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Climate Change in a Differential Equations Course: Using Bifurcation Diagrams to Explore Small Changes with Big Effects

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Abstract: The environmental phenomenon of climate change is of critical importance to today's science and global communities. Differential equations give a powerful lens onto this phenomenon, and so we should commit to discussing the mathematics of this environmental issue in differential equations courses. Doing so highlights the power of linking differential equations to environmental and social justice causes, and also brings important science to the forefront in the mathematics classroom. In this paper, we provide an extended problem, appropriate for a first course in differential equations, that uses bifurcation analysis to study climate change. Specifically, through studying hysteresis, this problem highlights how it may be the case that damage done to the environment by a small change cannot be reversed merely by undoing that small change. In addition to the problem itself, we elaborate on the mathematics, discuss implementation strategies, and provide examples of student work. Students in a mathematics classroom do not necessarily expect to confront such issues of social justice or environmental concerns, but we see it as our moral obligation as educators to include such lessons in our classes so that our students can become well-informed global citizens.

1 Introduction

In studying climate, scientists are often concerned about positive feedback loops: two or more processes that magnify each other, creating a system of amplification that leads to an enhanced cycle [4]. One example is the interaction of water vapor with global temperature. As the global temperature increases, the capacity of the atmosphere to contain evaporated water vapor also increases. Continued relative humidity levels would result in an increased amount of water vapor in the atmosphere. Water vapor is a greenhouse gas. Thus, if a climate system has more water vapor in the atmosphere, the global temperature will elevate due to the increased insulation of the atmosphere. These positive feedback loops will eventually equilibrate at a higher temperature. In a high emission scenario, scientists predict that a global increase in average temperature would be enough to kick off a system of positive feedback loops that would equilibrate, by the end of the 21st century, to a temperature between 2.6 and 4.8 degrees Celsius higher than in the years between 1986 to 2005 [7]. The result of this increase would be enough to melt ice caps, completely shift ecological systems, and contribute to species extinction due to significant changes in temperature, precipitation, and ocean acidification [7]. It may even redistribute the areas of the world that can support human life, making previously uninhabitable places like the northern reaches of Siberia and Canada habitable (though they may not support agriculture), and previously habitable places, like coastal zones [6] and southwest Asia [8], uninhabitable.

This environmental phenomenon can be studied in a first course in differential equations using bifurcation diagrams. In this paper, we provide an extended problem that has been implemented in an inquiry differential equations course. We also provide a discussion of the relevant mathematics, implementation strategies, and examples of student work. This extended problem has important mathematical concepts, namely bifurcation analysis (i.e., the effect of varying a parameter in a differential equation) and practical implications related to understanding societies' and governments' impact on the climate. Specifically, this problem highlights how it may be the case that damage done to the environment by a small change cannot be reversed merely by undoing that small change. Instead, reversing the damage may require dramatic changes in policy. The environmental phenomenon discussed here is of crucial importance to today's society. It is our ethical obligation to make clear that the rate at which the global temperature is rising is itself increasing [2] and after a certain point this will cause irreparable damage to our environment. Further to cultivate globally-minded citizens, we believe it is our moral obligation to engage students with these environmental issues in the mathematics classroom.

2 Introducing Climate Change to a Differential Equations Course

In this section, we provide our climate change problem, interspersed with connections to climate science research, the necessary mathematical background and experiences for students to be able to engage with these tasks, and a discussion of the mathematical solutions to the tasks. Following this section, we briefly describe implementation strategies for this problem and provide examples of student work.

2.1 Classroom Context

Importantly, prior to this extended problem, our students have reinvented a bifurcation diagram as a result of their work through modeling units [11]. These modeling units are part of a full course on differential equations taught from an inquiry-oriented perspective using our materials [12]. By inquiry-oriented we mean mathematics learning and instruction such that students are actively inquiring into the mathematics, while teachers, importantly, are inquiring into student thinking and are interested in using it to advance their mathematical agenda [9, 10]. An inquiry oriented differential equations (IODE) course is problem focused, with problems being experientially real, meaning students can utilize their existing ways of reasoning and experiences to make progress [3], and class time is devoted to a split of small group work and whole class discussion. Whole class discussion is facilitated by the instructor who focuses on generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation [5].

