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On Toric Symmetry of P1 x P2

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On Toric Symmetry of $\mathbb{P}^1 \times \mathbb{P}^2$

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Abstract

Toric varieties are a class of geometric objects with a combinatorial structure encoded in polytopes. $\mathbb{P}^1 \times \mathbb{P}^2$ is a well known variety and its polytope is the triangular prism. Studying the symmetries of the triangular prism and its truncations can lead to symmetries of the variety. Many of these symmetries permute the elements of the cohomology ring nontrivially and induce nontrivial relations. We discuss some toric symmetries of $\mathbb{P}^1 \times \mathbb{P}^2$, and describe the geometry of the polytope of the corresponding blowups, and analyze the induced action on the cohomology ring. We exhaustively compute the toric symmetries of $\mathbb{P}^1 \times \mathbb{P}^2$.

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Chapter 1

Introduction

1.1 Gromov Witten Theory, Enumerative Geometry, and Algebraic Geometry

String theory attempts to reconcile the incompatibilities of the theories of quantum mechanics and Einstein's general relativity, by stipulating that all particles are made up of tiny strings. These strings trace out surfaces called world sheets as they move through spacetime. So it is of physical significance to understand geometric properties of surfaces and curves.

Most problems of enumerative geometry are of the form "how many geometric objects satisfy some set of geometric properties"? An example of an enumerative geometry question is "how many conics in the plane intersect 5 specified points?". A more trivial example, which we will revisit later to provide motivation for projective space, is "at how many points do two distinct lines in the plane intersect?", which we know to be 0 or1 depending on which lines we specify. Gromov Witten theory is a branch of mathematics, also motivated by string theory, which can be used to study enumerative geometry and is concerned with the study of Gromov Witten invariants, which are rational numbers that enumerate certain properties of specific types of curves.

Enumerative geometry is a branch of algebraic geometry, which studies varieties, zero loci of polynomials.

Definition 1. *Let k be a field. Let $f_1, \ldots, f_m$ be in the polynomial ring $k[x_1, \ldots, x_n]$. An affine variety $V_{f_1,\ldots,f_m}$ is defined as the zero locus of a set of polynomials.*

$$V_{f_1,\ldots,f_m} = \{p \in k^n | f_i(p) = 0 \ \forall 1 \leq i \leq m\}$$

Many questions can be asked about these objects. For instance, many number theoretic problems concerning Diophantine equations can be realized as questions about varieties. If the field we consider is $\mathbb{Q}$, then asking whether a variety is nonempty is equivalent to asking whether a set of equations has rational solutions. A famous example is Fermat's last theorem, which is equivalent to asking whether $V = \{p \in \mathbb{Q}^3 | f(p) = 0\}$ where $f(x, y, z) = x^n + y^n - z^n$ for $n \geq 3$ is empty.

Affine varieties are embedded in $\mathbb{A}^n{}_k$, projective varieties in $\mathbb{P}^n$. These spaces have a topology called the Zariski topology, where the complements of the hypersurfaces, or zero loci of a single polynomial, are the basis elements. We can also discuss the behavior of structure-preserving maps between varieties, and a vast array of problems. The variety of interest in this project, $\mathbb{P}^1 \times \mathbb{P}^2$, belongs to a type of variety called toric varieties which have some combinatorial structure because of a group action called "the torus action", and we are interested in the structure of their automorphism groups.

1.1.1 Projective Space

Many questions in enumerative geometry do not have elegant solutions in k^n. For instance, the question mentioned above, "at how many points do two distinct lines in the plane intersect?" as we all know has two answers in the affine plane: zero if the two lines are parallel, one if they are not. This answer may be viewed as unsatisfactory because it is not simple; it might be nicer if there was just one answer. Often the answers to these questions are not as nice in affine space as in projective space, which is defined as th set of $n + 1$-tuples under the equivalence relation which sets two tuples which are non-zero multiples of each other equivalent. In more detail:

Definition 1.1.1 (Projective Space). *Let k be a field.*

$$\mathbb{P}_k{}^n = \{(X_0, X_1, \ldots, X_n) \in k^{n+1} \backslash (0, \ldots 0)\} / \sim$$

where
$$(X_0 : X_1 : \cdots : X_n) \sim (\lambda X_0 : \lambda X_1 : \cdots : \lambda X_n)$$
for all $\lambda \in k \backslash \{0\}$, $(X_0 : X_1 : \cdots : X_n) \in k^n \backslash (0, \ldots 0)$.

We will denote $\mathbb{P}_{\mathbb{C}}{}^n$ as simply $\mathbb{P}^n$. There are many ways to think about projective space. $\mathbb{P}_k{}^n$ can be viewed as the moduli space of lines through the origin in k^{n+1} by associating each $(X_0 : X_1 : \cdots : X_n) \in \mathbb{P}_k{}^n$ to the

line in k^{n+1} intersecting the origin and $(X_0, X_1, \ldots, X_n)$. $\mathbb{P}^n$ is also a compactification of $\mathbb{C}^n$. For instance, $\mathbb{P}^1$ can be viewed as $\mathbb{C}$ with one addition point "at infinity" via the correspondence $(0 : 1) \to \infty$, $(X_0 : X_1) \to \frac{X_1}{X_0}$ for $X_0 \neq 0$.

Polynomials in $\mathbb{P}^n$ must be homogenous, that is, all terms must have the same degree. Using homogenous polynomials, projective varieties can be defined.

Definition 2. *Let k be a field. Let $f_1, \ldots, f_m$ homogenous polynomials in $k[x_0, x_1, \ldots, x_n]$. A projective variety $V_{f_1, \ldots, f_m}$ is defined as the zero locus of a set of homogenous polynomials.*

$$V_{f_1, \ldots, f_m} = \{p \in k^n | f_i(p) = 0 \ \forall 1 \leq i \leq m\}$$

A line in $\mathbb{P}^2$ is defined by the points $(x : y : z)$ satisfying $ax + by + cz = 0$, for $(a : b : c) \in \mathbb{P}^2$. If we have two distinct lines defined by $ax + by + cz = 0$ and $a'x' + b'y' + c'z' = 0$, then they intersect at the points such that $\left(\begin{smallmatrix} a & b & c \\ a' & b' & c' \end{smallmatrix} \right) \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right)$. The rank of the matrix on the left is at most 2, so by the rank theorem it has nullity of at least 1, which means it must have a nonzero solution $(x : y : z) \in \mathbb{P}^2$ which lies in the intersection of the two lines.

So it is easy to check that any lines in $\mathbb{P}^2$ intersect at a single point. Bezout's theorem also illustrates how much more nicely enumerative properties are expressed in projective space than affine space.

Theorem 1.1.2 (Bezout). *If F and G are two curves in $\mathbb{P}^2$ of general position, then counting multiplicities, the number of their intersection points is the product of the degrees of F and G.*

So in $\mathbb{P}^2$, a very simple formula gives us the number of intersections of two curves. It is also easy to view any curve in affine space as embedded in projective space, and it is easy to translate between projective and affine coordinates. So it is useful in understanding properties of affine curves to understand properties of curves in $\mathbb{P}^n$, thus it is natural to ask questions about the algebraic properties and automorphisms of projective space, and a natural question from there is to ask about products such as $\mathbb{P}^1 \times \mathbb{P}^2$.

