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Cover Page Footnote

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Raphael's *School of Athens*: A Theorem in a Painting?

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Abstract

Raphael's famous painting *The School of Athens* includes a geometer, presumably Euclid himself, demonstrating a construction to his fascinated students. But what theorem are they all studying? This article first introduces the painting, and describes Raphael's lifelong friendship with the eminent mathematician Paulus of Middelburg. It then presents several conjectured explanations, notably a theorem about a hexagram (Fichtner), or alternatively that the construction may be architecturally symbolic (Valtieri). The author finally offers his own "null hypothesis": that the scene does not show any actual mathematics, but simply the fascination, excitement, and joy of mathematicians at their work.

Raphael's famous painting *The School of Athens* shows among the great Greek thinkers at work a geometer, presumably Euclid himself, demonstrating a construction to his students (Figure 1, front right).[1] But exactly what theorem is he proving? In this article I describe the best known candidate. I begin with an overview of Raphael's career and mathematical expertise, then survey the painting as a whole. Next, I focus down on Euclid's slate–in the painting tilted so sharply that the exact nature of its figure has been a source of controversy–and test whether that figure can in fact arise from an equilateral hexagram (six-pointed star).

[1]Raphael's figures in this article are images obtained through Wikimedia, and are in public domain. For readers who would like to see hard print versions, [18] contains well over a hundred beautiful photographs taken by photographers Felice Bono and Pietro

Figure 1: Raphael's *The School of Athens*, 1510–11. Fresco. Dimensions ca. 6 m × 9 m. *Stanza della Segnatura*, Vatican Palace, Rome. Image source: Raphael [PD-art], via Wikimedia Commons: http://commons.wikimedia.org/wiki/File:Raphael_School_of_Athens.jpg

I next present the result, apparently first proved by Richard Fichtner in 1984 [3, pages 21–22 and 102], then popularized in J. L. Heilbron's *Geometry Civilized* in 1998 [7], that for such a figure the diagonal of a centrosymmetric rectangle has the same length as a certain chord.[2] (My proof is slightly simpler and more general than theirs.) Fichtner's result may still fall short, though, of being "Raphael's theorem," and I describe also some rival theories, notably that Euclid's figure may be architecturally symbolic rather than geometric.

My final, and perhaps most important, goal is simply to acquaint the reader with the beauty and power of Raphael's painting. In this scene one of the greatest artists who ever lived has pictured the fascination and excitement of learning mathematics. I think the work deserves display in every mathematics department in the world.

1. Raphael

Raphael (1483-1520) was the youngest of the three great artists–the others being Leonardo da Vinci (1452-1519) and Michelangelo (1475-1564)–whose work defines the Italian High Renaissance.[3] The son of a minor painter, and orphaned at an early age, Raphael was apprenticed to the eminent painter Perugino who was famed for the sweetness of his figures, and by age twenty-one had equaled if not surpassed his master.

For the next four years Raphael then worked in the city of Florence, where Leonardo and Michelangelo were leading and rival artists, and absorbed much from Leonardo [1, pages 18-19]. For instance, Raphael's portrait paintings gained solidity and depth from the example of Leonardo's *Mona Lisa*, though retaining his own characteristic clarity and directness in contrast to the latter's ineffable mysteriousness.

In 1508 Raphael moved to Rome, where the pope Julius II (who was simultaneously employing Michelangelo to paint the Sistine Chapel ceiling)

Zigrossi following the 1995-1996 restoration of the painting.

[2] The review [14] notes in particular the "convincing identification of the theorem about star hexagons that Euclid appears to be enunciating there."

[3] The literature on Raphael is, of course, enormous. A convenient concise account, with forty color plates, is [1]. See its bibliography, and [9], for entry to the scholarly literature.

hired him to paint wall scenes including the *School of Athens*. Raphael plainly learned much here too from Michelangelo's example, regarding the power, grandeur, and expressiveness of the human figure. A strong brooding portrait of Michelangelo himself appears prominently in the *School of Athens*, illustrating the assimilative power and genius for friendship that made Raphael one of the most beloved artists of his time.

