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# Introducing Systems via Laplace Transforms

## Cover Page Footnote (optional)

I'd like to thank the two anonymous reviewers for improving my article.

# *Introducing Systems via Laplace Transforms*

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**Abstract**: The purpose of this note is to show how to move from Laplace Transforms to a brief introduction to two dimensional systems of linear differential equations with only basic matrix algebra.

## 1 Introduction

The purpose of this short note is to share a strategy of quickly pivoting from Laplace Transfroms to Linear Systems, while bypassing almost all of the usual prerequisite linear algebra background. Of course, some matrix manipulation is needed for this approach. The students are mostly sophomore or junior engineering majors and most were comfortable with some elementary matrix manipulation as many (most?) had used things like Cramer's rule in practice. When I used this approach in in Fall 2022, I gave a brief review of matrix multiplication (restricting to 2 by 2 matrices), multiplication of a 2 by 1 vector with a 2 by 2 matrix, the determinant of a 2 by 2, a too brief reminder of the role of the identity matrix and the adjoint method of calculating an inverse (for 2 by 2 matrices with non-zero determinant). I found that the students gained an adequate computational grasp of the matrix material with one lesson and homework assignment.

I hasten to point out that I am a strong advocate for the traditional linear algebra approach when time permits. For example, such an approach allows for the introduction of the matrix exponential and the derivation of solutions via the matrix exponential method. But the approach outlined here permits an introduction to systems when there isn't enough time to develop the linear algebra needed for the traditional approach.

## 2 The Approach

Prior to getting to systems, students had studied second order linear differential equations with constant coefficients and were familiar with the concept of the characteristic polynomial; that is, the differential equation $ay'' + by' + cy = f$ has characteristic polynomial $am^2 + bm + c$ and solutions to the related characteristic equation $am^2 + bm + c = 0$ completely determine the solutions to the related homogeneous equation $ay''+by'+cy = 0$.

Later, we had studied how to solve such an equation using the Laplace transform. In particular, students were familiar with the characteristic polynomial appearing in the following context:

$$as^2Y(s) + bsY(s) + cY + a(y'(0) + sy(0)) + by(0) = F(s) \tag{2.1}$$

which leads to

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2+bs+c} + \frac{F(s)}{as^2+bs+c} \tag{2.2}$$

with the first term being the *input free* part of the solution and the second being the *state free* part. It was then pointed out that the denominator of both parts is merely the usual characteristic polynomial in $s$ and the roots of this polynomial determine what the solution the input free part.

I should point out that we had also studied

$$ay'' + by' + cy = \delta(t) \tag{2.3}$$

and noted that

$$Y = \frac{1}{as^2+bs+c} \tag{2.4}$$

is the Laplace transform of the solution to $ay''+by+c = 0, y(0) = 0, y'(0) = 1$ but covering the Dirac delta distribution is not strictly necessary to undertake this approach to systems.

It is with this background, we began our approach to 2 dimensional linear systems. Notation: here, $x$ and $y$ represent functions $x(t), y(t)$ of a variable $t$ and $x'(t)$ and $y'(t)$ refers to $\frac{dx}{dt}, \frac{dy}{dt}$ respectfully. I'll start with the homogeneous system $\vec{x}' = A\vec{x}$ where

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \vec{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{2.5}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2.6}$$

We'll use the usual convention that

$$X = \mathcal{L}(x) = \int_0^\infty x(t)e^{-st}dt, Y = \mathcal{L}(y) = \int_0^\infty y(t)e^{-st}dt \tag{2.7}$$

$$\vec{X} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{2.8}$$

where $x, y$ are from a suitable restricted class of functions (say, piecewise smooth, exponential order; derivatives, where defined, are of exponential order as well).

Because this approach is designed for those who lack a linear algebra background but have been exposed to solving 2 equations with 2 unknowns by matrix methods, I started by writing the two differential equations separately:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \tag{2.9}$$

Now take the Laplace transform of both equations to obtain:

$$\begin{aligned} sX - x(0) &= aX + bY \\ sY - y(0) &= cX + dY \end{aligned} \tag{2.10}$$

At this point, we are in the $s$ domain and can now switch to a matrix formulation of this system of two equations and two unknowns; this is something the students have done in their previous technical classes.

