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Abstract: For thousands of years the population of Earth increased slowly, while per capita income remained essentially constant, at subsistence level. At the beginning of the industrial revolution around 1800, population began to increase very rapidly and income started to climb. Then in the second half of the twentieth century as a demographic transition began, the birth and death rates, as well as the world population growth rate, began to decline. The reasons for these transitions are hotly debated with no expert consensus yet emerging. It's the problem of economic growth. In this document we investigate a mathematical model of economic growth proposed by Michael Kremer in 1993.

1 The Malthus Model

Thomas Malthus, working in England in the 1790s, created the first mathematical model of economic growth [9]. The classic statement of his finding is that annual food production, Y, increases linearly and population p, unless checked, increases exponentially.

$$\begin{aligned} p &= me^{rt} \\ Y &= nt \end{aligned} \tag{Model 0}$$

The parameters m, n, and r are positive constants.

Malthus realized that the growth he described is unsustainable because it leads to ever-decreasing average food production that falls below the amount required for life.

$$\text{Food per person} = \frac{Y}{p} = kte^{-rt}.$$

Malthus's conclusion was that mass starvation on a continuing basis, or death by perpetual war or disease, is inevitable.

Malthus did not actually run the assumptions of Model 0 out to their logical conclusion (predicting zero living standards); his whole point was that positive and preventative checks (essentially, mortality and fertility changes) would ensure that the population growth slowed down as people had less to eat and in the long run we will all remain at constant, never increasing, near subsistence level incomes. This dismal situation has been called the Malthusian trap.

At just the time Malthus was working, development of technology took off, spurring increases in food production sufficient to support an ever more rapidly growing population. Thus technology became the way out of the trap. Can we produce a growth model that incorporates technology in a meaningful way?

2 Kremer's Model: The Variables

Kremer's theory of economic growth [8] is a mathematical model of the time evolution of three variables:

- p: population of a community
- A: level of technology of the community
- y: per capita income of the community

Note that $y = Y/p$, where Y is the income of the entire community, an annual gross national product (GNP). You can think of Y as equivalent to a quantity of food, perhaps measured in calories, since for most of human history all our income was hunted or gathered and eaten right away.

The three variables y, A and p are linked by a production function for Y. In the Kremer model the value of Y is determined by the population's ability to use two resources: (1) labor, which is the population p itself, and (2) a second resource X. Often X is interpreted as a fixed quantity of land. We model X as a constant and hence set it to 1. Assuming a standard Cobb-Douglas model[1] Kremer postulates that

$$Y = Ap^{1-\beta}X^{\beta} = Ap^{1-\beta}$$
$$y = \frac{Y}{p}.$$

The Cobb-Douglas exponent β is a constant parameter for the model, with $0 < \beta < 1$. Kremer suggests a very rough estimate of $\beta = 1/3$ based on tenants' shares in traditional sharecropping contracts. The level of technology, A, functions as a multiplier in the production function. If technology is greater, then the same number of people can produce more from the land.

Thus p, y and A are related via the yAp equations

$$y = Ap^{-\beta}$$
$$p = \left(\frac{A}{y}\right)^{1/\beta}.$$

Let's look at the implications for growth rates, by which we always mean relative growth rates. Taking logarithmic derivatives of the yAp equations we get

$$\frac{\dot{y}}{y} = \frac{\dot{A}}{A} - \beta\frac{\dot{p}}{p}.$$

[1] https://en.wikipedia.org/wiki/Cobb-Douglas_production_function

The growth rate of per capita income is a linear combination of the growth rates of technology and population. Per capita income y is dragged down by increasing population with fixed technology because the same wealth is divided among more people. But income grows more quickly as technology improves with fixed population as the same number of people can use the better technology to extract more wealth from the land.

We have specified one relation among the three variables y, A and p, so we need two more equations to complete a model. They will be growth equations. We will give equations for two of $\dot{y}/y$, $\dot{p}/p$ and $\dot{A}/A$. There are of course many ways to do this. Different choices can be appropriate for different populations, or for a single population in different stages of its growth.

