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An Epidemiological Math Model Approach to a Political System with Three Parties

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Abstract: The United States has proven to be and remains a dual political party system. Each party is associated to its own ideologies, yet work by Baldassarri and Goldberg in *Neither Ideologues Nor Agnostics* show that many Americans have positions on economic and social issues that don't fall into one of the two mainstream party platforms. Our interest lies in studying how recruitment from one party into another impacts an election. In particular, there was a growing third party presence in the 2000 and 2016 elections. Motivated by previous work, an epidemiological approach is taken to treat the spread of ideologies and political affiliations among three parties, analogous to the spread of an infectious disease. A nonlinear compartmental model is derived to study the movement between classes of voters with the assumption of a constant population that is homogeneously mixed. Numerical simulations are conducted with initial conditions from reported national data with varying parameters associated to the strengths of political ideologies. We determine the equilibria analytically and discuss the stability of the system both algebraically and through simulation, parameters are expressed to stabilize a co-existence between three parties, and numerical simulations are performed to verify and support analysis.

There is nothing which I dread so much as a division of the republic into two great parties, each arranged under its leader, and concerting measures in opposition to each other. This, in my humble apprehension, is to be dreaded as the greatest political evil under our Constitution.
— John Adams, 1780

1 Introduction

The United States has proven to be and remains a dual political party system. The 2000 and 2016 presidential elections have brought with them a growing number of voters moving into third parties. The goal of this research is to study the movement of voters between political parties, and the population dynamics amongst the voting class. To evaluate these attributes, we will approach the issue using a nonlinear mathematical model employing epidemiological methods with the assumption of a constant population that is homogeneously mixed. That is, the ideology and influence of a political party is treated as a disease that a person can contract.

A deterministic model of the movement of eligible voters within a constant population is presented. We determine the equilibria analytically and discuss the stability of the system. Parameters are expressed to stabilize a co-existence between three parties, and numerical simulations are presented to verify and support the analysis. This manuscript is concluded with a note to instructors interested in conducting this type of undergraduate research project.

2 Three-Party Voter Model

In [4], a deterministic model of the movement of voters between two political parties is developed with an epidemiological approach. The model presented here is expanded to include a third option that combines the "no party preference" and third parties. The total population size, N, is assumed to remain constant. That is, for every person that enters the system, another leaves the system. In the United States, citizens who turn 18 years of age or immigrants who become naturalized citizens enter the class of eligible voters. Voters enter the system at rate μN into the eligible voter class, V. The standard convention of registering as a member of a political party when registering to vote is followed. It is assumed that the choice an eligible voter makes is influenced by coming into contact with a member of a given party and the probability the eligible voter *contracts* their ideology. That is, an eligible voter enters a political party at a per-capita rate of $\beta = pk$ where k is the average number of eligible voters a member of a political party contacts and p is the probability an eligible voter is convinced to join that party. Hence, the rate at which eligible voters become members of Political Party A, is $\beta_1 V\left(\frac{A}{N}\right)$ where $\frac{A}{N}$ is the probability of coming into contact with a member of Party A. Similarly, the rate at which people join Political Party B from V is given by $\beta_2 V\left(\frac{B}{N}\right)$, and those who choose a third party will enter C from V at a rate of $\beta_3 V\left(\frac{C}{N}\right)$. It is important to note that voter registration or party affiliation is interchangeable here with votes to a presidential candidate from a given party in that the two acts are ultimately attributed to recruitment. As stated above, the population is assumed to be constant, therefore

$$N = V + A + B + C.$$

Variables and parameters in the model are summarized in Tables 1 and 2.

V	Population of eligible voters
A	Population of Political Party A
B	Population of Political Party B
C	Population of Political Party C
N	Total Population
v	Proportion of eligible voters, V/N
a	Proportion of Political Party A, A/N
b	Proportion of Political Party B, B/N
c	Proportion of Political Party C, C/N

Table 1: Description of variables in the model.

