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A Definition of Mathematical Beauty and Its History

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Synopsis

I define mathematical beauty as cognisability and trace the import of this notion through several episodes from the history of mathematics.

1. Beauty is cognisability.

The definition of mathematical beauty that I propose to defend in this essay is the following:

A beautiful proof is one which the mind can play its way through with a natural grace, as if it were created for this very purpose. We grasp a beautiful proof as a whole, yet see the role of every detail; it is vivid and transparent; we are its masters and its connoisseurs, like a conductor directing a symphony. I call this type of proof *cognisable* for short.

An ugly proof resorts to computations, algorithms, symbolic manipulation, ad hoc steps, trial-and-error, enumeration of cases, and various other forms of technicalities. The mind can neither predict the course nor grasp the whole; it is forced to cope with extra-cognitive contingencies. The mind's task is menial: it can only grasp one step at a time, checking it for logical adequacy. It can become convinced of the results but it is not happy since all the work was being done outside of it. Our memory is strained, our mind distorted to accommodate some artificial logic, like a student struggling with a foreign grammar.

1.1. The same otherwise, viz. by reference to Hilbertania

Hilbert, Minkowski, and Hurwitz used to discuss mathematics during walks "to the apple tree" every afternoon "precisely at five" [31, page 14]. This suggests an operational definition of cognisability: a beautiful proof is one which can be expounded and appreciated during such a walk.

1.2. The same otherwise, viz. by reference to antecedents in the literature

"Beauty is that which pleases in mere contemplation" (Aquinas, *Summa Theologica*, II.xxvii.1). It consists in "the play of the cognitive faculties" [18, §12], i.e., in the exercise of our universal human cognitive endowment, whence it is evidently non-subjective (§6) and disinterested, independent of worldly desires (§2). To understand beauty, then, is to understand the conditions under which "the mind thinks well of itself" [14, §3]. "In the beautiful, man postulates himself as the standard of perfection ... He thinks everything beautiful that throws his own image back at him" [29, §IX.19]. So, "what are the mathematic entities to which we attribute this character of beauty and elegance, and which are capable of developing in us a sort of esthetic emotion? They are those whose elements are harmoniously disposed so that the mind without effort can embrace their totality while realizing the details" [30, page 331].

2. Historical illustrations

I shall now discuss a few historical episodes in which cognisability played an important role. These case studies serve two purposes. Firstly, they provide empirical support for the cognisability thesis. Secondly, they are also applications of the cognisability thesis, showing that it is not an idle philosophical speculation but a useful tool for understanding the growth of science and mathematics. To lend force to this second aspect, I shall contrast my reconstructions of these historical episodes with other interpretations prominent in the literature.

2.1. Why Plato hated empiricism

"We will let be the things in the heavens, if we are to have a part in the true science of astronomy," says Plato in the *Republic* (530c, Shorey's translation). This is often taken to prove the backwardness of Plato's conception of science: empirical fact gives way to idealistic fantasy. But I say that Plato's point is, on the contrary, a straightforward statement of the cognisability thesis. For the sentence just quoted continues: "and so convert to right use from uselessness that natural indwelling intelligence of the soul" (530c). The point is clear: the purpose of science is not to amass truths but "to lead the best part of the soul up to the contemplation of what is best among realities" (532c).

Plato's condemnation of experimental and empirical science is not limited to the *Republic*. In the *Timaeus* we read:

> But if anyone who in considering these matters were to put them to an actual test, he would demonstrate his ignorance of the difference of the human and the divine. (68d)
>
> [Birds] descended from ... simpleminded men, men who studied the heavenly bodies but in their naiveté believed that the most reliable proofs concerning them could be based upon visual observation. (91d, Zeyl's translation)

Again Plato should not be scorned for having failed to grasp the empirical essence of science. The point he is making is simply this: it is good to think and come up with imaginative theories. Now, some people will try to deflate this enterprise by pointing out that these fancy theories fail to agree with actual, empirical observations. These people miss the point. Beautiful theories are an end in themselves; observations be damned. Fools with no imagination think they are being clever when they try to disprove theories by observations, but they prove nothing besides their own inability to understand the purpose of philosophy. What is the point in ruining a child's happy play by pointing out that the stick he is holding is not really a sword? This is all these fools do, and they imagine themselves superior for it. These are the people Plato is attacking in the *Timaeus*.