Problem 2.1. Consider Model 0, the naive Malthus model, which does not involve A.

(a) Show that it can be expressed in terms of growth rates as follows:

$$\frac{\dot{p}}{p} = r$$

$$\frac{\dot{Y}}{Y} = \frac{n}{Y}$$

(b) Show that

$$\frac{\dot{y}}{y} = \frac{n}{py} - r.$$

(c) How does the growth rate, $\dot{y}/y$, of per capita income y change when population p increases? When y increases?

(d) Explain how this model shows that per capita income eventually decreases.

(e) Assuming that $y = Ap^{-\beta}$, show that

$$\frac{\dot{A}}{A} = \frac{n}{py} - (1 - \beta)r.$$

3 A Modern Interpretation of Malthus

Malthus's main point is that in the long run population will grow but per capita income will be constant. To see this conclusion arise from a plausible differential equation model for A, p and y, consider Model 1, as follows:

$$\begin{aligned} p &= \left(\frac{A}{y}\right)^{1/\beta} \\ \frac{\dot{p}}{p} &= \theta(y - y_0) \\ \frac{\dot{A}}{A} &= k \end{aligned} \tag{Model 1}$$

The parameters θ, k and y_0 are positive constants.

We interpret y_0 as an acceptable per capita income. For most of history and prehistory y_0 may have been a subsistence level income, but it is probably higher than that now. When income is above the acceptable level, the population increases. When income is below the acceptable level, the population decreases, due for example to malnutrition or misery leading to increased mortality and decreased fertility. The parameter $\theta > 0$ governs how quickly p responds to deviations in per capita income from y_0. The level of technology A increases at a constant relative rate k as humans make discoveries, supporting a greater population at the same per capita income y. The model implies that A grows exponentially,

$$A = C_1 e^{kt}$$

where C_1 is a positive constant.

The implications of Model 1 for income y are most easily revealed by recasting the model as a system of differential equations for p and y. We have

$$\begin{aligned} A &= y p^{\beta} \\ \frac{\dot{p}}{p} &= \theta(y - y_0) \qquad\qquad \text{(Model 1, alt. form)} \\ \frac{\dot{y}}{y} &= k - \beta\theta(y - y_0). \end{aligned}$$

The equation for y is a logistic differential equation of the form

$$\begin{aligned} \frac{\dot{y}}{y} &= s\left(1 - \frac{y}{L}\right) \\ s &= k + \beta\theta y_0 \\ L &= \frac{s}{\beta\theta} = \frac{k}{\beta\theta} + y_0. \end{aligned}$$

with limiting value L, so

$$y(t) = \frac{L}{1 + C_2 e^{-st}}$$

where C_2 is constant. For all initial conditions, per capita income eventually becomes essentially constant, $y \approx L$. Indeed, one exact solution to Model 1 to which all others are asymptotic as $t \longrightarrow \infty$ is given by

$$\begin{aligned} p &= \left(\frac{C_1}{L}\right)^{1/\beta} e^{kt/\beta} \\ y &= L \\ A &= C_1 e^{kt}. \end{aligned}$$

The population ultimately grows exponentially, enabled to do so by the increasing level of technology. But there is no exit from the Malthusian trap. Per capita income does not grow at all. The ever greater populations remain forever at subsistence level.

4 The Malthusian Era

Kremer called the period from 1 million BCE to about 1800 the Malthusian era. It was characterized by three properties.

1. Population: Continual slow population growth.

   Relevant world population estimates are

Year	1 million BCE	0 CE	1800
Population	125,000	230 million	1 billion.

   Global population doubled fewer than 11 times in the first million years, on average about once every 100 thousand years. Contrast that with the present. World population doubled in just 47 years from 3.8 billion in 1971 to 7.6 billion in 2018.

   Problem 4.1. Assuming exponential growth at a constant rate, find the population growth rates for the periods from

   (a) 1 million BCE to 0 CE

   (b) 0 CE to 1800

   (c) 1971 to 2018.

2. Income: Nearly constant per capita income.

   Global per capita income, expressed in contemporary currency, is estimated to have been about \$450 per year from 1 million BCE to 1000 BCE only rising to \$670 per year by 1800. During this time per capita food consumption did not rise at all.