In short order (once Michelangelo had finished the ceiling and returned to his preferred art of sculpture) Raphael was the leading painter of Rome, and head of a very large workshop. It diminished his accomplishment scarcely at all to have assistants paint in his figures, because his supreme achievement was not the brushwork but the design. It is this natural, effortless, rightness of his designs, for instance how a Madonna and child should look, or a dignified apostle in his robes—designs that survive today as a permanent part of our culture despite 500 years of repetition and degradation at the hands of lesser artists—that constitutes the achievement and genius of Raphael. It is but idle speculation and regret to wonder what so gifted and assimilative an artist would have accomplished had he lived into his late sixties or eighties like Leonardo da Vinci and Michelangelo, rather than dying as he did on his thirty-seventh birthday.

2. Raphael and Mathematics

Substantial mathematics was available to Raphael through both his art and his acquaintances. In art, "scientific" perspective, allowing an artist to show three-dimensional space as realistically as in the *School of Athens*, was a newly-won triumph of early Renaissance mathematics. One of the pioneers was Piero della Francesca (c. 1410-1492), famed for his mathematically calm, solid, paintings, who spent many of his later years at Urbino (the court of Raphael's birth), and dedicated his treatises on mathematics and perspective to its dukes Federigo and Guidobaldo.[4]

[4] See pages 17–23 of [13] for the historical development of perspective painting in the Renaissance, pages 228–263 (especially 229–230) for Piero at Urbino, and pages 279–281 for his treatises *De prospectiva pingendi*, on perspective, and *Trattato d'Abaco* and *Libellus de quinque corporibus regularibus*, on mathematics. See also [24], especially pages 50–53 and Figure 57, for a more detailed (and controversial; cf. [13, pages 20–21, 23, 245]) mathematical analysis of Piero's paintings.

Raphael was also greatly interested in architecture, which provides much of the grandeur in the *School of Athens*. A massive, architecturally realistic building dominates even his early 1504 painting *Marriage of the Virgin* [1, pages 54-55]. And Raphael did not merely draw pictures of buildings; by 1514 he himself was named the papal architect, a position again suggesting he had a substantial knowledge of practical mathematics.[5]

Raphael's "mathematically humanistic" birth court of Urbino had in residence one of the leading mathematicians of the day, Paulus of Middelburg (1445-1533).[6] Paulus, who remained at that court with few interruptions for twenty-eight years, was physician and astrologer to Federigo, Duke of Urbino; he advised and conversed with the Duke about mathematics every day, earning thereby high praise in a poem by Giovanni Santi (Raphael's father, who besides being a painter was the court poet).

Born in Middelburg in Zealand (present-day Holland), Paulus had studied philosophy, theology, and medicine at Leuven, while apparently learning his mathematics on his own. He taught theology and dialectic at Middelburg and was ordained; around 1479 he accepted an invitation from the Republic of Venice to teach astronomy at the University of Padua, and a year or so later joined the court at Urbino. Federigo awarded him the abbey of Castello Durante, and in 1494 Paulus was appointed bishop at Fossombrone, a bishopric attached to Urbino. After Federigo's death in 1482, Paulus remained on with the new Duke Guidobaldo, and administered the last rites to him in 1508.

In those pre-Copernican days, astrology was a respectable, complex, and sophisticated enterprise, and Paulus issued many annual prognostications with some notable successes (for instance, in 1524 predicting that the world would *not* be ending in a flood that year). His prognostications for 1480-1482 include mathematical challenge questions so advanced they went unanswered,

[5] A comprehensive survey of Raphael's architectural work is [4].

[6] The fullest modern biography of Paulus of Middelburg is [27]. Focusing on (and slightly amplifying) the mathematics from this biography are Struik's two short Italian articles [28, 29]. All three articles, bound in one volume, are available in [30]. [5] is a short biography. Paulus's possible influence on Raphael is discussed in [10]. Paulus's influence at the court of Urbino, and the interesting though unsupported speculation that the old astronomer in Raphael's *School of Athens* may be his portrait, is discussed in [21, pages 38–43].

on topics like properties of the sphere and cylinder, the value of π, and the quadrature of the parabola, showing a good knowledge of the work of Archimedes. A 1518 publication by Paulus concerning compound interest and the number of atoms in the universe introduced an early form of decimals to notate the results.