2.2 Extended Problem Exposition and Introduction

In service of these aforementioned goals, our climate change problem begins with the following exposition and extended problem:

> *In studying climate, scientists are often concerned about positive feedback loops: two or more processes that amplify each other, creating a system of amplification that leads to a vicious cycle. One example is the interaction of water vapor with global temperature. If global temperature increases, the capacity of the atmosphere to contain evaporated water vapor also increases. If water resources are available, this would result in an increased amount of water vapor in the atmosphere. Water vapor is a greenhouse gas, thus if a climate system has more water vapor in the atmosphere, the global temperature will increase due to the increased insulation of the atmosphere. This positive feedback loop will eventually equilibrate at a higher temperature. Some scientists predict that a global increase in average temperature of just two degrees would be enough to kick off a system of positive feedback loops that would equilibrate at a temperature at least 6 degrees higher than we have now. This 6-degree increase would be enough to turn rainforests into deserts and melt ice caps. It may even redistribute the areas of the world that can support human life, i.e. making previously uninhabitable places, like the northern reaches of Siberia and Canada, inhabitable (though they may not support agriculture) and previously inhabitable places, like coastal cities, uninhabitable.*
>
> *1. A modern pre-industrial average temperature at the equator is about 20 degrees Celsius. Assuming that our current global climate system has not undergone this vicious cycle, model this system with a phase line. What are the essential features of that phase line?*

This enhanced water vapor greenhouse effect is only one example of the of known climate feedback loops, some of which will act to further enhance or somewhat counteract the modern warming. For example, as more terrestrial heat is reradiated to the surface by the increased levels of water vapor in the atmosphere, less of the earth will be covered by glaciers and sea ice. The reduced surface area of light colored surfaces leads to increased absorption of solar radiation into the surface, thus magnifying the warming. However, with the presence of more water vapor there may also be more clouds. Low-level clouds dampen warming since they reflect a portion of incoming solar radiation back to space. This complex system of feedback loops acts to maintain an equilibrium in the climate system, however when there is a large enough external perturbation a new state of equilibrium is achieved [1].

Phase lines are standard mathematical tools used in qualitative analysis of one-dimensional autonomous systems. Previously, our students worked through various modeling tasks to reinvent the phase line for themselves. The important features of the phase line associated with this problem are two stable equilibria at 20 and 26 degrees Celsius, and an unstable equilibria at 22 degrees Celsius.

The problem continues with the next two tasks:

> *2. What is a simple differential equation that corresponds to your above phase line?*
>
> *3. A group of scientists came up with the following model for this global climate system:* $\frac{dC}{dt} = \frac{1}{10}(C-20)(22-C)(C-26)-k$, *where* C *is the temperature, in Celsius, and* k *is a parameter that represents governmental regulation of greenhouse gas emissions. Assume the baseline regulation corresponds to* $k=0$, *increasing regulation corresponds to increasing* k, *and the current equatorial temperature is around* 20 *degrees. To what equatorial temperature will the global climate equilibrate?*

In task 2, students create their own differential equation to model this scenario. For the remainder of the extended problem, to facilitate classroom cohesion, students work with the equation given in task 3. In the context of climate science and government regulation, we specifically desired a *negative* k value to correspond to *less regulation*, that is, *deregulation*. Doing so necessitated the differential equation contain a "$-k$" so that a negative k results in a positive shift of the average equatorial temperature. While this differential equation does not capture the complexity of climate change science, it captures the long-term behavior described in the exposition to the extended problem.

2.3 Transition to Bifurcation Diagrams

After students have considered the long-term average equatorial temperature they identify important values of the bifurcation parameter:

> 4. Sketch a bifurcation diagram and use it to describe what happens to the global temperature for various values of k.