1.2   Maps between Varieties

We will establish here our notion of equivalence between varieties and our notion of automorphism. To start, we define

Definition 1.2.1 (Regular Function). *Let V be a variety. A regular function $f : V \to \mathbb{C}$ is a map such that for each $p \in V$, there is an open U containing p such that $f = \frac{g}{h}$ on U, where g and h are polynomials maps such that $h \neq 0$ on U.*

Definition 1.2.2 (Morphism). *Let V, W be varieties, and let $\phi : V \to W$ be a continuous map (for all open $U \subseteq W$, $\phi^{-1}(U)$ is open in V). Then ϕ is a morphism if for all open $U \subseteq W$, for all regular functions on U, $f \circ \phi$ is a regular function.*

The term morphism comes from category theory: the morphisms are the morphisms corresponding to the category of varieties.

An isomorphism between varieties V and W is a morphism with an inverse which is also a morphism, and an automorphism on V is just a isomorphism from V to V. We study a type of automorphism on toric varieties, toric symmetries, which will be described shortly.

A somewhat weaker notion of equivalence, birational equivalence, is also important to our discussions. A rational map $f : V \text{ -- > } W$ is a morphism on an open subset $U \subseteq V$, under the equivlence relation that two such morphisms are equivalent if they agree on the intersection of their domain. A birational map is a rational mapping with an inverse. The idea is that with rational maps, we do not require the map to be defined everywhere on V. The following example will appear later as a toric symmetry of $\mathbb{P}^2$.

Example 1.2.3 (The Cremona Transformation). *The map $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ defined by $\phi(x_0 : x_1 : x_2) = (x_1 x_2 : x_0 x_2 : x_1 x_2)$ is a birational map on $\mathbb{P}^2$. Note that it is defined everywhere on $\mathbb{P}^2$ except at the points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$.*

More generally, the n dimensional toric symmetry $\phi_n : \mathbb{P}^n \to \mathbb{P}^n$ is defined by $\phi_n(x_0 : x_1 : \cdots : x_n) = (\Pi_{1 \leq i \leq n} x_i : \Pi_{0 \leq i \leq n, i \neq 1} x_i : \cdots : \Pi_{0 \leq i \leq n, i \neq j} x_i : \cdots : \Pi_{0 \leq i \leq n, i \neq n} x_i)$.

The Cremona transformation is induced by a toric symmetry which will be discussed later.

1.3   Toric Varieties

The variety of interest, $\mathbb{P}^1 \times \mathbb{P}^2$, is an example of a toric variety, a class of varieties with a special structure, which we define in this section.

We define $\mathbb{C}^* := \mathbb{C} \backslash \{0\}$ and call $(\mathbb{C}^*)^n$ the *n*-dimensional torus. The torus acts on itself by coordinatewise multiplication.

A toric variety is an irreducible variety (which means it cannot be written as a union of two nonempty varieties) that contains a copy of the n-dimensional torus as an open dense subset such that the action of the torus on itself extends to an action on the entire variety.

One example of a toric variety is $\mathbb{P}^n$, where $(\mathbb{C}^*)^n$ is embedded by the mapping $(t_1, \ldots t_n) \rightarrow (1 : t_1 : \cdots : t_n)$, and the action of the torus on $\mathbb{P}^n$ is given by:

$$(t_1, \ldots, t_n)(X_0 : X_1 : \cdots : X_n) = (X_0 : t_1 X_1 : \cdots : t_n X_n).$$

Extending this, we see that our variety of interest, $\mathbb{P}^1 \times \mathbb{P}^2$, contains $(\mathbb{C}^*)^3$ by $(t_1, t_2, t_3) \rightarrow ((1 : t_1), (1 : t_2 : t_3))$, and the action of $(\mathbb{C}^*)^3$ on $\mathbb{P}^1 \times \mathbb{P}^2$ is given by

$$(t_1, t_2, t_3)((X_0 : X_1), (Y_0 : Y_1 : Y_2)) = ((X_0 : t_1 X_1), (Y_0 : t_2 Y_1 : t_3 Y_2))$$

The next section will describe how this torus action produces a simple combinatorial structure which is useful in studying automorphisms of the variety.

1.4   The fan of a toric variety

A key property of toric varieties is that each toric variety has an associated fan, which is a set of regions in $\mathbb{R}^n$ that combinatorially conveys all the information about the variety. The regions correspond to the torus fixed subvarieties of the toric variety.

First, we will define cones and fans.

Definition 1.4.1 (Cone). *The cone σ generated by $v_1, \ldots, v_k \in \mathbb{Z}^n$ is the region in $\mathbb{R}^n$ defined by*

$$\sigma = \{\alpha_1 v_1 + \cdots + \alpha_k v_k | \alpha_i \geq 0, 1 \leq i \leq k\}$$

We say that a cone σ is strongly convex if $\sigma \cap (-\sigma) = \{0\}$. A face of a cone σ is the intersection of σ with a linear form, that is a face is a cone of the form $\sigma \cap \{l = 0\}$, where l is a linear form non-negative on σ. A fan is a special collection of cones:

Definition 1.4.2 (Fan). *A fan Σ is a collection of cones such that each $\tau \in \Sigma$ is strongly convex, if $\tau_1, \tau_2 \in \Sigma$ then $\tau_1 \cap \tau_2 \in \Sigma$, and if $\tau \in \Sigma$ and σ is a face of τ, then $\sigma \in \Sigma$.*

For the fan of a toric variety, each cone corresponds to a subvariety fixed under the torus action. The key to finding the torus invariant subvarieties is to consider the orbits of the one parameter subgroups. The automorphisms of $\mathbb{C}^*$ are given by $\lambda_a(t) = t^a$, for $a \in \mathbb{Z}$, so the one parameter subvarieties of $\mathbb{P}^n$ are given by $\{(1, t^{a_1}, t^{a_2}, \ldots, t^{a_n}) \in \mathbb{P}^n | t \in \mathbb{C}^*\}$, setting $v = (a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$. We take the limit as the complex variable t approaches 0 of $\lambda_v(t) = (1, t^{a_1}, t^{a_2}, \ldots, t^{a_{n-1}})$, where $v = (a_1, \ldots, a_{n-1}) \in \mathbb{Z}^n$, and consider the closure of the orbit of this limit under the toric action. We do this for $\mathbb{P}^1$ in Table 1.1. To compute the second limit in Table 1.1, we simply notice that for $a < 0$, $(1 : t^a) \sim (t^{-a} : 1)$. The other limits are computed similarly.

a	$\lim_{t \to 0}(1 : t^a)$	closure of orbit of $\lim_{t \to 0}(1 : t^a)$
a > 0	(1:0)	$\{(1 : 0)\}$
a < 0	(0:1)	$\{(0:1)\}$
a = 0	(1 : 1)	$\mathbb{P}^1$

Table 1.1   Cones of $\mathbb{P}^1$

Notice that the limit depends on a, and left column of the table, which give the values of a determining the limit, correspond to regions in $\mathbb{R}^1$. Plotting these regions gives the picture of Figure 1.1.
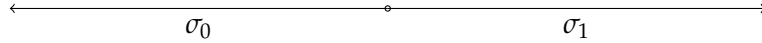
Figure 1.1   The Fan of $\mathbb{P}^1$

Figure 1.1 shows the fan of $\mathbb{P}^1$. Each cone corresponds to the torus fixed subvariety in the right column of Table 1.1. There are two one-dimensional cones, each corresponding to a fixed point, and a 0-dimensional cone corresponding to the entire space. The right column shows 3 torus fixed subvarieties of $\mathbb{P}^1$, the two fixed points (0:1) and (1:0), and the entire space $\mathbb{P}^1$.