Notably, not all people who are eligible voters register to vote. Hence, individuals leave the system with out joining any political party at a rate μV. Once having joined a party, voters may become ineligible due to inactivity, felony convictions, or death. Members of political Party A, B, and C, leave the system at rates μA, μB, and μC, respectively.

The model also accounts for the movement of registered voters between political parties. The per capita recruitment rate from one political party to another is denoted by θ_i, for $i = 1, \ldots 6$. Thus, the rate at which registered voters leave Political Party A and join Party B is $\theta_1 A \left(\frac{B}{N}\right)$. Similarly, the recruitment rate at which members of Political Party B join Political Party A is $\theta_2 B \left(\frac{A}{N}\right)$. Similar recruitment rates are defined for θ_i for $i = 3, \ldots 6$ and can be seen in Figure 1 in the diagram of the three-model system.

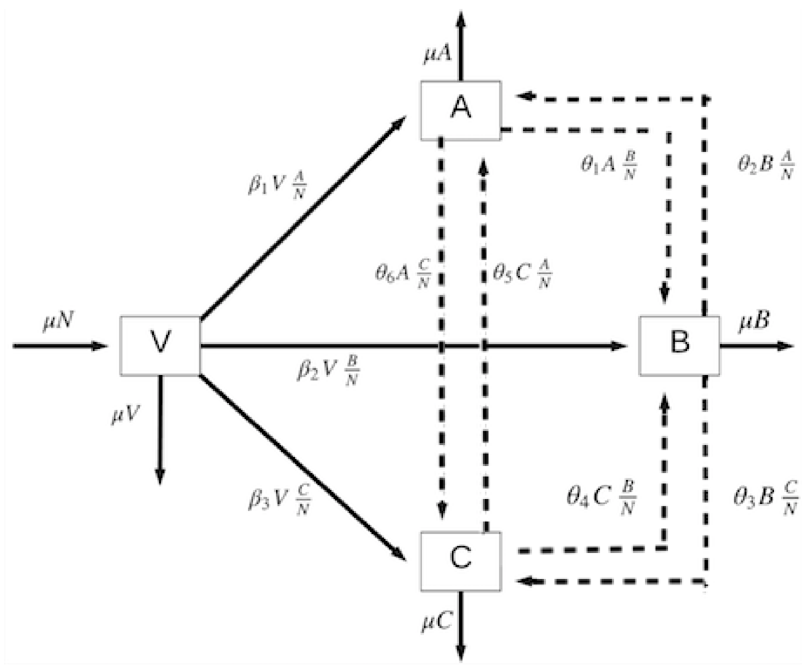

Figure 1: Diagram of the three-model system

μ	rate at which individuals enter and leave voting system
β_1	per capita recruitment rate of Party A from V
β_2	per capita recruitment rate of Party B from V
β_3	per capita recruitment rate of Party C from V
θ_1	per capita recruitment rate of Party B from Party A
θ_2	per capita recruitment rate of Party A from Party B
θ_3	per capita recruitment rate of Party C from Party B
θ_4	per capita recruitment rate of Party B from Party C
θ_5	per capita recruitment rate of Party A from Party C
θ_6	per capita recruitment rate of Party C from Party A
Ω_1	Net Shift between Party A and Party B
Ω_2	Net Shift between Party B and Party C
Ω_3	Net Shift between Party A and Party C

Table 2: Description of parameters in the model.

The solid lines in the diagram represent both the rates at which individuals enter and leave the system and the rates at which individuals initially choose their political parties. This initial choice will determine where most people stay within the system. The dashed lines represent the recruitment rates from one political party to another.