Briefly, for a one dimensional system, a bifurcation diagram is a plot of the equilibrium points as a parameter is varied. A bifurcation point is a value of the bifurcation parameter where a qualitative change in the structure of solution space occurs. In the climate change model there are two such points where the branch of unstable equilibria meets either branch of stable equilibria, resulting in saddle-node (or fold) bifurcations (see Figure 1). In

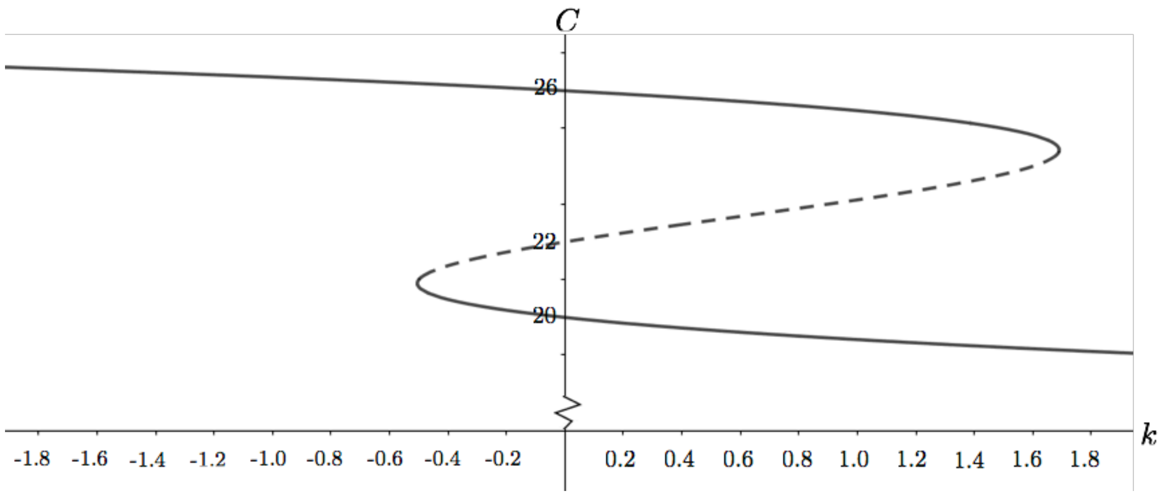

Figure 1: Bifurcation diagram for $\frac{dC}{dt} = \frac{1}{10}(C - 20)(22 - C)(C - 26) - k$ with solid lines for stable equilibria and dashed for unstable equilibria

an effort to provide an answer to what happens for various values of k, we can see that, for instance, for values of k between approximately -0.505 and 1.69, the system is bistable. As described above, at these special values of k the branch of unstable critical points meets the stable branches of equilibria in saddle-node bifurcations. For more extreme values of k the system has a globally attracting equilibrium point (approximately 19 or 26 degrees Celsius depending on the extreme value of k).

2.4 Small Changes with Big Effects

After orientation to the bifurcation diagram, the next task contains the pivotal moment of the problem, that is, how subtle parameter variation (in this case, government deregulation) can have dramatic impact that cannot be readily undone:

> *5. Suppose at the start of a new governmental administration, the temperature at the equator is about* 20 *degrees Celsius, and* $k = 0$. *Based on the model and other economic concerns, a government decides to deregulate emissions so that* $k = -0.5$. *Later, the Smokestack Association successfully lobbied for a* 5% *change, resulting in* $k = -0.525$. *Subsequently, a new administration undid that change, reverting to* $k = -0.5$, *and eventually back to* $k = 0$. *What is the equilibrium temperature at the equator after all of these changes?*

For this differential equation, we chose the k values of -0.5 and -0.525 because they straddle one of the bifurcation points in the diagram and have only a seemingly insignificant difference between them. Initially, for both $k = 0$ and $k = -0.5$, the global average equatorial temperature equilibrates at the lower of the two stable equilibria, namely 20 and approximately 21 degrees Celsius, respectively. However, the change to $k = -0.525$ moves the system beyond the bifurcation point, resulting in a globally attracting equilibrium corresponding to a significantly elevated average equatorial temperature, in this case approximately 26.2 degrees Celsius.