3. Technology: Slow but steady improvement.

   During this period tools were invented, language was developed, fire was tamed, agriculture and herding were invented, animals were domesticated, first settlements were created.

Let's convert the Malthusian Era characteristics into Kremer's first mathematical model, given by three equations:

$$
\begin{aligned}
p &= \left(\frac{A}{y}\right)^{1/\beta} \\
\frac{\dot{y}}{y} &= 0 \\
\frac{\dot{A}}{A} &= gp
\end{aligned}
\qquad \text{(Model 2)}
$$

The parameter g is a positive constant.

It is a model assumption that per capita income remains constant over time, as Malthus expected and as was the case for a million years. This assumption takes as a permanent feature the longterm stagnation in y from Model 1.

The technology equation for $\dot{A}/A$ in Model 2 differs from the analogous equation in Model 1. The new equation asserts that the growth rate of technology increases with population. This reflects the thought that with more people, it is more likely that someone will hit on a great idea that spurs technological development. We interpret g as research productivity. If g is higher then the same population improves technology faster.

The differential equation for y makes $y = y_0$, a constant. As for p, we have

$$\frac{\dot{p}}{p} = \frac{1}{\beta}\frac{\dot{A}}{A} - \frac{1}{\beta}\frac{\dot{y}}{y} = \frac{g}{\beta}p$$

Therefore,

$$\frac{dp}{dt} = \frac{g}{\beta}p^2.$$

In Model 2 population grows faster than exponentially. Increasing population p causes the growth rate of technology A to increase, which makes A climb ever faster. Since y is constant, the ever increasing rate of A drives a runaway increase in p. This may sound unsustainable, and it is, even mathematically. Solving explicitly we have

$$p(t) = \frac{p_0}{1 - gp_0 t/\beta}$$

where $p_0 = p(0)$. The solution for p runs off to infinity by finite time $t = \beta/(gp_0)$. This is not necessarily a flaw in the model, since the model was only designed to capture global population growth during the Malthusian Era, not for all time.

Postulating that per capita income is constant may seem a little drastic, since income eventually did begin to rise. Let's move on to a fancier model.

5 The Industrial Revolution

In the late 1700s and very early 1800s the Industrial Revolution began in England, then rapidly spread first to Europe and then to the whole world.[2] During this time period population and per capita income growth rates both exploded. Kremer's second model frees y to vary, as in Model 1, enables technology to grow at an increasing rate with increasing population, as in Model 2, and captures the transition from the Malthusian era to the industrial revolution.

$$\begin{aligned} p &= \left(\frac{A}{y}\right)^{1/\beta} \\ \frac{\dot{p}}{p} &= \theta(y - y_0) \\ \frac{\dot{A}}{A} &= gp \end{aligned} \tag{Model 3}$$

The parameters θ, g and y_0 are positive constants.

[2] https://en.wikipedia.org/wiki/Industrial_Revolution

We interpret y_0 as an acceptable per capita income, as in Model 1. In Model 3, if $y = y_0$, then p is momentarily constant, but A is still growing and hence so is y, which leads to further growth in p. The monotonic growth of technology enables population to continue growing, too.

We can recast Model 3 as a system of differential equations for p and y. We have

$$A = yp^{\beta}$$
$$\frac{\dot{p}}{p} = \theta(y - y_0) \qquad \text{(Model 3, alt. form)}$$
$$\frac{\dot{y}}{y} = gp - \beta\theta(y - y_0).$$

Figure 1 shows a nulllcline analysis in the py phase plane and solution trajectories.

The $\dot{p} = 0$ nullcline is the line $y = y_0$. The $\dot{y} = 0$ nullcline is the line $y = y_0 + gp/(\beta\theta)$. The nullclines separate the quadrant $p > 0$, $y > 0$ into three regions.