The governing equations to the model are

$$\begin{aligned}
\frac{dV}{dt} &= \mu N - \beta_1 V\left(\frac{A}{N}\right) - \beta_2 V\left(\frac{B}{N}\right) - \beta_3 V\left(\frac{C}{N}\right) - \mu V, \\
\frac{dA}{dt} &= \beta_1 V\left(\frac{A}{N}\right) + \theta_2 B\left(\frac{A}{N}\right) + \theta_5 C\left(\frac{A}{N}\right) - \theta_1 A\left(\frac{B}{N}\right) - \theta_6 A\left(\frac{C}{N}\right) - \mu A, \\
\frac{dB}{dt} &= \beta_2 V\left(\frac{B}{N}\right) + \theta_1 A\left(\frac{B}{N}\right) + \theta_4 C\left(\frac{B}{N}\right) - \theta_2 B\left(\frac{A}{N}\right) - \theta_3 B\left(\frac{C}{N}\right) - \mu B, \\
\frac{dC}{dt} &= \beta_3 V\left(\frac{C}{N}\right) + \theta_3 B\left(\frac{C}{N}\right) + \theta_6 A\left(\frac{C}{N}\right) - \theta_4 C\left(\frac{B}{N}\right) - \theta_5 C\left(\frac{A}{N}\right) - \mu C
\end{aligned} \tag{2.1}$$

where $V(0) > 0, A(0) \geq 0, B(0) \geq 0,$ and $C(0) \geq 0.$

2.1 Simplifying

To simplify the model, parameters are introduced to denote the net shift between political parties:

$$\begin{aligned}
\Omega_1 &= \theta_1 - \theta_2 \\
\Omega_2 &= \theta_3 - \theta_4 \\
\Omega_3 &= \theta_6 - \theta_5.
\end{aligned}$$

The system is rewritten to reflect proportion of populations as opposed to population size. The equation $N = V + A + B + C$ is rewritten as $1 = v + a + b + c$, where, for example, $a = \frac{A}{N}$ (See Table 1). System of equations (2.1) is rewritten as

$$\begin{aligned}
\frac{dv}{dt} &= \mu - \beta_1 va - \beta_2 vb - \beta_3 vc - \mu v \\
\frac{da}{dt} &= \beta_1 va - \Omega_1 ab - \Omega_3 ac - \mu a \\
\frac{db}{dt} &= \beta_2 vb + \Omega_1 ab - \Omega_2 bc - \mu b \\
\frac{dc}{dt} &= \beta_3 vc + \Omega_2 bc + \Omega_3 ac - \mu c.
\end{aligned} \tag{2.2}$$

The system is reduced to three differential equations by substituting $v = 1 - a - b - c$:

$$\begin{aligned}
\frac{da}{dt} &= a[\beta_1(1 - a - b - c) + \Omega_3 c - \Omega_1 b - \mu] \\
\frac{db}{dt} &= b[\beta_2(1 - a - b - c) + \Omega_1 a - \Omega_2 c - \mu] \\
\frac{dc}{dt} &= c[\beta_3(1 - a - b - c) + \Omega_2 b - \Omega_3 a - \mu].
\end{aligned} \tag{2.3}$$

3 Equilibria and Stability

The equilibrium of (2.3) are denoted by (a^*, b^*, c^*), where $a^* = A^*/N$, $b^* = B^*/N$, $c^* = C^*/N$, and (V^*, A^*, B^*, C^*) denotes the equilibrium of the unreduced system (2.1). Equilibria occur in the party-free system $E_0(0,0,0)$, single-party system $E_1(a^*, 0, 0)$, $E_2(0, b^*, 0)$, and $E_3(0, 0, c^*)$, dual-party system $E_4(a^*, b^*, 0)$, $E_5(a^*, 0, c^*)$, and $E_6(0, b^*, c^*)$, and in the interior case $E_7(a^*, b^*, c^*)$. We derive each equilibrium algebraically and include discussion on the stability algebraically in the dictatorship case (single-party equilibrium) and via simulation in the endemic case. The party-free equilibrium $E_0(0,0,0)$ always exists without any conditions on the associated parameters and corresponds to nonpartisan systems where no official political parties exist. Refer to [4] and references therein.