From 1484 on Paulus was intensely interested in calendar reform, summarizing the field in his major work *Paulina, de recte Paschae celebratione* (1513). Nowadays, century years like 1700, 1800, and 1900 not multiples of 400 are *not* leap years. But in the Middle Ages this correction was not known, and the calendar date had consequently drifted by ten days from its astronomical basis (e.g., solstices and equinoxes), leading to the embarrassment that the Catholic church could not correctly compute its own major holiday, Easter. By 1512, Paulus was in Rome, appointed by the pope to head the calendar reform committee of the fifth Lateran Council. (Unfortunately, uncertain astronomical data and political upheavals from the oncoming Reformation kept the Council from achieving its goal, and the corrected "Gregorian" calendar was instituted only in 1581.)

Raphael thus knew the eminent mathematician Paulus as a family friend since childhood. While there is no record of Paulus being in Rome around 1508 when Raphael was planning or beginning to paint the *School of Athens*, Urbino lies only about 125 miles from Rome. As a young artist in 1505, Raphael stated that he was painting in Urbino, Perugia, and Rome (as well as Siena, Florence, and Venice!), see [1, page 20]. And Paulus, as Bishop of Fossombrone, surely also had contacts with Rome prior to the Lateran Council. Thus Raphael probably could easily have consulted Paulus on any mathematical issues in his painting. Since Paulus was about sixty-four at the time, Maria Grazia Pernis has even made the suggestion, unsupported but intriguing and plausible, that the old astronomer standing behind Euclid holding a heavenly globe in the *School of Athens* might be Raphael's affectionate portrait of Paulus himself [21, pages 38–43].

3. *The School of Athens*

The School of Athens (Figure 1) is a fresco painting–a painting done in sections in the fresh plaster–on one of the four walls of the room, the *Stanza* *della Segnatura*, in the Vatican palace, chosen to house the papal library [9].[7] Raphael's painting made the library collection come alive by an imagined scene of the famous Greek philosophers in action—writing, thinking, and conversing; he copied the faces from classical statues if known, or else used his own contemporaries for models. Identifying the individual figures is an intricate, still-ongoing scholarly game; see Joost-Gaugier [9] for the state of the art. Such details hardly matter in comparison to Raphael's overall achievement here: a scene of nearly sixty figures, each individual and alive, that yet combine in a design that is harmonious, clear, and befitting the classic importance and dignity of its subject.

At the center of the picture are Plato (pointing up to his ideals in heaven) and Aristotle (gesturing down to the real world here on earth). The identities are unambiguous, because each man carries one of his own famous books, clearly labeled. But these are hardly needed: Raphael has brilliantly captured the essence of the two great men's philosophies, in instantly readable form, in their gestures alone.

The picture then divides neatly in half, the idealists with Plato on the left, the realists with Aristotle on the right. Busily arguing left of Plato is Socrates, with his circle of disciples (Phaedo, Crito, and in fancy armor maybe Xenophon or Alcibiades). Readers fond of English literature may recall the tribute to Socrates' gesture here in Laurence Sterne's novel *Tristram*

[7] In the technique of fresco the artist generally first prepares a full-size preliminary drawing called a *cartoon* [6]. A section of the wall suitable for one work session is then covered with smooth fresh plaster, and the outline of the design is transferred to it by pressing dry pigment through perforations in the cartoon or a copy. Following this outline the artist then paints the full design on the wet plaster using pigments suspended in water. The pigments become physically incorporated into the plaster when it dries, making the finished fresco stable for centuries. The cartoon for the lower, figural, half of Raphael's *School of Athens* is still extant; see [18, pages 15ff. and Figures 49 and 95] for comparison to the finished fresco. Oberhuber [19] likewise describes and compares the cartoon (pages 33–36), giving dozens of detail illustrations (Abb. 33–76, pages 79–110). Nesselrath [18] also provides a detailed analysis of the division and order of the fifty-nine one-day subsections (*giornate*) of the painting, so that, following along, one may sense the boldness and excitement of the brilliant young master Raphael painting here day by day in his first great commission.

Shandy [26].[8]

Leaning on the marble block at the lower left, wearing a crown of fig leaves and with a satisfied smirk on his pudgy face, is the arch-epicurean Epicurus. The face here is the portrait of the Pope's librarian Tommaso Inghirami, of whom Raphael also painted a fine oil portrait around 1510 ([1, Figure 38]; [9, color plate III]). Joost-Gaugier assembles an impressive argument that Inghirami was the brilliant Renaissance humanist whose learning underlay the design of the entire *Stanza della Segnatura*, including the *School of Athens* [9, pages 17-42].