Plato's disinterested attitude towards the nowadays all-pervasive conception that philosophy must be a sober quest for truth is also found in the *Meno*. Conventionally, the *Meno* is of course taken to be a profound work on epistemology: *Meno* puts forth a devastating "paradox" calling into question the very possibility of learning (80d), which Socrates can only defuse with a sophisticated and portentous theory of knowledge as recollection that is to become a linchpin of the mature Platonic worldview. But if we listen to Plato's own words we find that, far from claiming to have dealt with a momentous problem of epistemology, he is instead at pains to ridicule the problem and belittle his own solution:

> We ought not to listen to this sophistical argument about the impossibility of enquiry: for it will make us idle; and is sweet only to the sluggard; but the other saying will make us active and inquisitive (81d). Some things I have said of which I am not altogether confident. But that we shall be better and braver and less helpless if we think that we ought to enquire, than we should have been if we indulged in the idle fancy that there was no knowing and no use in seeking to know what we do not know—that is a theme upon which I am ready to fight, in word and deed, to the utmost of my power (86b-c, Jowett's translation).

In other words, the important thing is not epistemology or truth but the kind of life we ought to live. Thus Plato's aversion to empirical science is not a matter of epistemology but simply a practical judgement as to which type of philosophy leads to a richer and more satisfying life of the mind. Empiricism spoils many fun theories and offers nothing in their place but dry and boring catalogues of facts. But Plato likes beautiful theories, whether true or not, and he dislikes catalogues of facts. So he condemns empiricism for this reason—in order to stay true to the ideal laid down in the *Meno*: to be active, brave, and inquisitive.

Plato's dismissive attitude towards empiricism did not die with him, although some people wish it had. Philosophers of science are generally reluctant to address the role of beauty in science due to their conviction that science must ultimately be empirical. Indeed, the only substantive book on the subject—McAllister's *Beauty and Revolution in Science* [27]—argues that aesthetic judgements are induced from and determined by empirical considerations. Thus McAllister is in effect justifying the traditional neglect among philosophers of the role of beauty in science. But seeing as McAllister's stated goal is "to defuse the threat posed to the rationalist image of science" (page 9) by scientists' talk of beauty, one has the feeling that he is presupposing rather than concluding that beauty is subordinated to empiricism. This, at any rate, seems to be the fallacy committed by Todd in [33], when he claims that the idea that scientists put beauty before truth "can be quickly discounted, for it is simply implausible to think that beauty might be *more* important than empirical adequacy in evaluating theories, *qua* theories, for this simply ignores their primary function, namely, conforming to the relevant norms of empirical adequacy" (page 66).

The "rationalist" attitude of these philosophers is clearly undermined by the Platonic attitude towards empiricism. In fact, scientists often pursue cognisability even if it means sacrificing empiricism, in the manner of Plato. An example of this is the prevalence of drastic idealisations in classical physics. Such over-idealisations often severed all ties to empirical reality, but they remained useful for pursuing cognisability, for our mind is better equipped to grasp mechanical, visualisable, analogical theories than abstract differential equations. As Maxwell in [26, page 220] puts it, not many people can "retain in their minds the unembodied symbols of the pure mathematician"; rather, they prefer "the robust form and vivid colouring of a physical illustration" to "the tenuity and paleness of a symbolical expression." "To such men impetus, energy, mass are not mere abstract expressions of the results of scientific inquiry. They are words of power, which stir their souls like memories of childhood." This explains why one repeatedly finds that the differential equations chosen for study by scientists are not those distinguished by their empirical import but by their cognisability.