3.1 Single-party equilibrium.

We derive $E_1(a^*, 0, 0)$, the equilibrium in a dictatorship system. We seek $a > 0$ such that

$$\frac{da}{dt} = \beta_1(1 - a - b - c)a - \Omega_1 ab - \Omega_3 ac - \mu a = 0,$$

where $b, c \equiv 0$. That is,

$$\begin{aligned}\frac{da}{dt} &= \beta_1(1 - a^* - 0 - 0)a^* - \Omega_1 a^*(0) - \Omega_3 a^*(0) - \mu a^* \\ &= \beta_1(1 - a^*)a^* - \mu a^* = 0,\end{aligned}$$

which implies

$$a^* = 1 - \frac{\mu}{\beta_1}.$$

Hence, we find the equilibrium $E_1\left(1 - \frac{\mu}{\beta_1}, 0, 0\right)$. The equilibria $E_2(0, b^*, 0)$, and $E_3(0, 0, c^*)$ are solved using the same strategy. Note that these equilibria only exist if $\beta > \mu$.

3.2 Dual-party equilibrium.

Without the presence of a third-party, we can find the equilibrium $E_4(a^*, b^*, 0)$ by solving the following system using Cramer's rule,

$$\begin{bmatrix} -\beta_1 & -(\beta_1 + \Omega_1) \\ \Omega_1 - \beta_2 & -\beta_2 \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} \mu - \beta_1 \\ \mu - \beta_2 \end{bmatrix}.$$

Thus

$$E_4(a^*, b^*, 0) = E_4\left(\frac{\beta_2(\beta_1 - \mu) - (\beta_2 - \mu)(\beta_1 + \Omega_1)}{\beta_1\beta_2 + (\Omega_1 - \beta_2)(\beta_1 + \Omega_1)}, \frac{\beta_1(\beta_2 - \mu) + (\beta_1 - \mu)(\Omega_1 - \beta_2)}{\beta_1\beta_2 + (\Omega_1 - \beta_2)(\beta_1 + \Omega_1)}, 0\right).$$

This is consistent with the results in [4]. Similar computations yield E_5 and E_6,

$$E_5(a^*, 0, c^*) = E_5\left(\frac{\beta_3(\beta_1 - \mu) + (\beta_3 - \mu)(\Omega_3 - \beta_1)}{\beta_1\beta_3 + (\beta_3 + \Omega_3)(\Omega_3 - \beta_1)}, 0, \frac{\beta_1(\beta_3 - \mu) - (\beta_1 - \mu)(\Omega_3 + \beta_3)}{\beta_1\beta_3 + (\beta_3 + \Omega_3)(\Omega_3 - \beta_1)}\right),$$

and

$$E_6(0, b^*, c^*) = E_6\left(0, \frac{\beta_3(\beta_2 - \mu) - (\beta_3 - \mu)(\beta_2 + \Omega_2)}{\beta_2\beta_3 + (\beta_2 + \Omega_2)(\Omega_2 - \beta_3)}, \frac{\beta_2(\beta_3 - \mu) + (\beta_2 - \mu)(\Omega_2 - \beta_3)}{\beta_2\beta_3 + (\beta_2 + \Omega_2)(\Omega_2 - \beta_3)}\right).$$

3.3 Three-party equilibrium.

Finally, E_7 is the equilibrium solution of most interest. It is also found using Cramer's Rule. We express $E_7(a^*, b^*, c^*)$ as $E_7(p_1^*/q, p_2^*/q, p_3^*/q)$ where

$$\begin{aligned} q = &-\beta_1\beta_2\beta_3 - \beta_1(\beta_2 + \Omega_2)(\Omega_2 - \beta_3) - \beta_2(\Omega_3 - \beta_1)(\beta_3 + \Omega_3) - \beta_3(\beta_1 + \Omega_1)(\Omega_1 - \beta_2) \\ &- (\beta_1 + \Omega_1)(\beta_2 + \Omega_2)(\beta_3 + \Omega_3) + (\Omega_1 - \beta_2)(\Omega_2 - \beta_3)(\Omega_3 - \beta_1), \end{aligned}$$