So in general, the limit $\lim_{t \to 0} \lambda_v(t)$ depends on v, and the regions in $v \in \mathbb{R}^n$ for which $\lim_{t \to 0} \lambda_v(t)$ approaches the same limit define the cones of the fan. The subvariety corresponding to a cone is the closure of the limit of $\lim_{t \to 0} \lambda_v(t)$.

We compute the fan for $\mathbb{P}^2$. Plotting the regions in $\mathbb{R}^2$ specified by a, b gives the fan of $\mathbb{P}^2$, shown in figure 1.2.

In general, for $\mathbb{P}^n$, the primitive generators for the one dimensional cones are given by $e_1, \ldots, e_n$ and $e_0 := -e_1 - e_2 - \cdots - e_n$.

a,b	$\lim_{t\to 0}(1 : t^a : t^b)$	closure of orbit of $\lim_{t\to 0}(1 : t^a : t^b)$
a,b > 0	(1:0 :0)	$\{(1 : 0 : 0)\}$
$a < 0, a < b$	(0:1:0)	$\{(0 : 1 : 0)\}$
$b < 0, b < a$	(0:0:1)	$\{(0 : 0 : 1)\}$
$a = 0$, $b > 0$	(1 : 1 : 0)	$\{(x_0 : x_1 : 0)\}$
$b = 0$, $a > 0$	(1 : 0 : 1)	$\{(x_0 : 0 : x_2)\}$
$a = b < 0$	(0 : 1 :1)	$\{(0 : x_1 : x_2)\}$
$a = b = 0$	(1 : 1 :1)	$\mathbb{P}^2$

Table 1.2   Cones of $\mathbb{P}^2$
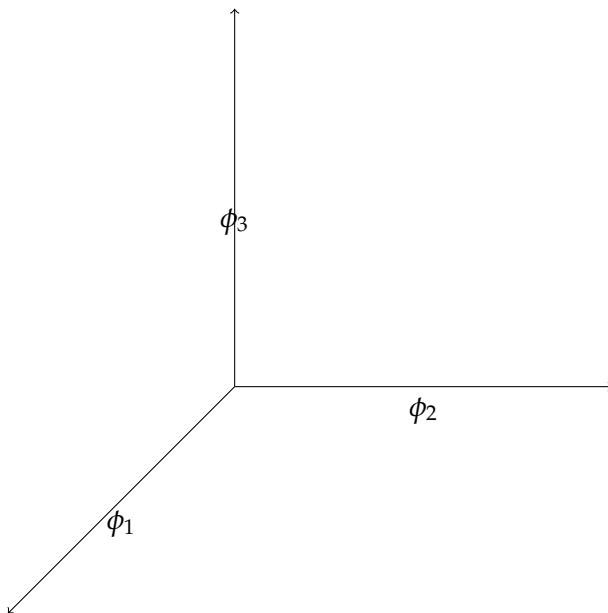
Figure 1.2   The Fan of $\mathbb{P}^2$

Notice that the dimension of the cone is the codimension of the corresponding subvariety.

It is also worth mentioning that it is possible to construct the toric variety from the fan Cox (1991), thus the fan encodes all information about the toric variety and vice versa.

We find 21 cones for the fan for the variety of interest, $\mathbb{P}^1 \times \mathbb{P}^2$, easily from the fans for $\mathbb{P}^1$ and $\mathbb{P}^2$. In particular, we have 5 toric fixed divisors (1 dimensional cones), 9 toric fixed lines (the two dimensional cones), 6 toric fixed points (three dimensional cones), and $\mathbb{P}^2$ (the origin). This can be

derived using the process above. As may be expected, the cones are embedded in $\mathbb{R}^3$, and the cones are found simply by taking cartesian products of the cones for $\mathbb{P}^1$ and $\mathbb{P}^2$.

It will be helpful if we establish some notation to denote the cones in the fan of $\mathbb{P}^1 \times \mathbb{P}^2$. The one-dimensional cones (corresponding to the toric fixed divisors of $\mathbb{P}^1 \times \mathbb{P}^2$ corresponds to either one of the one dimensional cones of $\mathbb{P}^2$, which we'll denote ϕ_1, ϕ_2, ϕ_3, generated by the primitive generators $v_1 = (-1, -1, 0)$, $v_2 = (1, 0, 0)$, and $v_3 = (0, 1, 0)$ respectively, or to one of the one dimensional cones of $\mathbb{P}^1$, which we will denote ϕ_4 and ϕ_5, which are generated by the primitive generators $v_4 = (0, 0, 1)$ and $v_5 = (0, 0, -1)$ respectively. The cone generated by ϕ_i, ϕ_j is denoted $\phi_{i,j}$, these correspond to the toric fixed lines of $\mathbb{P}^1 \times \mathbb{P}^2$, and the cone generated by ϕ_i, ϕ_j and ϕ_k is denoted $\phi_{i,j,k}$.

1.5   Blowing up

One method to deal with a singular point of a curve or space is to "blow up" at the singularity by replacing the singular point with the set of lines through that point. If the space is of complex dimension n, such as $\mathbb{P}^n$, we would replace the singularity with a copy of $\mathbb{P}^{n-1}$.

Let Δ be a disc in $\mathbb{C}^n$. We will blow up at the origin and can use this to blowup at any point. The blowup of Δ at the origin is given by

$$\tilde{\Delta} = \{(z, l) \in \Delta \times \mathbb{P}^n | z_i l_j = z_j l_i \; \forall i \neq j\}. \tag{1.1}$$

If we let $\pi : \tilde{\Delta} \to \Delta$ be the projection map $\pi(z, l) = z$, then one can easily verify that for $z = (z_1, z_2, \ldots, z_n) \neq$, $\pi^{-1}(z) = \{(z, (z_1 : z_2 : \cdots : z_n))\}$ and $\pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$.

A simple change of coordinates allows us to blow up at any point using this definition. The blow up at $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ is $\tilde{\Delta}_x = \{(z, l) \in \Delta \times \mathbb{P}^n | (z_i - x_i) l_j = (z_j - x_j) l_i \; \forall i \neq j | l \in \mathbb{P}^{n-1}\}$.

To blow up $\mathbb{P}^2$ at a point $(z_1 : z_2 : z_3)$, assume without loss of generality that $z_1 \neq 0$, and consider $\{(x_0 : x_1 : x_2) | x_0 \neq 0\} \subseteq \mathbb{P}^2$ which is equivalent to $\mathbb{C}^2$ via the mapping $(x_0 : x_1 : x_2) \to (\frac{x_1}{x_0}, \frac{x_2}{x_0})$, and blow up this set as before.