In vector-matrix form, the equations now are:

$$s\vec{X} - \vec{x}(0) = A\vec{X} \tag{2.11}$$

Now add $s\vec{X}$ to both sides. At this time I stress the importance of the relation: $s\vec{X} = sI\vec{X}$ (where $I$ is the 2 by 2 identity matrix). This leads to the equation

$$-\vec{x}(0) = (A - sI)\vec{X} \tag{2.12}$$

At this point, I pointed out that what is left is still a system of two equations and two unknowns, now in the $s$ domain.

Those familiar with the traditional linear algebra approach will notice that the $(A - sI)$ factor is what one uses to calculate eigenvectors and eigenvalues; in particular one looks for the values of $s$ that makes $A - sI$ a singular matrix, though I did not expect the students to know this. I did inform them of this connection in the final "a look ahead" remarks at the end of this material. Turning to the 2 by 2 adjoint method for inverse calculation (after noting that the matrix inverse exists for all but at most two different values of $s$):

$$-(A - sI)^{-1} = \frac{1}{det(A - sI)}\begin{bmatrix} s-d & b \\ c & s-a \end{bmatrix} = \frac{1}{(s-d)(s-a) - bc}\begin{bmatrix} s-d & b \\ c & s-a \end{bmatrix} \tag{2.13}$$

Those familiar with the linear algebra method will notice that the characteristic polynomial for $A$ is $(s-d)(s-a) - bc = 0$ and the roots of these determine the eigenvalues and the type of equilibrium the system has at the origin (e. g. node, spiral, center, etc.)

We now complete the process by multiplying $-(A - sI)^{-1}$ on both sides to obtain:

$$\vec{X} = \frac{1}{(s-d)(s-a)-bc} \begin{bmatrix} (s-d)x(0)+by(0) \\ cx(0)+(s-a)y(0) \end{bmatrix} = \begin{bmatrix} \frac{(s-d)x(0)+by(0)}{(s-d)(s-a)-bc} \\ \frac{cx(0)+(s-a)y(0)}{(s-d)(s-a)-bc} \end{bmatrix} \tag{2.14}$$

This is the Laplace transform of the solution, in vector form. Of course, one now has to take the inverse Laplace transform. That can be done, if desired, by writing this vector equation into a system of two equations, taking the inverse transform, and then rewriting in vector form. However, I found students do not object to taking the inverse Laplace transform of the vector equation, possibly because we did the "equation to vector/matrix equation" transition prior to taking the Laplace transform. Note that students in this class have had calculus 3 and many (most?) have had classes such as physics, statics and dynamics, and are therefore comfortable with vector valued functions and breaking down such functions into component vectors.

To drive home the relationship between the characteristic equations that the students have seen before and the characteristic polynomial in this setting, I found it useful to do the following example: given $ax''(t) + bx'(t) + cx(t) = 0, a \neq 0$ make the substitution $x = x, y = x' \longrightarrow x' = y, y' = x'' = -\frac{c}{a}x - \frac{b}{a}x' = -\frac{c}{a}x - \frac{b}{a}y$ which transforms this single second order differential equation into:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{2.15}$$

and here the characteristic polynomial is $s(s + \frac{b}{a}) + \frac{c}{a}$ which has the same roots as $as^2 + bs + c$.

We now return to the general case (Equation 2.14) and classify the types of solutions we can get, based on the type of roots that $(s-d)(s-a) - bc$ has:

- $(s-d)(s-a) - bc$ has real distinct roots: $r_1, r_2$ In this case, a partial fractions expansion has

$$\vec{X} = \begin{bmatrix} \frac{A}{s-r_1} + \frac{B}{s-r_2} \\ \frac{C}{s-r_1} + \frac{D}{s-r_2} \end{bmatrix} \longrightarrow \vec{x} = \vec{a}e^{r_1 t} + \vec{b}e^{r_2 t} \tag{2.16}$$

  where

$$\vec{a} = \begin{bmatrix} A \\ C \end{bmatrix} \text{and } \vec{b} = \begin{bmatrix} B \\ D \end{bmatrix} \tag{2.17}$$

  We can say a bit more: if both roots $r_1, r_2$ are negative, then note that $lim_{t\to\infty}||\vec{x}|| = 0$ and that $\vec{0}$ is the equilibrium solution which occurs when $x(0) = y(0) = 0$. The origin is said to be a *nodal sink*.