At the front left of the picture, engrossed in writing in his book, is Pythagoras, whose young disciple holds a slate diagramming musical intervals and the mystic formula $1 + 2 + 3 + 4 = 10$.[9] Pythagoras's face is that

[8]The passage (Book IV, Chapter 7, page 343, of [26]) reads:

> My father instantly exchanged the attitude he was in, for that in which Socrates is so finely painted by Raffael in his school of Athens; which your connoisseurship knows is so exquisitely imagined, that even the particular manner of the reasoning of Socrates is expressed by it—for he holds the fore-finger of his left-hand between the fore-finger and the thumb of his right, and seems as if he was saying to the libertine he is reclaiming—"You grant me this—and this: and this, and this, I don't ask of you—they follow of themselves in course."

[9]Wittkower [33, pages 104–137, especially page 119 on this slate] discusses the role of Pythagorean harmonies in Renaissance art; see also Heninger [8, pages 91–104]. The Roman numerals for 6, 8, 9, 12, in the upper part of the slate schematize dividing the musical interval of an octave (*diapason*, ratio $6{:}12 = 1/2$) either into a fourth (*diatesseron*, ratio $6{:}8 = 3/4$) plus a fifth (*diapente*, ratio $8{:}12 = 2/3$), or (commutatively!) into a fifth $(6{:}9 = 2/3)$ plus a fourth $(9{:}12 = 3/4)$.

Heninger [8, pages 71–145 and 151–152] describes Pythagorean number mysticism, in particular the relation $1 + 2 + 3 + 4 = 10$, the "tetractys." Four was a very important number in Pythagoreanism; e.g., there are four seasons, four elements (fire, air, water, and earth, in the creation account in Plato's book *Timaeus* that he holds in Raphael's painting), at least four points are required to determine a geometric solid, and so on. [8, pages 154-155] gives a table from Agrippa itemizing thirty-two such tetrads; see also [9, pages 44-45]. In consequence 10, the sum of the first four numbers, was highly revered. The cantos in the *Divine Comedy* of Dante Alighieri (1265-1321), for example, number one hundred, "the square of ten, regarded in the thought of the time as a perfect number" [17]. Dante appears twice, crowned with laurel leaves, in the frescos of Raphael's *Stanza della Segnatura*: among the poets in the *Parnassus*, and with the saints in the *Disputa*.

of Donato Bramante, who was the chief architect in Rome and first planner of the great new St. Peter's basilica, and who is thought to have been a distant relative of Raphael. Seated on the stairs immersed in his writing at left center front is Heraclitus (the face is Michelangelo's).

On the realist side of the painting, sprawled on the steps in front showing his contempt for both material possessions and other people, is the cynic Diogenes. Standing isolated at the top far right wrapped up in his cloak and his thoughts is the lawgiver Solon.

At the far right front two painters enter; the left one, staring out at the viewer asking "Do you like my work?", is Raphael's face, thus cleverly signing his painting. These men presumably represent great classical Greek painters like Apelles, whose name and fame alone remain now that their works have been lost. By including painters in the scene Raphael here also pays tribute to Leonardo da Vinci, who campaigned to elevate the status of painting from a mere manual craft to a liberal art comparable to poetry and music.

Beside the painters, holding globes of the earth and sky, are the two astronomer / geographers Ptolemy (wearing a crown) and Strabo (some critics, incongruously, say it is Zoroaster). The latter, as noted above, might be a portrait of Paulus of Middelburg. And finally, next to them at the front, is Euclid (Bramante's face again) demonstrating a geometric construction to his fascinated disciples.