Historical evidence for this claim exists in abundance. For example, Grattan-Guinness in a study of mathematical physics in France from 1800 to 1840 [15] noted the influence of a type of mathematics he described as follows:

> Mathematics pursued with an interpretation in physics initially in mind, but where the guidance of the physical problem has been lost. The result is, for example, a mathematical expression for a physical effect which cannot possibly be currently detected, or be computed from that expression with an error less than the effect itself; or the use of mathematical relationships between physical constants which make no physical sense; or the use of a hopelessly oversimplified physical model in the first place; or methods of reasoning in which physical concepts play little or no role [15, page 110].

Mathematics "where the guidance of the physical problem provides adequate control of the mathematics produced" was of course also important, but many scientists were frequently found "slipping into the 'notional' mode of mathematical physics," including major physicists such as Ampère, Biot, Fresnel, Laplace and Malus [15, pages 110, 113, 115]. Archibald [1] reported similar findings in a study of German mathematical physics from 1855 to 1875, pointing to the following features as characteristic of a large body of these works:

> They provide a mathematical description of some portion of physical reality which is consciously simplified or idealized. ... Further, results may be obtained concerning objects which are completely fictitious (e.g. infinitely thin plates, ellipsoids with vanishingly small mass). ... The description is based on principles (starting points) which are considered by the writer to be simple ... and evident ... There is little or no interest in producing experimentally testable results [1, page 52].

Again, this was certainly not just a game for esoteric mathematicians. It included major physicists such as Helmholtz, Boltzmann, and Kirchhoff, who, "in the preface to his lectures on mechanics [21], ... expressly disavowed the connection between the basic objects of mechanical theory and the physical world, positing for experimental concordances the status almost of happy coincidence" [1, page 65].

Now some will say that these examples prove nothing since the empirically groundless speculations were only at the mathematical extreme, so to speak, of otherwise empirically sound theories. So one may claim that scientists pursued these problems merely to "sharpen their teeth" in preparation for the true problems. But history has provided us with a clear test case that speaks against such an interpretation.

In the *Principia*, Newton gave the world two theories: his general mechanics, which was a remarkable empirical success, and his hydrodynamics, which was an equally remarkable empirical failure. Scientists chose to pursue both with equal enthusiasm. "In the eighteenth century hydromechanics was notorious for the discrepancy between its theories and experimental results." Nevertheless, "Newton's *Principia* [28], Daniel Bernoulli's *Hydrodynamics* [2], Lagrange's *Analytical Mechanics* [24], and a number of other books and articles in the learned journals promoted theoretical hydromechanics magnificently" [3, pages 119–120], despite the fact that still well into the 19^{th} century it was cheerfully admitted that hydrodynamical theory "virtually do not agree at all with the practical phenomena" (Prandtl, quoted from [13, page 240]).

Thus these two theories of Newton's were diametrical opposites so far as empiricism was concerned, and yet they were treated as equals. In other words: a huge empirical difference failed to leave any noticeable footprint in scientific practice. This is hard to reconcile with any attempt to construe science as ultimately empirical. But the mystery dissolves when we realise that so far as cognisability is concerned the two theories are indeed equals, so there is no wonder that they were treated as such.

2.2. Why Kepler was a Copernican

At its inception, Copernicus' heliocentric theory of the solar system could not be justified on straightforward empirical grounds. "Contemporary empiricists, had they lived in the sixteenth century, would have been the first to scoff out of court the new philosophy of the universe," as Burtt in [6, page 38] observes. Copernicus' theory was neither more accurate nor computationally simpler than competing theories, nor did it make any new predictions that could be confirmed at the time. Nevertheless its proponents championed it on essentially aesthetic grounds. This has led even historically sensitive authors such as Thomas Kuhn to speak of Copernicus as "strange" and Kepler as "irrational":

> "Harmony" seems a *strange* basis on which to argue for the earth's motion ... Copernicus' arguments are not pragmatic. They appeal, if at all, not to the utilitarian sense of the practising astronomer but to his aesthetic sense and to that alone. ... New harmonies did not increase accuracy or simplicity. Therefore they could and did appeal primarily to that limited and perhaps *irrational* subgroup of mathematical astronomers whose Neoplatonic ear for mathematical harmonies could not be obstructed by page after page of complex mathematics leading finally to numerical predictions scarcely better than those they had before ([23, page 180], italics added).