and with

$$\begin{aligned} p_1 &= -(\beta_1 - \mu)[\beta_2\beta_3 + (\beta_2 + \Omega_2)(\Omega_2 - \beta_3)] + (\beta_2 - \mu)[\beta_3(\beta_1 + \Omega_1) - (\Omega_3 - \beta_1)(\Omega_2 - \beta_3)] \\ &\quad - (\beta_3 - \mu)(\beta_2(\beta_1 - \Omega_3) - (\beta_1 + \Omega_1)(\beta_2 + \Omega_2)), \\ p_2 &= \beta_1[\beta_3(\mu - \beta_2) - (\beta_2 + \Omega_2)(\mu - \beta_3)] + (\Omega_1 - \beta_2)[\beta_3(\mu - \beta_1) + (\Omega_3 - \beta_1)(\mu - \beta_3)] \\ &\quad + (\beta_3 + \Omega_3)[(\beta_2 + \Omega_2)(\mu - \beta_1) + (\Omega_3 - \beta_1)(\mu - \beta_3)], \\ p_3 &= \beta_1[\beta_2(\mu - \beta_3) + (\mu - \beta_2)(\Omega_2 - \beta_3)] + (\Omega_1 - \beta_2)[(\beta_1 + \Omega_1)(\mu - \beta_3) + (\mu - \beta_1)(\Omega_2 - \beta_3)] \\ &\quad + (\beta_3 + \Omega_3)[(\beta_1 + \Omega_1)(\mu - \beta_2) - \beta_2(\mu - \beta_1)]. \end{aligned}$$

3.4 Stability Analysis.

The Jacobian of system (2.3) is given by

$$J = \begin{pmatrix} \Theta_1 & -(\Omega_1 + \beta_1)a & (\Omega_3 - \beta_1)a \\ (\Omega_1 - \beta_2)b & \Theta_2 & -(\Omega_2 + \beta_2)b \\ -(\Omega_3 + \beta_3)c & (\Omega_2 - \beta_3)c & \Theta_3 \end{pmatrix},$$

where

$$\begin{aligned} \Theta_1 &= \beta_1(1 - 2a - b - c) + \Omega_3 c - \Omega_1 b - \mu, \\ \Theta_2 &= \beta_2(1 - a - 2b - c) + \Omega_1 a - \Omega_2 c - \mu, \\ \Theta_3 &= \beta_3(1 - a - b - 2c) + \Omega_2 b - \Omega_3 a - \mu. \end{aligned}$$

We can now find the eigenvalues of J evaluated at each E_i to determine the stability. For instance, in the non-partisan system, the eigenvalues of $J(E_0(0, 0, 0))$ are given by $\lambda_1 = \beta_1 - \mu$, $\lambda_2 = \beta_2 - \mu$, and $\lambda_3 = \beta_3 - \mu$. Relative to the restriction for E_1, E_2, and E_2 to exist, namely $\beta_i > \mu$ for $i = 1, 2, 3$; we conclude that E_0 is unstable. Alternatively, the goal of this manuscript will be to understand the stability of equilibrium using a simulation approach.

3.5 Simulation approach.

Below we take a simulation approach to analyze stability of the equilibrium $E_7(a^*, b^*, c^*)$. The simulations were run using ODE solvers in the SciPy library and plotted with the Matplotlib library.