To blow up along a codimension-k submanifold, if the submanifold is given by $x_1 = x_2 = \cdots = x_k = 0$, then the blowup is contained in $\mathbb{C}^n \times \mathbb{P}^{k-1}$ and is given by

$$\{(z, l) \in \mathbb{C}^n \times \mathbb{P}^{k-1} | z_i l_j = z_j l_i\}.$$
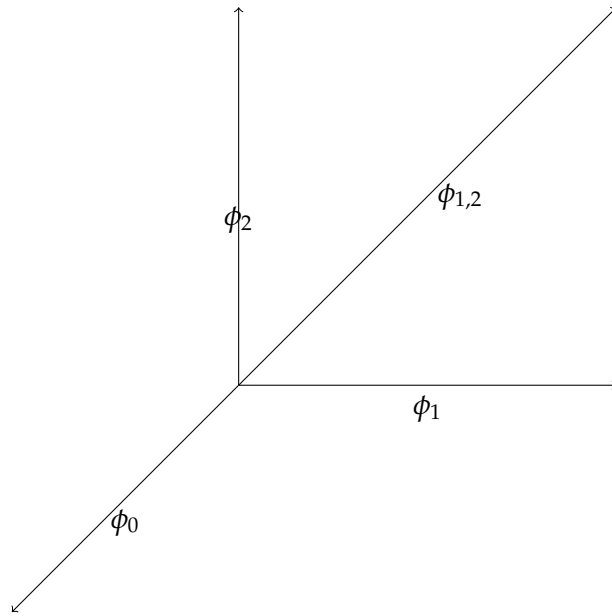
Figure 1.3   The Fan of $\mathbb{P}^2$ Blown up at the point $(1:0:0)$

When we blow up a toric fixed point in a toric variety, this corresponds to subdividing the fan on the cone corresponding to the point we're blowing up. For example, figure 1.3 depicts the fan of $\mathbb{P}^2$ blown up at the point $(1:0:0)$, which we denote $\mathbb{P}^2(1)$.

In the diagram, we have blown up $\mathbb{P}^2$ at the point $(1:0:0)$, and this corresponds to creating a new one dimensional cone $\sigma_{(2,3)}$, where the primitive generator of this fan is the sum of the primitive generators of ϕ_2 and ϕ_3, so $\sigma_{2,3} = \langle (1,1) \rangle$. We replace the cone generated by ϕ_2 and ϕ_3 with three new cones: $\sigma_{(2,3)}$, a cone generated by ϕ_2 and $\sigma_{(2,3)}$, and a cone generated by ϕ_3 and $\sigma_{(2,3)}$. In general, to blow up a toric invariant subvariety whose cone is generated by one dimensional cones with primitive generators $v_1, \ldots, v_k$, we add a one dimensional cone generated by $v = v_1 + \cdots + v_k$, and replace the cone generated by $v_1, \ldots, v_k$ with k new cones generated by $v, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$ for each $1 \leq i \leq k$.

1.6   Cohomology

Homology and cohomology are covariant and contravariant functors from the category of topological spaces to the category of commutative algebras.

For example, De Rham cohomology defines a sequence of rings on a differential manifold using the kernel and image of exterior derivative on the sets of differential k-forms.

For toric varieties, the elements of the cohomology ring correspond to classes of subvarieties of each dimension. Multiplication in this ring corresponds to intersection. For example, in $\mathbb{P}^2$, for the element corresponding to a line, call it H, the element H^2 corresponds to a point, because two lines intersect at a point.

A theorem due to Fulton Fulton (1993) allows us to compute the cohomology classes easily using the one skeleton of the fan. The theorem says that for any "nice" toric variety, we get the cohomology from the hyperplanes of the toric variety, the one dimensional cones of the fan. In other words cohomology is generated by divisors.

Theorem 1.6.1. *(from page 106 of Fulton) Fulton (1993) : For a nonsingular projective toric variety X, whose fan $\Sigma \subseteq \mathbb{R}^k$ has primitive generators $v_1, \ldots v_d$, $H^*X \cong \mathbb{Z}[t_1, t_2, \ldots t_d]/I$, where I is the ideal generated by all*

$$(i) t_{i_1} \ldots t_{i_k}, \text{ for } v_{i_1}, \ldots v_{i_k} \text{ not in a cone of } X.$$

$$(ii) \sum_{i=1}^{d} < u, v_i > t_i \text{ for } u \text{ in } M,$$

where M is the dual of the lattice on which the fan is defined, that is the set of $v \in \mathbb{Z}^k$ such that $\langle u, v \rangle \geq 0$ for all $v \in \mathbb{Z}^k \cap \Sigma$.

Applying this theorem to $\mathbb{P}^n$, we find that

$$H^*(\mathbb{P}^n) = \mathbb{Z}[H]/H^{n+1}.$$

Because codimension k subvarieties are generated from intersections of lower codimension subvarieties, the degree of an element in the ring corresponds to its codimension. For instance, $H^*(\mathbb{P}^2) = \mathbb{Z}[H]/H^2$. In this ring H corresponds to a line, H^2 corresponds to the intersection of 2 general lines which is a point, and is also the generating element of $H^2(\mathbb{P}^n)$. Then $2H$ corresponds to a conic, the coefficient of H corresponds to the degree of the curve. Then $H * 2H = 2H^2$, so the intersection of the class of a conic with the class of a line is the class of two points.

To compute $H^*(\mathbb{P}^1 \times \mathbb{P}^2)$ we can use Theorem 2.1.1. Here we use H_i's instead of t_i's to be consistent with later notation.

$$H^*(\mathbb{P}^1 \times \mathbb{P}^2) = \mathbb{Z}[H_{(2,1)}, H_{(2,2)}, H_{(2,0)}, H_{(1,1)}, H_{(1,0)}]/$$
$$(H_{(1,1)}H_{(1,0)}, H_{(2,1)}H_{(2,2)}H_{(2,0)}, H_{(1,1)} - H_{(1,0)}, H_{(2,2)} - H_{(2,0)}, H_{(2,1)} - H_{(2,0)})$$
$$= \mathbb{Z}[H_1, H_2]/(H_1^2, H_2^3) \quad (1.2)$$

So the ring is generated by H_1, H_2, the generating divisor classes. H_1 represents the class of the divisors $\{x\} \times \mathbb{P}^2$, where $x \in \mathbb{P}^1$, which is isomorphic to $\mathbb{P}^2$. D_2 represents the divisors $\mathbb{P}^1 \times l$, where l is a line in $\mathbb{P}^2$, and is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

The curve classes are generated by $h = H_2^2$, which corresponds to the class of the fibers $\mathbb{P}^1 \times \{y\}$, for $y \in \mathbb{P}^2$. Also $g = H_1 H_2$ is the curve class of a line in $\mathbb{P}^2$, that is, the class of horizontal fibers $\{x\} \times l$, where $x \in \mathbb{P}^1$ and l is a line in $\mathbb{P}^2$.