- Region I: per capita income y is less than the threshold y_0. Population therefore declines and income rises.
- Region II: income y is greater than y_0. Population increases and income falls.
- Region III: income y is greater than y_0. Population increases and so does income. This is the region of the post-Malthusian boom, the Industrial Revolution. Population is so large that technology increases fast enough to keep production ahead of population growth and thus provides a way out of the Malthusian trap. All solutions eventually cross one of the nullclines into Region III after which both population and income rise forever. The solutions move through Region III at an ever increasing speed, which means faster and faster increases in both y and p.

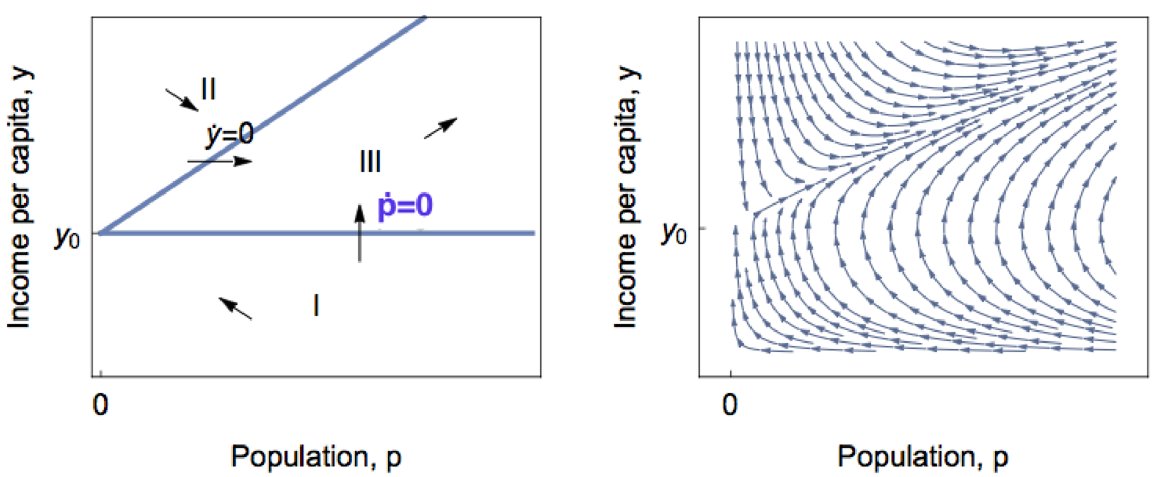

Figure 1: Model 3: Nullclines and trajectories. If per capita income is below y_0 then population decreases. If income is above y_0 then population increases. Increasing population causes a decrease in income when population is low (Region II) but an increase in income when population is high (Region III). The larger population spurs more technological growth. The equations of the nullclines are $y = y_0$ and $y = y_0 + gp/(\beta\theta)$.

Perhaps the most important thing to say about Model 3 is that it predicts that all populations, if left long enough, will eventually enter Region III of continual population and income growth. In this model, something like the Industrial Revolution was inevitable, the result of the human proclivity to improve their technology. Whether this simple explanation truly solves the question of the causes of the Industrial Revolution is a matter of debate.

The numerical solutions of an example of Model 3 with $y_0 = 1$ and $y(0) = 1.5$ is shown in Figure 2. Initially income and population are in Region II. The initial income is above the minimal acceptable level so supports a rapid population increase. Once the income drops near y_0, we enter Region III. Population slowly increases due to slowly improving technology then begins to increase faster. At a later time income begins to explode. The increase in income signals that the transition from the Malthusian Era to the period of the Industrial Revolution has been achieved.