The bifurcation diagram indicates that, to undo the change to the average equatorial temperature, it is not mathematically viable to simply undo the change from k=-0.525 back to k=-0.5. This is known as a hysteresis effect. For $k = -0.5$, an initial condition near 26.2 degrees Celsius would be in the basin of attraction for the upper equilibria. Indeed, for $k = -0.5$, any initial condition exceeding 21 degrees Celsius would equilibrate near 26.2 degrees Celsius. Moreover, returning to the baseline regulation ($k = 0$) also fails to return average equatorial temperature to the initial 20 degrees Celsius. Therefore, at the end of these changes to k, the global average equatorial temperature equilibrates to 26 degrees Celsius.

In the course of this exploration, students may have already discussed the final task:

> *6. Use your bifurcation diagram to propose a plan that will return the temperature at the equator to 20 degrees Celsius.*

To have the global average equatorial temperature return to 20 degrees Celsius requires extreme government regulation (i.e., a k in excess of approximately 1.69). Such a value of k would shift the autonomous derivative graph down far enough to result in a globally attracting equilibria near 19.1 degrees Celsius. After reaching this equilibrium, k could be returned to the baseline of $k = 0$, so that the global average equatorial temperature would equilibrate to 20 degrees Celsius.

3 Implementation and Student Work

In this section, we provide a general discussion on the implementation of the extended problem that we have used in our classrooms. Further, we provide examples of student work to highlight the power of this extended problem and its impact on student understanding. We also highlight how students come to interpret the problem in the significant moral landscape of climate change.

Two of the authors implemented this extended problem at their home universities, during an inquiry-oriented differential equations course as described above. The extended problem appeared about halfway through the semester and served as a capstone on the study of one-dimensional differential equations.

Students formed small groups and worked through series of problems. The tasks were spread across several handouts with tasks 3, 5, and 6, strategically placed to avoid spoiling answers to previous tasks. To cover the problem, one author devoted two class days to the climate change problem while the other author devoted one class day and assigned the remaining tasks as homework with a written report. During class time the instructors circulated the room working with small groups to facilitate deep engagement in the tasks. For example, one student supplemented their bifurcation diagram with a sequence of autonomous derivative graphs with values of k corresponding to task 5, and the instructor built on that student contribution to develop a shared understanding of the hysteresis effect among the whole class.

Importantly, we highlight examples of student work to showcase the rich potential that students have to engage with a culturally and ethically significant problem such as climate change. For example, when drawing bifurcation diagrams for task 4, Figure 2 shows a carefully labeled, qualitatively accurate, bifurcation diagram. Similarly, Figure 3 shows a contextualized understanding of "safe and unsafe zones" for temperature. Note, however, the idea of safe and unsafe zones should be restricted to the bistable region, as opposed to the shading shown that suggests, for instance, that a temperature of 19 degrees Celsius is unsafe when $k = 0$. When discussing task 5, one student wrote:

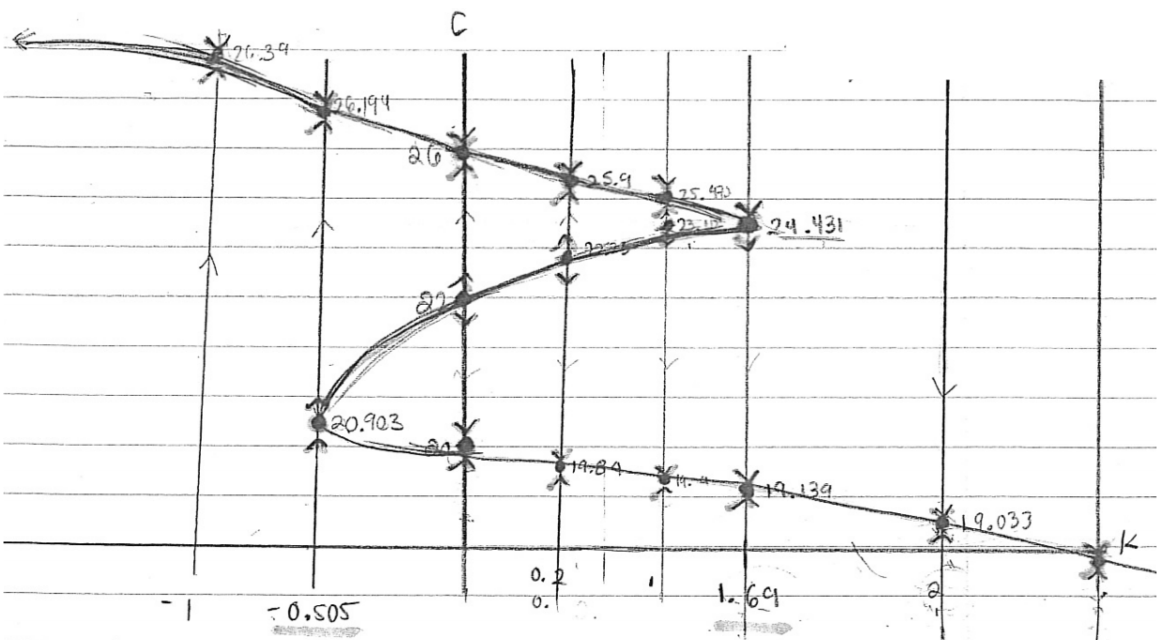

Figure 2: Example of student work.

> After we get $k = -0.525$, we lose two equilibrium points and are forced to jump to the highest equilibrium point $y = 26$. Even after going back to $k = 0$ we still have to settle at $y = 26$ because we already jumped passed the repeller $y = 22$, which is sort of the "over-the-hill" point.

From an expert's point of view, this student is discussing the hysteresis effect. The small change to $k = -0.525$ (i.e., deregulation) causes the global average equatorial temperature to equilibrate near 26 degrees Celsius as previous equilibrium points no longer exist. Further, the student highlights an understanding of the way in which the unstable equilibrium (22 degrees Celsius) serves as a threshold between the other equilibria, providing an opportunity for the instructor to introduce the formal term hysteresis.