Before focusing in on Euclid's group in the next section, it seems worthwhile to note some of the poetic license Raphael has allowed himself in his picture. The historical Pythagoras (fl. c. 530 B.C.), Plato (428/427–348/347 B.C.), and Epicurus (341–270 B.C.) lived hundreds of years apart; Socrates (c. 470–399 B.C.) was about forty years older than his student Plato; the Greeks would have carried their works on scrolls rather than books; and the arched, vaulted, and domed buildings are not Greek, but rather reflect Bramante's designs for St. Peter's. The whole tableau, furthermore, cannot represent a real scene—or otherwise, fifteen seconds later, when Plato and Aristotle, deep in conversation, come forward and start down the steps, Aristotle might well trip over Diogenes and go flying to the right (realist) side, and Plato trip over Heraclitus with his marble block incongruously built into the stairs, and go flying to the left (idealist) side! Against such quibbles one can only reply that Raphael is here painting not a historical gathering, but a Renaissance library collection (of books, housed in Renaissance architecture), from which each author does forever come toward each reader in the eternal moment of his prime.

One other point that might trouble a twenty-first century viewer is that the *School of Athens* contains no women. What a pity that Raphael did not include Hypatia, or Aspasia, or the wise woman Diotima of Mantineia who was Socrates' teacher. In extenuation though, Raphael paints the great Greek woman poet Sappho, prominently positioned and labeled, on the adjacent wall fresco *Parnassus* celebrating the arts and music. She appears there in company with Apollo, the nine Muses, great poets like Homer, Dante, Virgil, Petrarch, and Boccaccio, and another self-portrait of Raphael.

Raphael's decorations for the *Stanza della Segnatura* include fresco paintings on all four walls, and intricate ceiling paintings and a gorgeously inlaid marble floor besides. The wall opposite the *School of Athens* is a fresco of comparable size and complexity, *La Disputa*, illustrating the Christian religion, from the Eucharist celebrated by popes and saints on earth, up to apostles and biblical characters on the clouds in heaven, and then God the Father, Son, and Holy Ghost.[10] It has been persuasively argued [32] that the *Stanza* frescoes here form a unified program: the great Greek thinkers in the *School* bring forward truth to the limit achievable by intellect alone, truth which then finds its completion and fulfillment on the opposite wall in faith. The totality impressively demonstrates Raphael's ability to bring complex concepts to pictured life.

4. Euclid's Group

At the front of the *School of Athens* are Euclid and his four students, absorbed in their mathematical demonstration (Figure 2). The youngest student, apparently seeing the proof for the first time, kneels at the left with his head closest to his teacher's, all rapt attention. The two older students standing hovering further back seem to know the argument already, but that only increases their delighted anticipation, "Now watch what happens!" Meanwhile the youth kneeling at the center seems caught in a flash

[10] For "Pythagorean" harmonies in the *Stanza* see [9, pages 164–172].

of insight, an "Aha! moment".[11] Teachers of mathematics work their whole lives hoping to find students like those Raphael has given Euclid here.

What theorem are they all studying? Their slate shows a six-pointed star with two parallel lines and a diagonal at its center. The figure, it has been noted, is neither a Euclidean nor an Archimedean problem.[12] Interestingly, Raphael drew the star only after the fresco plaster had been painted and dried [18, page 17]; his cartoon leaves the slate blank ([18, Figure 95]; [3, Figure 12, page 21]). This two-step procedure could have permitted more deliberation and geometric precision in drawing the star than possible under time pressure from the wet plaster.[13] On the other hand, though, it might also explain why Euclid's painted compass might not synchronize perfectly with the star.

The slate is tilted so sharply from view that it has been a controversial point whether the star is symmetrical or not. Joost-Gaugier believes no, while Bellori (1695) and Reale (1997) think yes [9, page 82]. Calculations to undo the tilt depend critically on the assumption that Euclid's slate is truly rectangular. The slate seems small enough that its own perspective recession should be negligible. But its remote right-hand edge, contrary to perspective, actually measures about 12% longer than the nearer parallel left

[11] One might recall John Keats' "On First Looking into Chapman's Homer" (1816): "Then felt I like some watcher of the skies / When a new planet swims into his ken; / Or like stout Cortez when with eagle eyes / He stared at the Pacific—and all his men / Looked at each other with a wild surmise—"

[12] See [19, page 57]. [12, pages 203–206] remarks that, by adding some lines and forming solids of revolution, the figure could recall Archimedes' famous theorem about volumes of the cone, hemisphere, and cylinder standing in 1:2:3 proportion; but he thinks this connection unlikely.