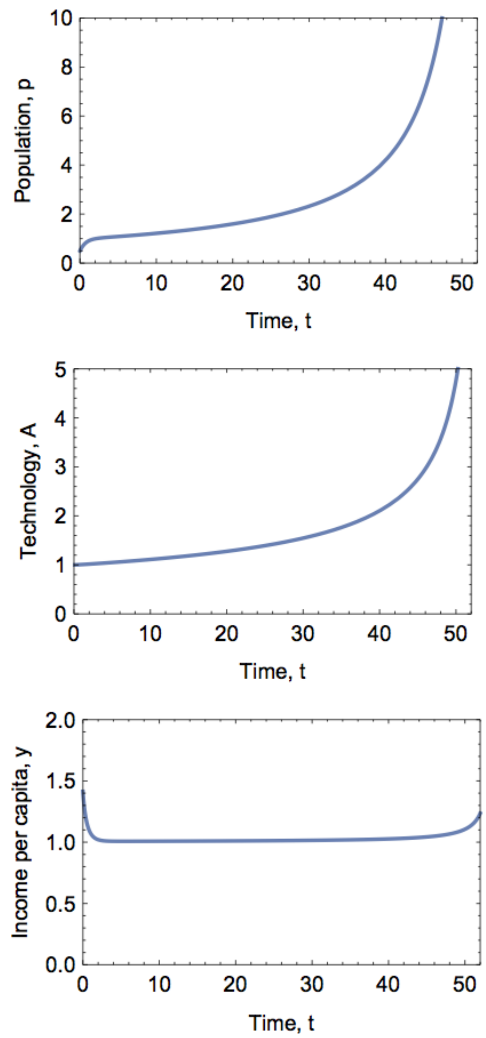

Figure 2: Model 3: After a long period of nearly constant per capita income, income begins to rise and both population and technology surge. This is a possible model for the transition from the Malthusian Era to the Industrial Revolution. Parameter values are $\beta = 0.5, \theta = 3, g = 0.01, y_0 = 1, p(0) = 0.5, A(0) = 1$.

6 The Demographic Transition

The world population growth rate has not continued to increase. It had begun to decline by 1970, at which point global population was around 3 billion. No one knows the reason for sure. One line of thought is that increasing wealth changes the economics of the decision about whether to have more children or to have educated children, an issue referred to in the literature as a quantity versus quality tradeoff. Perhaps when individual incomes become high enough, people are opting for fewer but educated children. Whatever the reasons may be, all the world over birth rates have declined as incomes have risen.

Model 3 does not model this demographic transition. The reason is that the population equation for Model 3

$$\frac{\dot{p}}{p} = \theta(y - y_0)$$

describes the population growth rate $\dot{p}/p$ as a monotonically increasing (linear) function of income. It does not describe a drop in the growth rate as the response to sufficiently high y. To capture this effect Kremer postulates a new population equation, which can be described by a function $f(y)$. We have

$$\begin{aligned} p &= \left(\frac{A}{y}\right)^{1/\beta} \\ \frac{\dot{p}}{p} &= f(y) \\ \frac{\dot{A}}{A} &= gp \end{aligned} \qquad \text{(Model 4)}$$

or equivalently

$$\begin{aligned} A &= yp^{\beta} \\ \frac{\dot{p}}{p} &= f(y) \\ \frac{\dot{y}}{y} &= gp - \beta f(y). \end{aligned} \qquad \text{(Model 4, alt. form)}$$

The function $f(y)$ is to be a function whose graph resembles that in Figure 3. The population growth rate increases for low incomes, just as in Model 3. As income continues to increase, $\dot{p}/p$ reaches a maximum then decreases, approaching a constant value that could be either positive or zero.

The nullclines in the py phase plane with some streamlines for Model 4 are shown in Figure 4. As was the case for Model 3 there are three regions. If income is below y_0 then population decreases, and if it is above y_0 it increases. But the bend in the $\dot{y} = 0$ nullcline causes an effect not seen in Model 3. It is possible for a sudden increase in income to move a population vertically in the phase plane from Region II, where income is declining, to Region III, where income is increasing. Such a jump could be, for example, a technological breakthrough such as the discovery of inexpensive nuclear fusion power, artificial efficient photosynthesis or some other source of cheap power.