We can use the Kunneth formula to confirm this, using the fact that $H^*(\mathbb{P}^1) = \mathbb{Z}[H_1]/(H_1^2)$ and $H^*(\mathbb{P}^2) = \mathbb{Z}[H_2]/(H_2^3)$ which we showed using Theorem 2.1.1.

$$H^*(\mathbb{P}^1 \times \mathbb{P}^2) = H^*(\mathbb{P}^1) \otimes H^*(\mathbb{P}^2)$$
$$= \{\sum_{i=1}^{k} \alpha_i e_{(a_i H_1 + b_i),(x_i H_2^2 + y_i H_2 + z_i)} | \alpha_i, a_i, b_i, c_i, x_i, y_i, z_i \in \mathbb{Z}\}/R$$
$$= \mathbb{Z}[e_{1,1}, e_{1,H_2}, e_{1,H_2^2}, e_{H_1,1}, e_{H_1,H_2}, e_{H_1,H_2^2}].$$

By the isomorphism $H_1^n H_2^m \to e_{H_1^n, H_2^m}$, we see that this expression agrees with equation 1.2.

Of course, blowing up $\mathbb{P}^1 \times \mathbb{P}^2$ introduces more classes. For instance, blowing up $\mathbb{P}^1 \times \mathbb{P}^2$ at a point introduces the class of exceptional divisor E of the blowup, which is a copy of $\mathbb{P}^2$, and of the fibers of lines inside the blown up space, e. The exceptional divisors obtained when blowing up a line in $\mathbb{P}^1 \times \mathbb{P}^2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and therefore has two classes of fibers, f, g.

Chapter 2

Toric Symmetries

We can now describe more explicitly the idea of a toric symmetry.

Definition 3. *A toric symmetry of a toric variety X is an isomorphism of X which is induced by an automorphism of its fan.*

An automorphism on the fan is an automorphism on $\mathbb{Z}^3$ which permutes the cones of the fan.

We are interested in birational maps on toric varieties, and in particular $\mathbb{P}^1 \times \mathbb{P}^2$, which are induced by automorphisms of toric blow-ups. The induced action of the maps on the cohomology classes can reveal enumerative properties of the subvarieties. Next we'll discuss a well known example to illustrate what kinds of maps we are interested in.

2.1 The Cremona Transformation

We discuss one illuminating example of a toric symmetry of $\mathbb{P}^2$ blown up at three points, which we denote $\mathbb{P}^2(3)$, to illustrate how we can use properties of the fan of a toric variety to find automorphisms that act nontrivially on cohomology.

We will blow up all three fixed points and consider the automorphism on the fan which is reflection across the origin, that is, $\tau(a, b) = (-a, -b)$. The corresponding birational map in $\mathbb{P}^2$, which was mentioned earlier, is

$$\tau(X_0 : X_1 : X_2) = (X_1 X_2 : X_0 X_2 : X_1 X_2).$$

When we blow up the three points, the one skeleton consists of the three previous one dimensional cones, denoted ρ_1, ρ_2, ρ_0 with primitive

generators $v_1 = (1,0)$, $v_2 = (0,1)$, and $v_0 = (-1,-1)$, and the new one dimensional cones $\rho_{1,2}$, $\rho_{0,1}$, $\rho_{0,2}$ with primitive generators $v_{1,2} = (1,1)$, $v_{0,1} = (-1,0)$, and $v_{0,2} = (0,-1)$, shown in Figure 1.4.
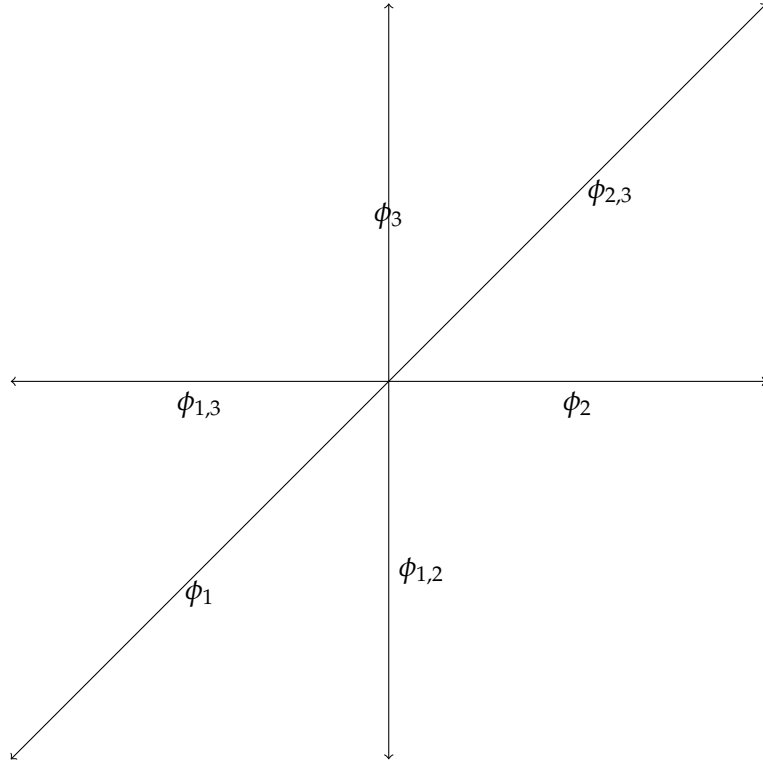
Figure 2.1   The Fan of $\mathbb{P}^2$ Blown up at the three points

To calculate the homology $H^*(\mathbb{P}^2(3))$ we use the theorem above.

$H^*(\mathbb{P}^2(3)) = \mathbb{Z}[t_1, t_2, t_3, t_{1,2}, t_{1,3}, t_{2,3}]/I$, where I is the ideal generated by the equations listed in the theorem. We map this group via the isomorphism which sends $t_i \rightarrow H - E_j - E_k$, where $\{i, j, k\} = \{0, 1, 2\}$, and $t_{j,k} \rightarrow E_i$, for any $\{i, j, k\} = \{0, 1, 2\}$.

This shows that

$H^*(\mathbb{P}^2(3)) = <H, E_0, E_1, E_2> / <E_i^2 = -H^2, E_i E_j = 0, E_i H = 0>.$

Written this way, the E_i's correspond to the exceptional divisors at the blown up points, and H corresponds to a line of $\mathbb{P}^2$.

Now looking at the symmetry of the fan of $\mathbb{P}^2(3)$, $\tau(a, b) = (-a, -b)$, this sends $\tau * (H) = 2H - E_0 - E_1 - E_2$, so this automorphism sends the

cohomology class of line to the class of conics passing through the 3 fixed points. The automorphism on the blow-up space can be pulled back to an birational map on $\mathbb{P}^2$, so the Cremona transform provides an automorphism on $\mathbb{P}^2$ which maps lines to conics passing through three fixed points. By the way, the birational map on $\mathbb{P}^2$ is given by mapping $(X_0 : X_1 : X_2)$ to $(X_1 X_2 : X_0 X_2 : X_0 X_1)$. This map has as its singularities the toric fixed points of $\mathbb{P}^2$, which is why we resolved these singularities by blowng up.

An intuitive question is "does this generalize to $\mathbb{P}^3$?". Blowing up the 4 fixed points in $\mathbb{P}^3$ does not give reflectional symmetry, so to generalize this automorphism, we must blow up the 6 fixed lines as well. The generalized map of reflection across the axis would be the rational map which maps $(X_0 : X_1 : X_2 : X_3)$ to $(X_1 X_2 X_3 : X_0 X_2 X_3 : X_0 X_1 X_3 : X_0 X_1 X_2 X_3)$, and its singularities are not only the fixed toric points, but the toric fixed lines as well. So they must all be blown up.