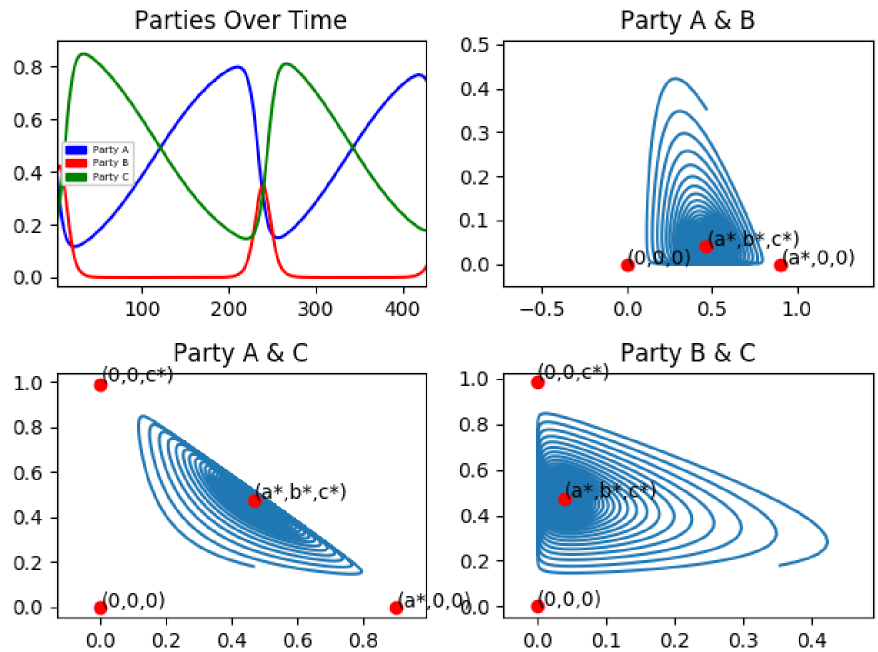

Figure 2: Three-party simulation with $\mu = 0.01$, $\beta_1 = 0.1$, $\beta_2 = 1$, $\beta_3 = 1$, $\Omega_1 = 1$, $\Omega_2 = 1$, $\Omega_3 = 0.1$

In Figure 2 the net-shift parameters are $\Omega_1 = 1.00$, $\Omega_2 = 0.10$ and $\Omega_3 = 1.00$. Hence, the net shift between between Party A and Party B is to Party A, Party C dominates in recruitment between Party A and Party C, and the net shift is to Party C between Party B and Party C.

By setting the net-shift recruitment rate between Party B and Party C low at 0.10 and having the recruitment rates between Party A and Party B, and between Party A and Party C be the same at 1.00, we are simulating a system in which two political parties are initially dominant with the third political party, namely, Party B staying close to 0. Over time, we see oscillatory behavior as the parties tend towards the endemic equilibrium. That is, there is arise in the third party. This choice in particular seemed to capture the recent pattern in which third-parties rise in American politics. Again, for our model there are only three political parties and we consider the third one being made up of many small political parties.

Over a longer simulation, the graph in the top left of Figure 2 shows how all of those parties can chip away at the two major ones as the system approaches $E7(a^*, b^*, c^*)$. The trajectory observed over time and projected in the figures labeled Party A & B, Party A & C, and Party B & C, produces asymptotic behavior towards $E7(a^*, b^*, c^*)$. This could provide insight into they dynamics that were observed in the 2016 Presidential Election. An alternate dynamic is captured in Figure 3, with different parameters. This figure shows that there are limit cycle solutions and has implications about the type of stability displayed by $E7(a^*, b^*, c^*)$.

