactly one partial row. The front maximal SYT is such that the first k numbers inhabit exactly this partition. Thus, the front maximal SYT will take on the maximal eigenvalue for any front sum F_k.

For the back sums B_k, note that $B_k = K_n - F_{n-k}$. Thus, it is equivalent to minimizing F_{n-k}. This is done by finding the shape within σ which is minimal in the dominance order. This corresponds to some number of full columns, plus one partial column, which is what the described tableau gives.

The front sums give a simple proof of the following theorem:

Theorem 5.4 (Lexicographic Suborder Theorem)**.** *The Aldous order is a suborder of the lexicographic.*

Proof. Since the lexicographic order is a total order, it suffices to show for partitions $\rho, \sigma \dashv n$ that if $\rho < \sigma$ in the lexicographic, then $\sigma \prec \rho$; that is, the Aldous order never contradicts the lexicographic. This is analogous to showing that there is some A such that

$$\lambda_{\max}(\beta_A; \rho) < \lambda_{\max}(\beta_A; \sigma).$$

To find such an a, suppose that the first $r - 1$ rows of ρ and σ are equal, with $\rho_r < \sigma_r$. Then, let $k = \sum_{i=1}^{r} \rho_r$, the number of boxes in the first r rows of σ. Then, it will be the case that

$$\lambda_{\max}(F_k; \rho) < \lambda_{\max}(F_k; \sigma).$$

To see this, note that the maximal eigenvalues are achieved by the respective partition's front maximal tableaux. Then, the k-shape of σ will strictly dominate that of ρ by construction, giving the desired strict inequality. $\square$

Since the dominance order is also a suborder of the lexicographic order, this also has as a corollary Theorem 5.2. In the next chapter we will see the crucial role played by the back sums, in proving some of our most important results.

Chapter 6

Two Theorems in the YJM Aldous Order

This chapter centers on proving two theorems, which together describe much of the YJM Aldous order. The first can be attacked with little machinery apart from that covered in Chapter 5. The second will require the additional definition of *strict dominance*.

6.1 First Row Incomparability

Computations such as those described in Section 5.1 led to the observation that, even within the YJM Aldous order, partitions whose first row was the same length were incomparable. In fact, the incomparability could be shown only using the back sums $\{B_k\}$. Thus, we have the following theorem, and the accompanying proof.

Theorem 6.1 (First Row Incomparability Theorem). *For partitions $\rho, \sigma \dashv n$, if $\rho_1 = \sigma_1 = k$ and $\rho \neq \sigma$, then there exists $m_1, m_2 \leq n$ such that*

$$\lambda_{\max}(B_{m_1}; \rho) < \lambda_{\max}(B_{m_1}; \sigma) \text{ and } \lambda_{\max}(B_{m_2}; \rho) > \lambda_{\max}(B_{m_2}; \sigma).$$

Thus, ρ and σ are incomparable under the Aldous order.

Proof. First, if $\rho'_k = \sigma'_k$ (the last columns are of equal length), then we can proceed by induction by removing the last column of each partition, and taking the $m_i = \rho'_k + \bar{m}_i$ from the reduced shapes. The content sums for the new back sums will just be increased by the constant factor of the contents of the last column. These will be the same for both, so the inequalities will carry over.

Otherwise, assume without loss of generality that $\rho'_k > \sigma'_k$. Then, setting $m_1 = 1$ gives us the desired inequality immediately, as

$$\lambda_{\max}(B_1; \rho) = k - \rho'_k < k - \sigma'_k = \lambda_{\max}(B_1; \sigma).$$

Now, it remains to show that a valid m_2 must exist. To do so, consider the inequality

$$\sum_{i=r}^{k} \rho'_i \leq \sum_{i=r}^{k} \sigma'_i.$$

Clearly this inequality is satisfied if $r = 1$, so set r to be equal to the greatest such r. Now, set

$$m_2 = \sum_{i=r}^{k} \rho'_i.$$

To see that this gives the desired result, consider the subpartitions $\bar{\rho}$ and $\bar{\sigma}$ which consist of the columns from r to k. If we can establish our inequality on the last $\sum_{i=r}^{k} \rho'_i$ of these, then we will be done. Now, it is possible that $\bar{\sigma}$ has more elements than $\bar{\rho}$, so remove boxes off of the first column until we have equality. Then, they are both partitions of m_2, and we have

$$\sum_{i=m}^{\infty} (\bar{\rho})'_i \geq \sum_{i=m}^{\infty} (\bar{\sigma})'_i,$$

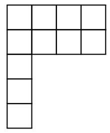
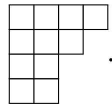
for all $m > 0$. This follows from the maximality of r, and we have equality when $m = 1$ now. Thus, this implies that $\bar{\rho}' \lhd \bar{\sigma}'$ by an earlier description of the dominance order. Thus, $\bar{\sigma} \lhd \bar{\rho}$ because the transpose inverts the operator. Thus, the sum of the last $\sum_{i=r}^{k} \rho'_i$ YJM will have a greater value on $\bar{\rho}$ then $\bar{\sigma}$ by the dominance order theorem. Recall that we had to remove some elements from the first column of $\bar{\sigma}$. Adding these back on, however, will only *decrease* the content sum, because it is replacing boxes with those farther down the column. Finally, going back to σ, ρ involves shifting all the counted boxes by the same number of boxes to the right, so the change in content sum will be the same in both. Thus, we get

$$\lambda_{\max}(B_{m_2}; \rho) > \lambda_{\max}(B_{m_2}; \sigma),$$

as desired. □

This can be seen in the following example:

Example 6.1. Consider the partitions $\rho = [4, 4, 1, 1, 1]$ and $\sigma = [4, 3, 2, 2]$. Then, the Young diagrams are

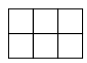
$\rho =$  $\sigma =$  .

Now, we take $r = 2$, as

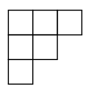
$$\sum_{i=2}^{4} \rho'_i = 2 + 2 + 2 = 6,$$

$$\sum_{i=2}^{4} \sigma'_i = 1 + 2 + 4 = 7.$$

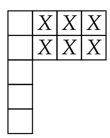
Then, $m_2 = 6$. Thus, the subshapes will be

$\bar{\rho} = [3, 3] =$ 

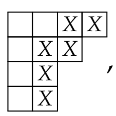
and

$\bar{\sigma} = [3, 2, 1] =$  .

Thus, $\bar{\rho} \lhd \bar{\sigma}$ by one up move. Then, the boxes that will be included in calculating the back sum B_6 will be



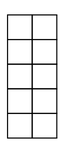
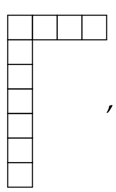
and



,

with ρ having the larger content sum.

This may lead to a natural supposition that the length of the first row provides a rank function, and that the Aldous order is thus a graded partial order. While this could be true for the general order, counterexamples arise in the case of the YJM Aldous order. In this order, we have

 $\prec_Y$  ,

but every partition with first row 3 is incomparable to one or the other. The general order could still be graded, if cases like this proved to be incomparable in the general order.

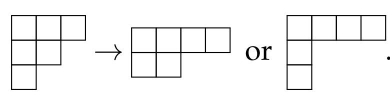
6.2 Strict Dominance

While the previous result deals with incomparability, and thus carries over directly into the full order, this section deals with conditions under which we can easily say that two elements are comparable in the YJM order. To do this, we will need to explicate the notion of *strong dominance*, which produces a suborder of the dominance order.

To this end, define a strong up-move be defined as follows.

Definition 6.1 (Strong Up-Moves)**.** Let a *strong up-move* on a Young diagram correspond to removing the last boxes from some intermediary k rows, and placing them on the end of the first k rows.

Example 6.2. For instance, if $\sigma = [3, 2, 1]$, then there are two shapes resulting from strong up-moves:



Note that every strong up-move increases the length of the first row.

Then, in the same way that the usual up-moves correspond to the dominance order, the strong up-moves define an order on partitions as well.

Definition 6.2 (Strong Dominance)**.** For two partitions $\rho, \sigma \dashv n$, σ strongly dominates ρ, written $\sigma >> \rho$, if σ can be produced from ρ via a series of strong up-moves.

Strong dominance is a graded order, graded by the length of the first row. This is a consequence of the property noted earlier, that every strong up-move increases the length of the first row by one.

Then, we have the following theorem, with accompanying proof, and its immediate corollary.

Theorem 6.2 (Strong Dominance Theorem)**.** *For partitions $\rho, \sigma \dashv n$, if σ strongly dominates ρ, then, for any tableau T of ρ, there exists a tableau T' of ρ with content greater than that of T for any value.*

Proof. In this proof, we provide an algorithm such that given a SYT of some partition ρ and a strong up-move performed on row, you can find a SYT of the new partition where each number has content equal to or greater than that of the old.

The first assumption that we make is that there is no overlap between the rows which are losing a box and those gaining a box. If there is, then the same transformation can be effected by a strong up-move which moves fewer boxes. Then, for a strong up-move which moves k boxes, the process is as follows: let $\{n_1, n_2, \ldots, n_k\}$ be the numbers inside the k boxes being moved. Next, order those numbers from least to greatest, so that $n_1 < n_2 < \ldots < n_k$. Then, place n_1 in column 1, n_2 in column 2, and so forth, placing them so that the each column is increasing left to right. This may result in shifting some boxes of the other boxes over to the right by one.

This will give the new shape with the numbers 1 through n placed in the boxes. It remains to show two things. First, that the content of no box decreases. Secondly, that the resulting distribution is in fact a SYT.

For the first part, we first consider the content of the $\{n_1, \ldots, n_k\}$. If n_i was in column r, then its new position will be in a column greater than r. This is because the number in row i column r had to be less than n_i, so when row i is sorted by order, n_i will be to the right of that number. Additionally, since there is no overlap, every n_i is moved to a lower row. Both of these moves increase the content of the box, so overall it will increase that content. For the boxes which don't change row, they either stay in the exact same spot, or move to the right one square. Either way, their contents don't decrease.