From the point of view of the cognisability thesis there is nothing "strange" or "irrational" about the Copernican revolution. There is no need to invoke eccentric "Neoplatonic ears" and the like; the aesthetic dividend of the new astronomy is clear as day in cognitive terms. After all, Kepler expressly traced his interest in astronomy to the fact that "the heavens, the first of God's works, were laid out much more beautifully than the remaining small and common things" (title page of the second edition of the *Mysterium Cosmographicum*). Copernicus makes the same point in the very first sentence of his *magnum opus*:

> Among the many and varied literary and artistic studies upon which the natural talents of man are nourished, I think that those above all should be embraced and pursued with the most loving care which have to do with things that are very beautiful and very worthy of knowledge ([8], Book I, first sentence, C. G. Wallis' translation).

More generally, I propose that the bulk of Kepler's alleged "mysticism" was not a relic of pre-scientific traditions, as it is commonly interpreted, but rather a derivative of a straightforward pursuit of cognisability. Witness the following quotation, which indicates the causal priority of mathematical beauty over religious overtures:

> I consider it my duty and task ... to advocate ... what I ... have recognized as true and whose beauty fills me with unbelievable rapture on contemplation ... Whenever I consider in my thoughts the beautiful order [of the universe] then it is as though I had read a divine text, written onto the world itself ... saying: Man, stretch thy reason hither, so that thou mayest comprehend these things (Kepler, quoted from [7, pages 298, 152]).

The point of genesis is the pursuit of things "whose beauty fills me with rapture on contemplation." All else is secondary. As it happens that man can indeed "stretch his reason thither"—i.e., find beauty in the world—one *concludes* that "the intellect of man ... is an image of the Creator" (Kepler, quoted from [7, page 94]).

The same hierarchy of inferences is found in the following passage.

> As we do not ask what hope or gain makes a little bird warble, since we know that it takes delight in singing because it is for that very singing that a bird was made, so there is no need to ask why the human mind undertakes such toil in seeking out these secrets of the heavens. ... [T]he reason why there is such a great variety of things, and treasuries so well concealed in the fabric of

> the heavens, is so that fresh nourishment should never be lacking for the human mind, and it ... should have in this universe an inexhaustible workshop in which to busy itself [20, page 55].

"There is no need to ask why": mathematics is beautiful; that much we know. This is "Kepler's *cogito* argument," if you like, from which his world view follows. In fact, Descartes may not have been far from agreeing. One is reminded of his remark that "you can substitute the mathematical order of nature for 'God' whenever I use the latter term" (Descartes, quoted from [16, page 74]). Again note the asymmetry: it is God who is the definiendum and mathematical beauty that is the primitive known.

2.3. Why Descartes invented analytic geometry

While philosophers have been insistent that the growth of science is dictated by empirical concerns, this can obviously not be the case in mathematics. Nevertheless, philosophers have been eager to dismiss the role of beauty in mathematics as well. This is often done by taking problem-solving instead of empiricism as the ultimate test for rationality. The history of mathematics is thus rationalised in terms of a conception of mathematical research as an autonomous and rational problem-solving enterprise. A prominent proponent of this view is Kitcher, who devotes a chapter of his book on *The Nature of Mathematical Knowledge* [22] to "expos[ing] some rational patterns of mathematical change in a way that will make it clear that they are rational" [22, page 193].