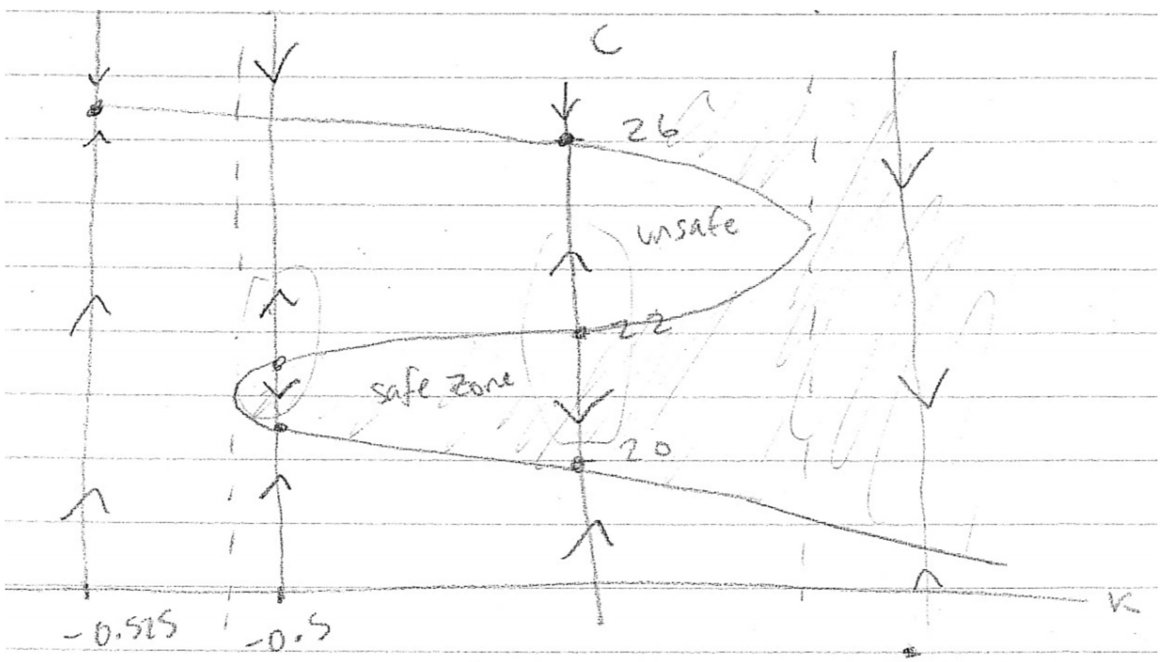

Figure 3: Example of student work.

4 Discussion and Conclusion

In this paper, we presented an extended problem that orients students to the hysteresis effect, one in which a subtle change in a parameter can have drastic implications for long-term behavior of solutions to a differential equation. We framed this mathematical lesson within a moral imperative to engage mathematics students with important modern and global environmental issues.

Our students appreciated the relevance and immediacy of this extended problem. Many were surprised in task 5 that the temperature would not return to 20 degrees Celsius when baseline regulation is reestablished. They were also concerned about the implications of needing a much larger, in absolute value, k to undo the damage done by the deregulation in task 5. One student was particularly excited that, as a dual major in mathematics and political science, this was the first time his two majors ever coincided. Another student wrote the following as a reflection on their work in the climate change problem:

> I found the climate change problem particularly interesting because it shows how bifurcation diagrams can be used in real-world scenarios. Initially I thought the phase line for this problem would only have 2 or 1 equilibrium solutions because I could not comprehend how we could stray away from an attractor far enough to jump to another attractor point, but this problem helped me see that it actually could happen. Also, I like this problem because the context makes sense. If there is too much government deregulation, we would lose our "safe zone" and our repelling equilibrium solution, forcing us to jump up to high average temperatures. This is what I would expect to happen in the real world. However, this problem also shows me if this

> were to happen, it is not necessarily irreversible. Initially, I figured if we were to lose our safe zone, we would jump up to the highest equilibrium point and we would be screwed, but this problem showed me we actually could reset temperatures with very strict government regulation of pollution. This problem is a nice illustration of bifurcation diagrams and bifurcation values in the real world.

Of course, the model in task 3 is simple and could not possibly capture the complexity of climate science. However, it is a powerful model because in task 2, students actually develop a very similar model for themselves. This affords the opportunity to fully understand the model and facilitates engagement with the surrounding lesson. Predictions about global average temperature at the equator made sense to our students. Therefore, the drastic changes to these temperatures, resulting from small changes in the parameter, were surprising but also easy to grasp as exemplified in the student quote above. Therein lies the power of the model, it's simplicity allowed students to grasp the consequences of the regulation/deregulation scenario and also to imagine how such consequences could arise in more complicated and scientifically accurate models.

The environmental phenomenon of climate change science discussed in this paper is of immediate concern to our society. It is our ethical obligation to make clear that the rate at which the global temperature is rising is itself increasing [2] and after a certain point that will cause irreparable damage to our environment. In particular, such damage may redistribute habitable areas in such a way that raises serious concerns regarding equity and social justice. We used differential equations to explore this phenomenon and what it might take to undo certain damages. Students in a mathematics classroom do not necessarily expect to confront such issues of social justice or environmental concerns, but we contend that it is our moral obligation as instructors to include such lessons in our classes so that our students will be well-informed global citizens.

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