Now, it remains to show that the produced object is in fact a SYT. That is, we must show that the numbers increase along the rows and down the columns. The rows condition is satisfied by construction. For the columns, it is a little trickier. First, consider n_k. We wish to show that n_k will be less than the number below it. If n_k is on the end of the row, there will be no number below it and we are done. Otherwise, the k and $k+1$th rows will look like

$$\begin{array}{|c|c|c|c|c|c|} \hline \cdots & a_{i-1} & a_i & n_k & a_{i+1} & \cdots \\ \hline \cdots & b_{i-1} & b_i & b_{i+1} & b_{i+2} & \cdots \\ \hline \end{array}.$$

Now, since $a_j < b_j$ for any j since the original shape was a SYT, we have that $n_k < a_{i+1} < b_{i+1}$, as desired. In fact, the whole row works, because the inequalities elsewhere are either $a_j < b_j$, or $a_j < b_{j+1}$, both of which are true.

Next we turn to the intermediate columns. Again, use the same a, b

column designation, and this time let the inserted terms be n_a and n_b. Then, there are three possibilities based on the relative positionings of n_a and n_b. Most straightforward is when they are in the same column, so that it looks like

$$\begin{array}{|c|c|c|c|c|c|}\hline \cdots & a_{i-1} & a_i & n_a & a_{i+1} & \cdots \\ \hline \cdots & b_{i-1} & b_i & n_b & b_{i+1} & \cdots \\ \hline \end{array}.$$

Then, the necessary inequalities are the same as before. In general, we'll only be worried about the terms between the columns of n_a and n_b, as every other row a-row b pair will have existed in the original shape, either in their original position (boxes to the left of n_a and n_b) or shifted to the right (boxes to the right of n_a and n_b). Now, if n_a is to the left of n_b, then it looks like

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & n_a & a_i & \cdots & a_{j-1} & a_j & \cdots \\ \hline \cdots & b_i & b_{i+1} & \cdots & b_j & n_b & \cdots \\ \hline \end{array}.$$

In this case, $n_a < a_i < b_i$ and $a_j < b_j < n_b$, so n_a and n_b. The only other worry is about the terms in the middle, but $a_l < b_{l+1}$ in general, so each element in row b is less than the one above it.

The final case is

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & a_i & a_{i+1} & \cdots & a_j & n_a & \cdots \\ \hline \cdots & n_b & b_i & \cdots & b_{j-1} & b_j & \cdots \\ \hline \end{array}.$$

For this, recall that $n_a < n_b$. Thus, $a_l < n_a < b_{l-1}$ for all the intermediary l, satisfying the necssary inequalities. Now, the increasing down columns condition is really a pairwise condition, so this suffices to show that the produced tableau is in fact a SYT, proving the theorem. □

As an immediate corollary, we have

Corallary 6.1. *The $\preceq_Y$ is a suborder of the strong dominance order.*

Unfortunately, strong dominance doesn't provide a full description of $\preceq_Y$. As alluded to earlier, strong dominance is a graded poset, while $\preceq_Y$ is not graded. More generally, however, there exist $\sigma \preceq_Y \rho$ where for each SYT of σ T there is *no* SYT of ρ whose contents are all greater than or equal to T. The simplest example is $\sigma = [3,3], \rho = [4,1,1]$, but $\sigma \preceq_Y \rho$. This also shows that $\preceq_Y$ is not a suborder of the dominance order, although it is possible that $\preceq$ is.

6.3 Conjectures and Future Work

The main conjecture about the YJM order that went unproven was about the relationship between the backsums and the YJM order.

Conjecture 6.1 (Back Sum Sufficiency)**.** *If $B = \{B_k | k = 1, \ldots, n\}$, then $\preceq_Y$ is the same as $\preceq_B$.*

This is a surprising conjecture, as it narrows the family Y, which is an n-dimensional cone, down to a finite set of n elements. Intermediate to proving this would be showing that the binary sums sufficed, that is the linear combinations of YJM elements with coefficients 0 or 1. The LP formulation of $\preceq_Y$ might provide a promising avenue as exploration, as the binary coefficients can be seen as corner points of the region defined by $0 \leq c_k \leq 1$ (A constraint which we know doesn't affect the order).

Other open questions include some sort of combinatorial description of the YJM order. As of now, the only (conjectured) description is to compare the maximal content sums of all the back sums. These numbers seem to arise arbitrarily, and we were unable to attach any other combinatorial significance. Instead, it would be interesting to find some rule similar to the up-moves or strong up-moves which, through repeated application, produces all shapes greater than a given Young diagram.

A related question that went unsolved was how to, given a YJM element sum $\sum c_k \chi_k$, algorithmically construct a SYT with maximal eigenvalue. While we were able to do it for the back and front sums, we weren't able to find any general form, even when restricting to binary sums.

Altogether, there is much exploration left to be done, both in the YJM order and the general order. The YJM order is interesting because it plays very well with the combinatorics of standard Young tableaux and their content, while the general is a deep statement relating the representations of S_n in a new way.

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