If both roots $r_1, r_2$ are positive, then $lim_{t \to -\infty} ||\vec{x}|| = 0$ and the origin is said to be a *nodal source.*

If one root is positive and the other is negative, then there are solutions that do not tend to the origin as $t \longrightarrow \pm\infty$. In this case, the origin is called a *saddle point.*

Of course, is possible that one root, say $r_1$, is zero. In this case $\vec{x} = \vec{a} + \vec{b}e^{r_2 t}$ and the is not an equilibrium point unless $\vec{a} = \vec{0}$.

In the traditional linear algebra approach: these are the cases that arise when $A$ has real, distinct eigenvalues.

- $(s - d)(s - a) - bc$ has a single, repeated real root $r$. In this case, a partial fractions expansion has

$$\vec{X} = \begin{bmatrix} \frac{A}{s-r} + \frac{B}{(s-r)^2} \\ \frac{C}{s-r} + \frac{D}{(s-r)^2} \end{bmatrix} \longrightarrow \vec{x} = \vec{a}e^{rt} + \vec{b}te^{rt} \tag{2.18}$$

where

$$\vec{a} = \begin{bmatrix} A \\ C \end{bmatrix} \text{and } \vec{b} = \begin{bmatrix} B \\ D \end{bmatrix} \tag{2.19}$$

Here, the origin is an equilibrium, and is a source if $r > 0$ because $lim_{t \to -\infty} ||\vec{x}|| = 0$ and a sink if $r < 0$ because $lim_{t \to \infty} ||\vec{x}|| = 0$. But this sort of equilibrium is called *non-generic.*

In the traditional linear algebra approach, this occurs when $A$ has an eigenvalue of algebraic multiplicity 2.

- $(s - d)(s - a) - bc$ has pure imaginary roots $\pm\beta i$. In this case, a partial fractions expansion has

$$\vec{X} = \begin{bmatrix} \frac{A}{s^2+\beta^2} + \frac{Bs}{s^2+\beta^2} \\ \frac{C}{s^2+\beta^2} + \frac{Ds}{s^2+\beta^2} \end{bmatrix} \longrightarrow \vec{x} = \vec{a}sin(\beta t) + \vec{b}cos(\beta t) \tag{2.20}$$

where

$$\vec{a} = \begin{bmatrix} \frac{A}{\beta} \\ \frac{C}{\beta} \end{bmatrix} \text{and } \vec{b} = \begin{bmatrix} B \\ D \end{bmatrix} \tag{2.21}$$

Note: $x(\vec{t})$ is periodic in $t$ and it can be shown that the solutions curves, except for the equilibrium $\vec{x} = \vec{0}$ solution, are ellipses or circles. In this case, the origin is called a *center.*

In the traditional linear algebra approach, this occurs when $A$ has pure imaginary eigenvalues.

- $(s-d)(s-a) - bc$ has complex conjugate roots with non-zero real part $r \pm \beta i$. In this case, a modified partial fractions expansion has

$$\vec{X} = \begin{bmatrix} \frac{A}{(s-r)^2+\beta^2} + \frac{B(s-r)}{(s-r)^2+\beta^2} \\ \frac{C}{(s-r)^2+\beta^2} + \frac{D(s-r)}{(s-r)^2+\beta^2} \end{bmatrix} \longrightarrow \vec{x} = \vec{a}e^{rt}sin(\beta t) + \vec{b}e^{rt}cos(\beta t) \tag{2.22}$$

where

$$\vec{a} = \begin{bmatrix} \frac{A}{\beta} \\ \frac{C}{\beta} \end{bmatrix} \text{and } \vec{b} = \begin{bmatrix} B \\ D \end{bmatrix} \tag{2.23}$$

If $r > 0$ then $lim_{t\to-\infty}||\vec{x}|| = 0$ But the solution curves form a spiral and the origin is called a *spiral source.* If $r < 0$ then $lim_{t\to\infty}||\vec{x}|| = 0$ and the origin is called a *spiral sink.*

In the traditional linear algebra approach, this occurs when $A$ has complex eigenvalues with non-zero real part.

This is the classification of 2 dimensional homogeneous linear systems with constant coefficients. Note that there is no difficulty with the “real double root" case that leads to long discussion in some texts (pp. 386-389 of [1]).

# References


[1] David Arnold John Polking, Albert Boggess. *Differential Equations.* Prentice Hall, 2’nd edition, 2006.

[2] Virginia W. Noonburg. *Differential Equations: From Calculus to Dynamical Systems.* MAA Press, 2’nd edition, 2019.