[13] [6, pages 36–37] describes the fresco artist's tension:

> Herein lies the great fascination of fresco. Instead of the dead and sometimes discouragingly inanimate surface of a canvas, you are struggling with a living thing. Your plaster, born so to speak in the morning, must have lived its life before night. Every instant it has its requirements and at moments when time grows short, its desperate needs. The tendency to work by the watch should make it the most modern of mediums. But for the artist it has the greatest quality of all–it forces him into a state of passionate absorption in which he must be utterly oblivious of everything but his work.

Figure 2: Detail from Raphael's *The School of Athens*, 1510–11, group around Euclid. Image source: Raphael [PD-art], via Wikimedia Commons: http://commons.wikimedia.org/wiki/File:Raphael_School_of_Athens.jpg

edge, making the whole recalculation somewhat moot.[14] Cézanne tabletops have, of course, taught us that a figure may be far more solid or expressive when it does not rigidly follow the rules of perspective. Renaissance artists were also more flexible or subtle in such matters than one might give them credit. It has been noted, for instance, that while Raphael drew the *School of Athens* architecture from single-point perspective, the figures frequently have their own individual centers of projection. Likewise, the globes that Ptolemy and Strabo hold are drawn as perfect circles, simply because they look better that way, even though strict perspective would require them to be ellipses; (see [22, pages 116–123] and [11, pages 112–116]).

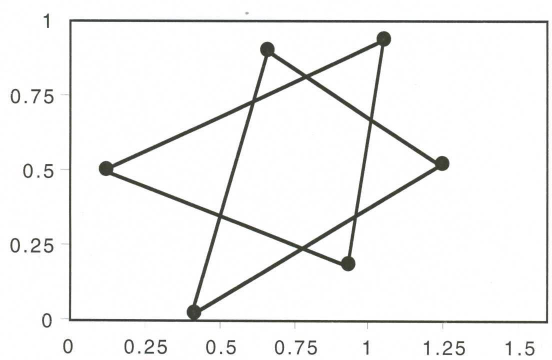

Figure 3: Planar reconstruction of the star: Projective transformation with $c = 1.60$.

Figure 3 is my "near optimal" attempt to reverse the diagram's tilt. Beginning with a large clear picture of the slate [12, page 19], I measured the coordinates of the four corners of the slate (taking the lower left one as the origin) and of the six vertices of the hexagram. I then transformed these coordinates by the projective geometric methods used in reconstruction of aerial photographs [20, pages 121–122 and 341–346] so that the four corners would map to a rectangle with corners $(0, 0)$, $(0, 1)$, $(c, 0)$, and $(c, 1)$, where c is a constant to be determined. The hexagram vertices similarly transformed to points with x-coordinates proportional to c. [15]

[14] Also, while four of the five visible shorter edges of the frame are nearly parallel, the top front one closest to the viewer is considerably skewed (cf. [12, page 23]).

[15] Measured coordinates (in centimeters) of the slate corners were $A(0, 0)$,

A perfect hexagram would have all three of its long corner-to-corner diameter distances the same length. Accordingly, I sought to optimize c by maximizing the ratio of the shortest to the longest of these three diameters. But this is not a strong condition, as the ratio is nearly constant for $1.07 \leq c \leq 1.60$ (it is 7% greater at 1.07 than at 1.60). Figure 3 shows the transformed hexagram, choosing the slightly suboptimal value $c = 1.60$ to give the slate a more plausible rectangular shape. The three diameters are 0.77, 1.11, and 1.11, and the six edges of the figure's two "equilateral" triangles have average length 0.87 ± 0.12 S.D. (range $0.70 - 1.02$). With their coefficient of variation thus $0.12/0.87 = 14\%$ (25% similarly determined from the reconstruction of Mazzola *et al.* discussed below [16, page 16], the hexagram is somewhat irregular. But of course geometric proofs do not depend on the precision of their diagrams. It is the reader's choice whether or not, given the artistic license of a fresco painter, and a mathematician's license to reason perfectly from an imperfect diagram, to call this figure symmetric.