A toric symmetry is an autmorphism of a toric variety X induced by an automorphism of the fan Σ_X of X. An automorphism of Σ_X is an automorphism of $\mathbb{Z}^3$ which permutes the cones of Σ_X. The Cremona transform is a toric symmetry of $\mathbb{P}^2$ blown up at three points, and in fact it is a nontrivial toric symmetry because it induces a nontrivial action on the cohomology ring. The goal of this project is to study non trivial toric symmetries of blow ups of $\mathbb{P}^1 \times \mathbb{P}^2$. We are interested in finding automorphisms of $\mathbb{Z}^3$ that permute the cones of the fan of a blow up of $\mathbb{P}^1 \times \mathbb{P}^2$ such that the elements of the cohomology ring of $\mathbb{P}^1 \times \mathbb{P}^2$ are permuted non trivially. Such automorphisms are nontrivial toric symmetries.
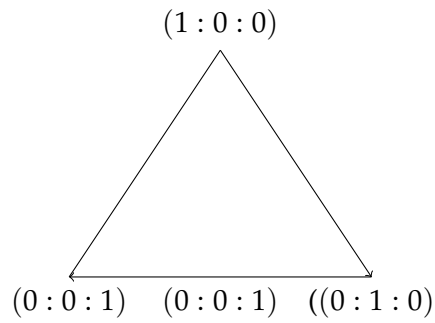
2.2   Polytopes

A projective toric variety also has a corresponding polytope. The fan will be our main tool for computing symmetries, but the polytope is useful for visualizing and understanding the blown ups of $\mathbb{P}^1 \times \mathbb{P}^2$. A face of dimension k on the polytope corresponds to a cone of codimension k in the fan.

Example 2.2.1. *The polytope for $\mathbb{P}^2$ is a triangle, in which the three corners correspond to the fixed points.*

*When we blow up at the point $\mathbb{P}^2$, we subdivide the cone generated by ρ_1, ρ_2, removing one maximal cone and replacing it with 2 new maximal cones and a 1 dimensional cones. On the polytope of $\mathbb{P}^2$ blown up at a point, the two new maximal cones correspond to two new points, and the new 1 dimensional cones corresponds to a new 1 dimensional face. The result is depicted below.*
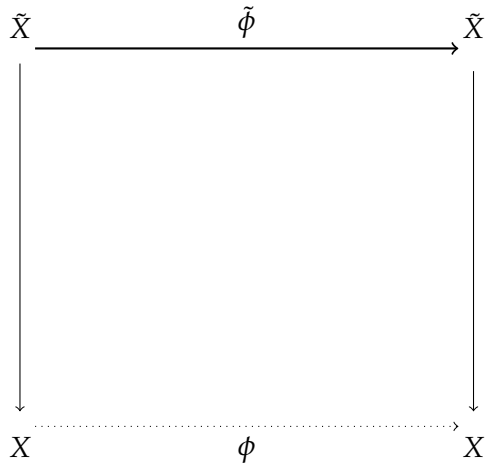
Figure 2.2   A toric symmetry $\tilde{\phi}$ of a toric blowup $\tilde{X}$ induces a birational map on X

$(1:0:0)$

$(0:0:1)$ $(0:0:1)$ $((0:1:0)$
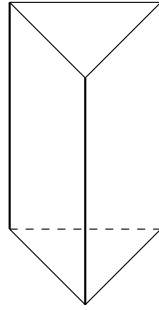
Figure 2.3   The Polytope of $\mathbb{P}^2$

$(0:0:1)$ $(0:1:0)$
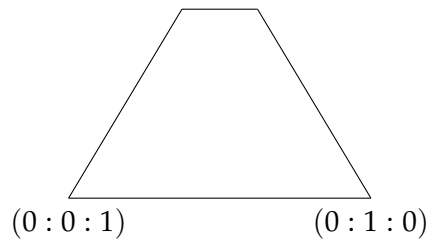
Figure 2.4   The Polytope of $\mathbb{P}^2$ Blown up at $(1:0:0)$

Example 2.2.2. *Polytope of $\mathbb{P}^1 \times \mathbb{P}^2$ Since $\mathbb{P}^1 \times \mathbb{P}^2$ is the space we are interested in, we will discuss the geometry of the polytope and its blown up varieties. For*

$\mathbb{P}^1 \times \mathbb{P}^2$, *the corresponding polytope is a triangular prism. The corners correspond to the generating cones, the three dimensional cones, which correspond to fixed points, the edges are the two dimensional cones corresponding to fixed lines, and the two dimensional faces which are the divisor classes* $\phi_1, \ldots, \phi_5$.

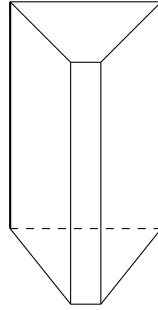**Figure 2.5** The Polytope of $\mathbb{P}^1 \times \mathbb{P}^2$

The divisor classes described in the previous section can be visualized as vertical and horizontal slices of the polytope. The horizontal slices are triangles, which are the $\mathbb{P}^2$ *divisors, described above as* H_1. *Vertical slices are rectangles, which are copies of* $\mathbb{P}^1 \times \mathbb{P}^1$, *described above as the class* H_2.

What does the polytope of $\mathbb{P}^1 \times \mathbb{P}^2$ *blown up at a point look like? The point is replaced by a copy of* $\mathbb{P}^2$ *when we blow up, so the corner is replaced by a triangle, the polytope of* $\mathbb{P}^2$. *We can also blow up lines, in which case we replace a line in* $\mathbb{P}^1 \times \mathbb{P}^2$ *with a copy of* $\mathbb{P}^1 \times \mathbb{P}^1$, *whose polytope is given by a rectangle.*
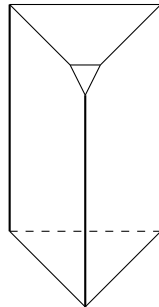
Figure 2.6 The Polytope of $\mathbb{P}^1 \times \mathbb{P}^2$ Blown Up at a Point

It is perhaps easy to understand by considering these pictures why the order in which we blow up certain lines is important, while the order in which we blow up non-intersecting lines or pairs of points is not. Blow-up is local, so the order in which we blow up objects matters only when they intersect. Also it is impossible

Figure 2.7 The Polytope of $\mathbb{P}^1 \times \mathbb{P}^2$ Blown Up on a Line

to blow up a point after a line which contains it has been blown up, because that point no longer exists.