Kitcher offers Descartes's invention of coordinate geometry as a prime example of a mathematical discovery "justified by pointing out that [it] enables one to answer questions antecedently identified as important" [22, page 198]. Kitcher uses this point of view to explain away mathematicians' talk of beauty:

> We may view the mathematicians as responding to the fact that some questions are rationally generated, in the ways I have identified above, and characterizing their responses in terms of interest or beauty. To simplify, what makes a question interesting or gives it aesthetic appeal is its focussing of the project of advancing mathematical understanding, in light of the concepts and system of beliefs already achieved [22, page 206].

I believe that historical reconstructions based on viewing mathematics as dictated by problem-solving often gain an aura of plausibility by exploiting an ambiguity between cause and correlation. It may be that new inventions are applied to old problems not because that was the driving force motivating them, but simply because human ingenuity is insufficient to invent new problems even if we wanted to. There are only so many reasonable mathematical problems, as history shows in the form of strong commonalities and codiscoveries even among mathematicians working independently. There is reason to think that Descartes's treatment of classical problems has more to do with such factors than with a Kitcherian conception of rationality. For while it is true that Descartes rather proudly applied his new methods to certain classical problems, he likewise sharply condemned other well-established problems:

> I would not make so much of these rules [of analytic geometry], if they sufficed only to resolve the inane problems with which Calculators and Geometers amuse themselves to pass time, for in that case all I could credit myself with achieving would be to drabble in trifles with greater subtlety than they ([10, §IV], quoted from [17, page 35]).

These hardly sound like the words of someone who defines the rationality of his work in terms of "questions antecedently identified as important," as Kitcher would have it.

In contrast to this, Descartes states clearly that the "principal utility" of mathematics is to "cultivate your mind":

> I stop before explaining all this in more detail because I would take away the pleasure of learning it yourself and the utility of cultivating your mind in exercising yourself on these problems, which is, in my opinion, the principal utility that one can take away from this science ([11, page 301], quoted from [17, page 24]).

Descartes's "principal utility" is thus an allusion to cognisability. This is a particularly striking instantiation of the cognisability thesis since coordinate geometry is today often thought of as a paradigmatic example of extra-cognitive mathematics: it relieves us of having to use our imagination by replacing ingenious geometrical arguments by straightforward formula-crunching. But Descartes did not look at it this way at all. On the contrary, he saw his new invention as eminently pro-cognitive. "Descartes rejected the symbolic and the formal as memory arts appropriate to imitation, not true self-cultivation"; instead he "defended the use of algebra as a temporary means to help focus the attention on discovering the interconnection among a set of geometrical objects. Putting 'down on paper whatever we have to retain' allows 'the imagination to devote itself freely and completely to the ideas immediately before it' [helping it to] 'intuit as many as possible at the same time'" ([17, pages 38, 33], quoting [10, §XVI]).

Thus cognisability was crucially and explicitly important to Descartes. But what about the link between this concept and beauty that the cognisability thesis requires? In fact, Descartes made this link explicit as well:

> The sciences are now masked; once the masks are removed, they will appear very beautiful. To him that sees completely the entire chain of knowleges, it will not seem more difficult to retain them in the soul than to retain a series of numbers (Descartes, writing in 1619, quoted in [17, page 58]).

This was no mere pet topic of Descartes's; Leibniz also evaluated algebra on the grounds of beauty and cognisability, although he reached the opposite conclusion:

> I look for almost nothing more in geometry than the art of finding first of all the most beautiful constructions. I feel more and more that algebra is not the natural means of doing so ... Algebra leads to a solution which is not always the shortest and whose method of calculation is not the most natural, and which does not enlighten the mind [25, pages 183–184].

3. The "phenomenology of mathematical beauty" today

The phrase in quotes in the heading for this section is the title of an article by Gian-Carlo Rota [32]. This authoritative "phenomenology" aspires to be an accurate and unbiased description of the views of the modern mathematical community on the subject of beauty. I shall regard it as a kind of empirical data against which to test the cognisability thesis, and show that a number of conspicuous phenomena noted by Rota can be readily interpreted as corollaries of it.