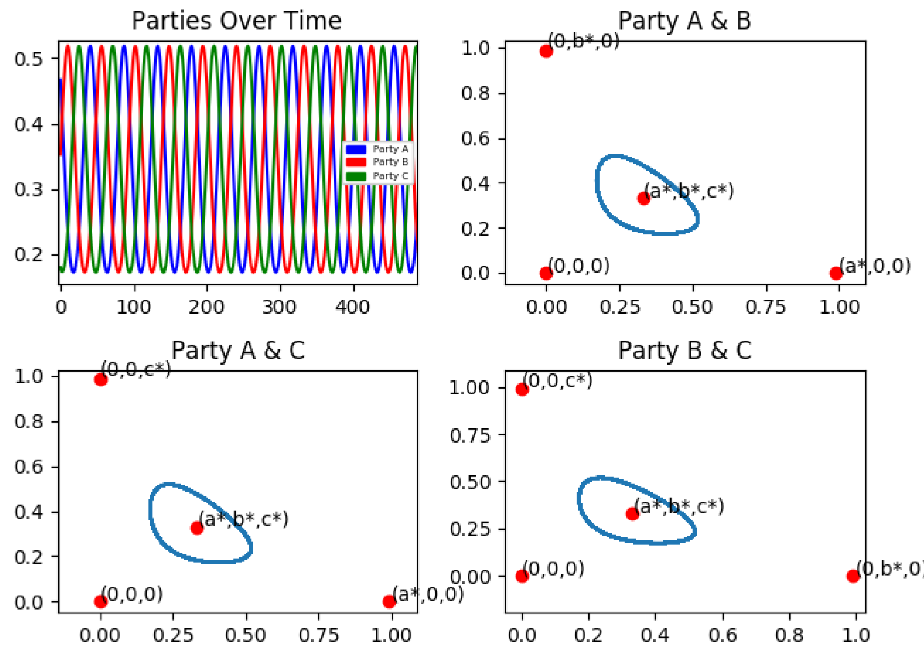

Figure 3: Three-party simulation with $\mu = .01$, $\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = 1$, $\Omega_1 = -1$, $\Omega_2 = 1$, $\Omega_3 = 1$

4 Discussion

By including an influential third party to the system we were able to study how votes can sway between the two major political parties. According to the Federal Election Commission the 2016 and the 2012 Presidential Elections had very similar voting age population size (N) with 2016 having a 0.9% increase in voting age population. The Republican party attained approximately 2 million more votes in 2016 than in 2012 and the Democratic Party lost 0.1 million votes. However, there was an estimated 7 million increase in the three voting classes. Therefore, the largest increase in population was the third party class who rose from 2 million votes to more than 8 million votes. Hillary Clinton did not visit swing states like Wisconsin, Pennsylvania, and Michigan, implying a lower recruitment rate. However, Donald Trump had a heavy recruitment strategy in these states. Interestingly, the third parties having such an increase in their population suggests they had the highest per capita recruitment from the other two parties.

5 Note to Instructors

The work in this paper was conducted during a one-year long experience with undergraduates during academic semesters with a team of two faculty research supervisors and seven undergraduates with varying previous knowledge of ODE's and programming in MATLAB and Python. The faculty approach is as follows:

1. Build a team with diverse skills; some have seen ODEs before, some have computing skills, others have only seen linear algebra and the calculus sequence,

2. Mini-lectures on MATLAB and math modeling; SIR article from MAA [5],
3. Gather voter data from different counties as well as nationally [2, 3],
4. Once students understand the two-party voter dynamical system [4], allow students to build their own models – to answer their own questions which they formulate,
5. Provide feedback as they *experiment* with their models; i.e., run numerical simulations and interpret outcomes,
6. Compare simulated results with collected data,
7. Write official reports, posters and talks for presenting; faculty period of reflection.

Expanding on point (4), the faculty mentors found that students were very passionate about this project which was conducted during the 2016-2017 academic year. This lead to much subjective writing from the students which faculty members had to channel towards mathematical reasoning. For instance, students often conjectured the motivations of the voter population before analyzing the model mathematically. Ultimately, we found that this student driven interest in the project kept the experience fruitful. One cautionary note: faculty are responsible for providing feedback so that model designs balance complexity and functionality.

The results presented in this article lead to a starting point for a future academic research experience with undergraduates focused mainly on analyzing the stability of the remaining equilibrium both analytically and via simulation. Open questions remain around any chaos present within the model, or present through the numerical methods implemented in studying the model.

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