Note that while we did not explicitly derive the construction of the polytope here, the polytope is constructed as the dual of the fan. Each cone of the fan of dimension k, has a corresponding dual face on the polytope which has codimension k and is normal to the cone. The polytope is constructed by gluing together faces. If σ_1 and σ_2 are cones with corresponding faces F_1, F_2, and if σ_1, σ_2 generate a cone, that is, $\sigma_{1,2} = \{\alpha v + \beta u | v \in \sigma_1, u \in \sigma_2, \alpha, \beta \geq 0\}$ is a cone, then F_1 and F_2 are glues together along the face corresponding to $\sigma_{1,2}$. For instance, a few of the faces of $\mathbb{P}^1 \times \mathbb{P}^2$ are labeled below with their corresponding cones. The intersection of the faces for ϕ_1 and ϕ_4 corresponds to the cone they generate, which is the line $\phi_{1,4}$.
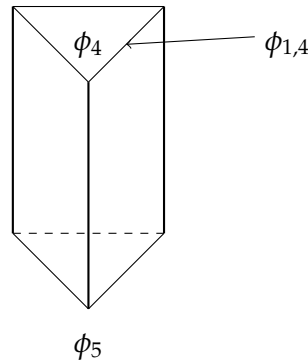
Figure 2.8 The Polytope of $\mathbb{P}^1 \times \mathbb{P}^2$

It is also possible to, given a polytope, construct the associated fan and variety using a method described by Cox Cox (1991).

Chapter 3

The Symmetries of $\mathbb{P}^1 \times \mathbb{P}^2$

3.1 Exhaustive Computation of the symmetries of $\mathbb{P}^1 \times \mathbb{P}^2$

We provide code which computes the exhaustive list of toric symmetries of $\mathbb{P}^1 \times \mathbb{P}^2$, based on a previous program used to compute the symmetries of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ Karp et al. (2010). Here we include some of the output of that code. The algorithm takes as input the list of primitive generators of the 1-skeleton of the fan, which for $\mathbb{P}^1 \times \mathbb{P}^2$ is the ordered set $\{(-1,-1,0),(1,0,0),(0,1,0),(0,0,1),(0,0,-1)\}$, labeled $\phi_1, \ldots, \phi_5$, and the list of generating cones, that is, the cones of maximal dimension:

$$\phi_{124}, \phi_{125}, \phi_{134}, \phi_{135}, \phi_{234}, \phi_{235},$$

where ϕ_{ijk} is the cone generated by the primitives ϕ_i, ϕ_j, ϕ_k,

$$\phi_{ijk} = \{\alpha v_i + \beta v_j + \gamma v_k | \alpha, \beta, \gamma \geq 0\}.$$

A list of things to blow up is defined as the set of generating cones (described above, these correspond to the six fixed points of the torus action) and two dimensional cones (these correspond to the 9 torus fixed lines and are given by $L_{12}, L_{23}, L_{13}, L_{14}, L_{15}, L_{24}, L_{25}, L_{34}, L_{35}$. A list is made of the possible ordered combinations of points and lines which can be blown up to isomorphism. There are 2^{15} subsets of these objects, and, because blow-up is local, the order in which we blow up these objects matters whenever two objects intersect nontrivially. The points may be interchanged with other points, that is, if a combination includes P_{ijk} and P_{abc}, they can be exchanged isomorphically. For two lines, the order in which we blow up

matters if they intersect at a point, that is, if we include L_{ij}, L_{ab} in our combination, then their order may be exchanged if either both $\{i, j\}, \{a, b\}$ are contained in $\{1, 2, 3\}$, or if $\{4, 5\} \cap \{i, j\} \neq \varnothing$, and $\{4, 5\} \cap \{a, b\} \neq \varnothing$, and $\{4, 5\} \cap \{i, j\} \cap \{a, b\} = \varnothing$. These are the cases in which two lines do not intersect. As long as the points are blown up before the lines, these are the only restrictions on ordering. Symmetries which arise only from symmetries of $\mathbb{P}^2$ are also removed at this point.

Once the algorithm has created a list of nonisomorphic combinations of points and lines to blow up, it creates a list of 3x3 invertible matrices with coefficients 0,1,-1. These matrices represent the only automorphisms of $\mathbb{Z}^3$ which might permute the cones of a blowup of $\mathbb{P}^1 \times \mathbb{P}^2$. For each combination, we check each transformation and sees if it permutes the one skeleton, by applying the potential symmetry to each primitive generator for the blow-up space and seeing if it maps to another primitive generator. If it does, and if the map cannot be trivially reduced to a symmetry of $\mathbb{P}^2$ or $\mathbb{P}^1$, then the transformation is put into a list of likely symmetries. Finally the algorithm checks if each transformation is a symmetry by checking if it permutes the generating cones. If it does, then it is a symmetry, and if the map is not the identity map, it is returned as a nontrivial symmetry.

In the output given below, for each symmetry, the first line lists the objects to be blown up, in the order in which they will be blown up. For instance, for the first symmetry, first the point generated by ϕ_1, ϕ_2, ϕ_4 is blown up. Then the point generated by ϕ_2, ϕ_3, ϕ_5 is blown up, then the line generated by ϕ_2, ϕ_3, and so forth. These objects are labeled with an index corresponding to the objects corresponding ray. There are already 5 rays from $\mathbb{P}^1 \times \mathbb{P}^2$, so P124 is labeled 6, P325 is labeled 7, and so forth. The next line gives the permutation of the one-skeleton, for instance, $(2, 4)$ means that ϕ_2 and ϕ_4 are swapped. The third line gives the matrix of the automorphism on the fan of the blow up. The 2s in the matrices are actually -1s, but are left as 2s for readability.

Below is a truncated segment of the output of the algorithm.

```
  ['P124', 'P235', 'L23', 'L24', 'L35', 'L13']
There are 1 nontrivial automorphisms of this fan.
There are 2 automorphisms of this fan:


1 (nontrivial!):
(2,4)(3,7)(5,11)
[0 1 1]
```

```
[0 1 0]
[1 2 0]
```

['P124', 'P234', 'L12', 'L35', 'L13', 'L14', 'L15', 'L23']
There are 1 nontrivial automorphisms of this fan.
There are 2 automorphisms of this fan:

1 (nontrivial!):
(1,4)(2,9)(3,8)(5,13)(6,10)(7,12)
```
[0 0 2]
[1 2 2]
[2 0 0]
```

['P124', 'P234', 'L12', 'L15', 'L23', 'L35', 'L13', 'L34']
There are 1 nontrivial automorphisms of this fan.
There are 2 automorphisms of this fan:

1 (nontrivial!):
(1,10)(2,9)(3,4)(5,8)(6,11)(7,12)
```
[2 0 0]
[2 0 1]
[2 1 0]
```

['P125', 'L15', 'L34', 'L13', 'L24', 'L23']
There are 1 nontrivial automorphisms of this fan.
There are 2 automorphisms of this fan:

1 (nontrivial!):
(1,8)(3,7)(6,11)
```
[1 2 0]
[0 2 0]
```

```
[0 2 1]
```

```
['P134', 'L12', 'L14', 'L23', 'L25']
There are 1 nontrivial automorphisms of this fan.
There are 2 automorphisms of this fan:
```

```
...
```

```
['L25', 'L34', 'L23']
There are 2 nontrivial automorphisms of this fan.
There are 4 automorphisms of this fan:
```

```
1 (nontrivial!):
(2,6)(3,7)(4,5)
[1 0 0]
[0 1 0]
[2 1 2]
```

```
2 (nontrivial!):
(2,7)(3,6)
[0 1 0]
[1 0 0]
[1 2 1]
```

```
['P125', 'P134', 'L13', 'L25', 'L34']
There are 1 nontrivial automorphisms of this fan.
There are 2 automorphisms of this fan:
```

```
1 (nontrivial!):
(1,3)(4,9)(5,7)(6,10)
[1 2 1]
[0 2 0]
[0 0 2]
```

```
['P135', 'L12', 'L34', 'L13', 'L24']
There are 1 nontrivial automorphisms of this fan.
There are 2 automorphisms of this fan:
```

```
1 (nontrivial!):
(1,8)(2,5)(3,7)(4,9)(6,10)
[0 0 2]
[0 2 0]
[2 0 0]
```

```
['P124', 'P235', 'L23', 'L35', 'L13', 'L24']
There are 1 nontrivial automorphisms of this fan.
There are 2 automorphisms of this fan:
```

```
1 (nontrivial!):
(2,4)(3,7)(5,10)
[0 1 1]
[0 1 0]
[1 2 0]
```

3.2   Action on Divisor Classes

In the following theorems we describe a nontrivial symmetry of $\mathbb{P}^1 \times \mathbb{P}^2$ and the induced action on cohomology.