3.1. Mathematical beauty is associated with "light-bulb" insights.

Rota writes:

> We think back to instances of mathematical beauty as if they had been perceived by an instantaneous realization, in a moment of truth, like a light-bulb suddenly being lit. All the effort that went in understanding the proof of a beautiful theorem, all the background material that is needed if the statement is to make any sense, all the difficulties we met in following an intricate sequence of logical inferences, all these features disappear once we become aware of the beauty of a mathematical theorem, and what will remain in our memory of our process of learning is the image of an instant flash of insight, of a sudden light in the darkness [32, page 179].

This is precisely what the cognisability thesis predicts. Beauty is precisely the elevation of a proof from "an intricate sequence of logical inferences" to a cognitively coherent whole. It is precisely when our cognitive faculties are capable of performing this transformation that we experience beauty.

3.2. Mathematical beauty is independent of formal aspects of mathematics.

Today it is popular to characterize mathematics in terms of its rigor and axiomatic structure. The cognisability thesis, however, relegates this aspect of mathematics to a secondary role at best, so far as mathematical beauty is concerned. This prediction is strikingly confirmed.

> The beauty of a mathematical theory is independent of the aesthetic qualities, or the lack of them, of any of the theory's rigorous expositions. [...] Mathematical beauty and mathematical elegance are distinct phenomena ... Mathematical elegance has to do with the presentation of mathematics, and only tangentially does it relate to content [32, pages 173, 178].

Similarly, distinctions between entities such as proofs, theorems, definitions, etc., are, while formally strong, cognitively weak, so the cognisability thesis predicts that such distinctions should not be relevant to matters of mathematical beauty. This, again, we find confirmed.

> The most common instance of beauty in mathematics is a brilliant step in an otherwise undistinguished proof. . . . [But] Most frequently, the *word* 'beautiful' is applied to theorems. In the second place we find proofs; . . . Axiom systems can also be beautiful. [There can also be] beauty in a definition [32, pages 172, 171, 173].

3.3. Mathematical beauty is nonsubjective.

It follows from the cognisability thesis that aesthetic judgements should be universal among everyone to the (presumably very considerable) extent that human cognitive endowments are so, except insofar as informed judgement is precluded by lack of study and training. "Appreciation of mathematical beauty requires thorough familiarity with a mathematical theory," as Rota observes [32, page 177]. But the cognisability thesis predicts that there should be no great differences in aesthetic judgement that cannot be so explained. This is confirmed by Rota:

> The beauty of a piece of mathematics does not consist merely in subjective feelings experienced by an observing mathematician. The beauty of a theorem is a property of the theorem, on a par with its truth or falsehood. . . . Both the truth of a theorem and its beauty are equally objective qualities, equally observable characteristics of a piece of mathematics which are equally shared and agreed upon by the community of mathematicians [32, page 175].

4. Epilogue

By way of conclusion I put it to the readers of this journal of humanistic mathematics that the cognisability thesis makes the study of beautiful mathematics an eminently humanistic enterprise. As birds are made to sing, so human beings are made to reason mathematically, as we saw Kepler reason above. Beautiful mathematics, therefore, is the expression of the "the best part of the soul," in Plato's phrase.

This provides an answer to those who think of mathematics and science along the lines of the horrifying vision of Keats:

Philosophy will clip an angel's wings,
Conquer all mysteries by rule and line,
Empty the haunted air, and gnomed mine—
Unweave a rainbow...

(John Keats, *Lamia*, part II. in [19, page 307])

I for my part have urged in this essay that the moss be cleared off beauty:

I died for beauty, but was scarce
Adjusted in the tomb,
When one who died for truth was lain
In an adjoining room.

He questioned softly why I failed?
"For beauty," I replied.
"And I for truth,—the two are one;
We brethren are," he said.

And so, as kinsmen met a night,
We talked between the rooms.
Until the moss had reached our lips,
And covered up our names.

(Emily Dickinson, untitled, in [12, page 119])

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