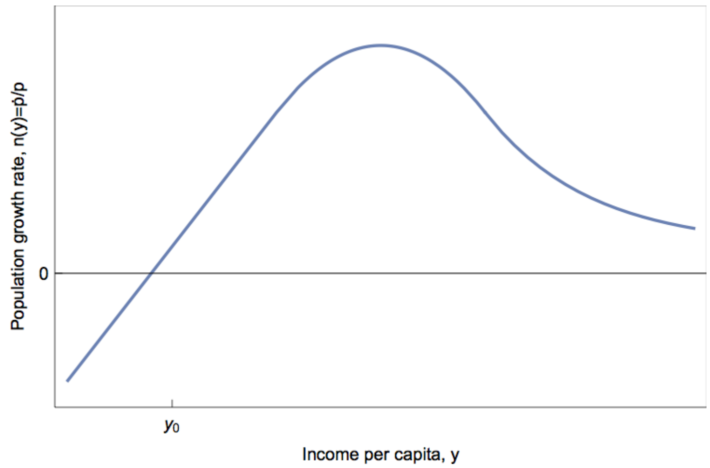

Figure 3: Model 4: When per capita income is high enough, the population growth rate declines with increasing income.

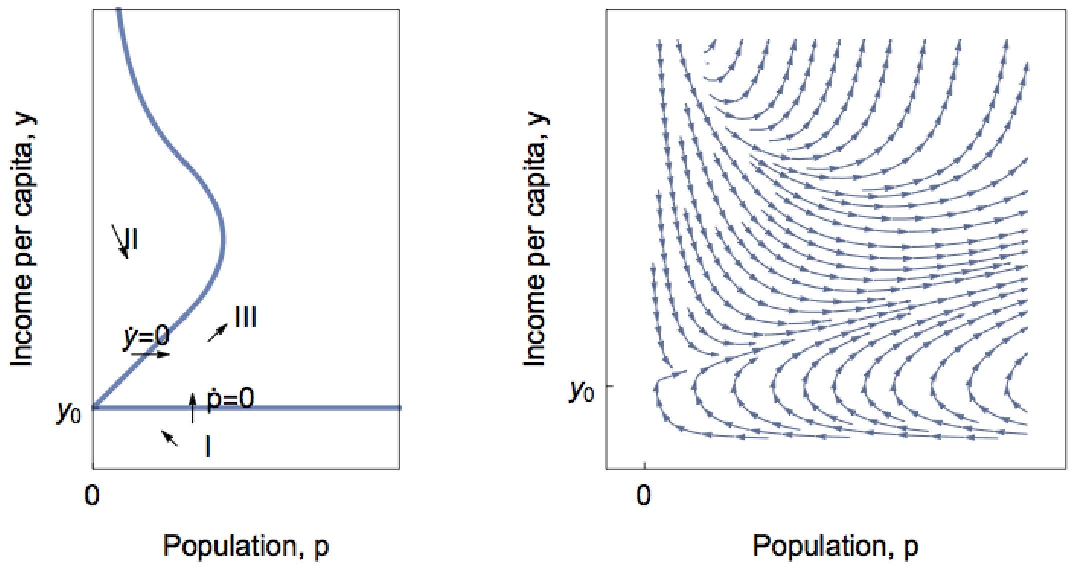

Figure 4: Model 4: Nullclines and trajectories. Population falls in the low income Region I and grows in the high income Regions II and III. A technological breakthrough could move a population overnight from Region II to Region III, thus reversing a downward trend in per capita income. The equations of the nullclines are $y = y_0$ and $gp = \beta f(y)$.

A population drop at a moderate income may take the population from Region III to Region II, where income stops growing and begins decreasing. On the other hand, the same population drop in a wealthier level may stay in Region III, so income continues to grow. This phenomenon was not seen in Model 3.

In Model 4 all populations eventually enter Region III after which their incomes grow forever. If the incomes get high enough, the rate of population growth declines and a demographic transition begins. A sample solution is shown in Figure 5. The figure is meant to be qualitatively correct, but not numerically so. The function $f(y)$ and model