Theorem 3.2.1. *There is a toric symmetry τ of $\mathbb{P}^1 \times \mathbb{P}^2$ on the blow up space given by blowing up the points p_{124}, p_{235}, and then the lines $L_{23}, L_{24}, L_{35}, L_{13}$ in that order, such that $\tau(H_1) = H_2 - E_{235} - F_{23}$, and $\tau(H_2) = H_1 + H_2 - E_{234} - F_{35} - F_{23}$.*

This is the first symmetry in the output list. The blow up space, $\mathbb{P}^1 \times \mathbb{P}^2$ blown up at 2 points and 4 lines, is shown below as its polytope. The cohomology ring is given by Theorem 2.1.1, and is $Z[t_1, t_2, t_3, \ldots, t_{11}]/R$, where $t_6, \ldots t_{11}$ are the primitive generators corresponding to the objections blown up, in the order by which they are blown up, so that t_6 corresponds to p_{124} and t_{11} corresponds to L_{13}, and R is the set of relations given by the Fulton theorem. To write the t_i in terms of exceptional divisor classes and divisor classes of $\mathbb{P}^1 \times \mathbb{P}^2$, we use an isomorphism, which maps the t_is that are subdivided to their corresponding exceptional divisor, and the other t_is are mapped according to which divisors they intersect. The isomorphism is given in the following table.

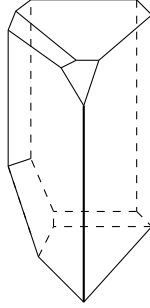| t_i	$f(t_i)$
t_1	$H_2 - E_{124} - F_{13}$
t_2	$H_2 - E_{124} - E_{235} - F_{23} - F_{24}$
t_3	$H_2 - E_{235} - F_{13} - F_{23} - F_{35}$
t_4	$H_1 - F_{24} - E_{124}$
t_5	$H_1 - E_{235} - F_{35}$
t_6	E_{124}
t_7	E_{235}
t_8	F_{23}
t_9	F_{24}
t_{10}	F_{35}
t_{11}	F_{13}

Table 3.1 Mapping

One easily verify that the adjunction formula is satisfied, $\sum t_i = 2H_1 + 3H_2 - 2\sum E_{\alpha\beta\gamma} - \sum F_{\alpha\beta}$ and one can further verify that this is an isomorphism by constructing the inverse map, by mapping $H_2 \rightarrow t_1 + t_6 + t_{11}$, $H_1 \rightarrow t_4 + t_9 + t_6$.

If τ is the automorphism given by the algorithm,then $\tau(H_2) = \tau(t_1) + \tau(t_6) + \tau(t_{11}) = t_1 + t_6 + t_5 = H_1 + H_2 - E_{235} - F_{13} - F_{35}$, and $\tau(H_1) = \tau(t_4) + \tau(t_9) + \tau(t_6) = t_2 + t_9 + t_6 = H_2 - E_{235} - F_{23}$.

The birational map for this particular symmetry, is given by $((y_1 : y_2), (x_0 : x_1 : x_2),) --> ((x_2y_2 : x_1y_1), (x_0y_1 : x_2y_2 : x_2y_1))$. One can easily ver-

ify that the singularities of this map are the lines and points on which are blown up, and that the cones of $\mathbb{P}^1 \times \mathbb{P}^2$ are mapped to each other in the way we would expect.

In general, when analyzing a symmetry, the isomorphism goes as follows: For $1 \leq i \leq 3$, $t_i \to H_2 - \sum_{i \in \{\alpha, \beta, \gamma\}} E_{\alpha\beta\gamma} - \sum_{i \in \{\alpha, \beta\}} F_{\alpha\beta}$.
For $4 \leq i \leq 5$, $t_i \to H_1 - \sum_{i \in \{\alpha, \beta, \gamma\}} E_{\alpha\beta\gamma} - \sum_{i \in \{\alpha, \beta\}} F_{\alpha\beta}$.

Figure 3.1   The Polytope of the Blow-Up Space for Theorem 1

In general for nontrivial toric symmetries, we find that the generating divisor classes of $\mathbb{P}^1 \times \mathbb{P}^2$, which are H_1, H_2 are mapped in one of a few ways. If $\{i, j\} = \{1, 2\}$, then either H_1 is mapped to $H_i - \sum E_{hjk} - \sum F_{lm}$ and H_2 is mapped to $H_i + H_j - \sum E_{ghk} - \sum F_{lm}$, or H_1 is mapped to $H_i + H_j - \sum E_h jk - \sum F_{lm}$ and H_2 is mapped to $H_i - \sum E_{hjk} - \sum F_{lm}$.

Chapter 4

Future Work

There are many questions to be explored about curve classes of $\mathbb{P}^1 \times \mathbb{P}^2$. One way to proceed is to analyze the action of the symmetry on cohomology classes corresponding to a specific curve when pulled back to $\mathbb{P}^1 \times \mathbb{P}^2$, for example, for the symmetry described in theorem 3.2.1, the class $H_2^2 - E_{124}^2$.

The curves of $\mathbb{P}^1 \times \mathbb{P}^2$ and its blow-ups are given by the intersections of divisor classes, which in the cohomology ring are generated by the products of divisor classes. Using the relations given by Theorem 2.1.1, we can describe these curve classes by describing products of the generating divisor classes. Several pairs of elements, which have trivial intersections, have a product of 0. We find that $H_1^2 = 0$, because the divisors of the form $\{x\} \times \mathbb{P}^2$ do not intersect each other, and similarly $H_1 F_{24} = H_1 F_{35} = H_1 E_{ijk} = 0$. We also find that $E_{124} E_{235} = 0$, because there is no cone containing both of these exceptional divisors, similarly other classes of exceptional divisors with trivial intersection have a product of 0. Of course, many $E_{ab}^2 = -e_{\alpha_b eta}$, the class of a line E_{ab}. Also, $E_{abc} F_{ab}$ is a fiber a class of F_{ab}, and if the exceptional divisors for two blown up lines intersect, the product of their classes in the cohomology ring is a fiber class. So properties likes these of the generators of the curve classes and help study the image under τ of specific curves.

Further investigation may uncover enumerative properties of the curves of $\mathbb{P}^1 \times \mathbb{P}^2$.

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