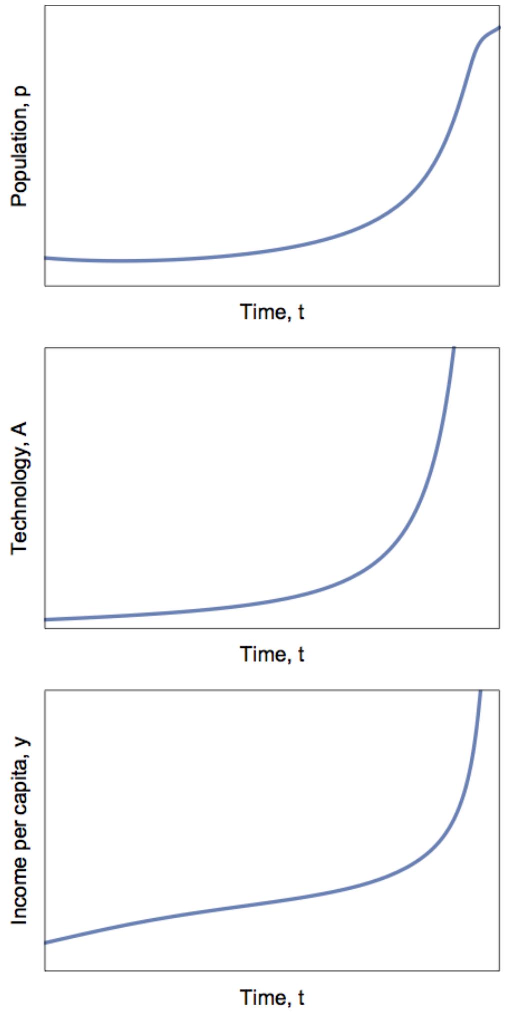

Figure 5: Model 4: When income is high enough, population growth slows, though income and technology continue to increase dramatically.

parameters were not fit to real data.

Model 4 is intended to capture the demographic transition, not to model the future forever. As $f(y)$ decreases with increasing y, per capita income y continues to increase without bound, an unsustainable trajectory. As always, building and refining a model is an iterative never-ending process.

7 Additional Reading

Malthus wrote, "Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will shew the immensity of the first power in comparison of the second." For this and more, see [9].

For Kremer's original very clear article (with an awe-inspiring title) see [8]. An elementary textbook exposition appears in Chapter 8 of [7]. An extension to the controversial Unified Growth Model of Oder Galor is given in [5].

In this article we do not tackle directly the question of why birth rates decline in sufficiently wealthy populations. We merely postulate the general shape of the graph of the modeling function $f(y)$. But economists have proposed models to explain declining fertility with increased income, for which see [3] and [4].

There has been a great deal of discussion about the technology equation $\dot{A}/A = gp$. Is the research productivity parameter g really a constant? It's partly an empirical question. Some evidence is that large technologically advanced populations innovate rapidly, and some large less developed populations (for example China in the 1980s) innovated slowly. Perhaps g should depend on A. This led Jones to propose an alternative technology equation: $\dot{A}/A = gp^{\psi}A^{\phi}$ where g is again a constant parameter. See [6].

For alternative explanations of the cause of the Industrial Revolution see [1], [2] and [10].

Acknowledgments

Robert Borrelli and I first crossed paths years ago when I began teaching differential equations from *Differential Equations: A Modeling Perspective* by Borrelli and Coleman. About a month into the course it hit me just how much I loved teaching from his book. Borrelli made it easy to keep the focus on mathematical modeling, with differential equations in a supporting role. I sent Bob some fan-mail, he responded and attracted me into the CODEE orbit. At the next JMM I met Bob and was soon working on the presentation of CODEE sponsored MAA minicourses "Teaching differential equations with modeling." I will forever be grateful for my friendship with Bob. It is a joy for me to contribute to a volume in honor of all that he gave to so many people for so many years.

Borrelli was always on the lookout for interesting mathematical models for students to explore. In this paper we suggest Kremer's model of economic growth as a good way to lead students beyond the exponential and logistic population models that they may have seen in earlier courses. The exponential and logistic models do not capture the global transition to rapid growth at the time of the industrial revolution. Nor do they model the more recent demographic transition to declining population growth rates due to reduced birth rates associated with increasing wealth. Kremer's economic growth model tackles both of these transitions. His model includes a time dependent variable that represents the level of technology of a population. The Kremer model is a 2-dimensional continuous dynamical system appropriate for a first course in differential equations.

I want to thank economist Dietrich Vollrath (University of Houston) for helpful expert comments on the first draft of this paper.

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