$B(-2.17, 0.48)$, $C(2.84, 1.28)$, $D(5.27, 0.75)$, and of the hexagram $P_1(-0.76, 0.30)$, $P_2(1.17, 0.19)$, $P_3(2.50, 0.51)$, $P_4(2.72, 0.84)$, $P_5(1.14, 0.95)$, $P_6(-0.05, 0.74)$. These were immersed in projective space by adjoining third coordinates of 1. The fundamental theorem of projective geometry guarantees the existence of a unique projective transformation mapping any four distinct ordered points of one plane (no three collinear) to any four distinct ordered points of another plane (no three collinear). Accordingly, one constructs a transformation T_1 from the standard plane to the slate mapping $I_x[1, 0, 0]$, $I_y[0, 1, 0]$, $U[1, 1, 1]$, $O[0, 0, 1]$ respectively to A, B, C, D, and similarly T_2 mapping I_x, I_y, U, O respectively to $P[0, 0, 1]$, $Q[0, 1, 1]$, $R[c, 1, 1]$, $S[c, 0, 1]$. The desired reconstruction transformation from the slate to the rectangle is then $T = T_1^{-1} \circ T_2$ with matrix:

$$\begin{pmatrix} 0.59c & -0.83 & 0.06 \\ 2.68c & 5.85 & 0.27 \\ 0 & 0 & 4.60 \end{pmatrix}.$$

The hexagram corner points, transformed by multiplication by T, then viewed in Euclidean space by choosing a representative with third coordinate 1, are $P_1T(0.0767c, 0.51)$, $P_2T(0.254c, 0.03)$, $P_3T(0.581c, 0.19)$, $P_4T(0.773c, 0.53)$, $P_5T(0.654c, 0.94)$, and $P_6T(0.407c, 0.91)$. The three hexagram diameters are then (P_1T to P_4T) which we label D_{14}, which measures $0.696c$, (P_2T to P_5T) which we label D_{25}, which measures $(0.160c^2 + 0.828)^{1/2}$, and (P_3T to P_6T) which we label D_{36}, which measures $(0.030c^2 + 0.518)^{1/2}$. Since the ratio of smallest to largest diameter is D_{14}/D_{25} (rising with c) for $0 \leq c \leq 1.07$, D_{36}/D_{25} (falling slowly) for $1.07 \leq c \leq 1.60$, and D_{36}/D_{14} (falling) for $1.60 \leq c$, it is maximal at $c = 1.07$ and near-maximal for $1.07 \leq c \leq 1.60$.

5. A Proof

Figure 4 shows the theorem that Fichtner suggests Euclid is proving. Given a regular six-pointed star crossed by an arbitrary centrally-symmetric

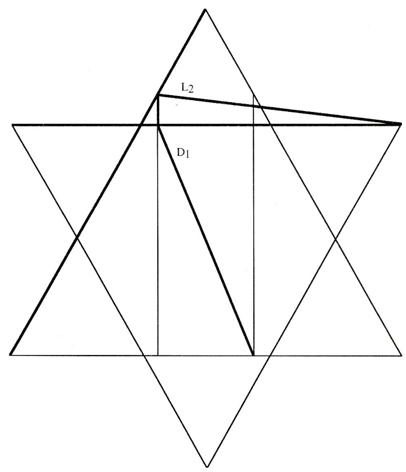

Figure 4: Fichtner's theorem: $D_1 = L_2$.

pair of parallel lines, the diagonal D_1 has equal length as the chord L_2 (which is not drawn on Euclid's slate).

The proof, as Fichtner shows, follows easily from the Pythagorean theorem; Figure 5 below is my simplified and generalized form, adding point names for the analysis. Aside from a central vertical line BD, this has just the heavier lines from Figure 4, including the upper horizontal $APDC$, the left diagonal AQB inclined at angle θ ($60°$ in Figure 4), and $L_2 = QC$. The diagonal D_1 has been replaced with its half length L_1, reflected upwards (PB). Thus the theorem of Figure 4 to be proved is $2L_1 = L_2$, generalized in Figure 5 to $(k/a)L_1 = L_2$.

Letting line segment names also denote their lengths, assign $AD = a^2$, $AP = x$, $AB = ak$, and $AC = k^2$. This generalizes $a = 1$, $k = 2$ in the six-pointed star of Figure 4; thus $\theta = \text{arcsec}(k/a)$. Consider the three triangles ABC, ADB, and APQ. Each contains an angle θ; the latter two also each contain a right angle, hence match in all three angles, and are thus similar. Since the corresponding sides around angle θ in triangles ABC and ADB are

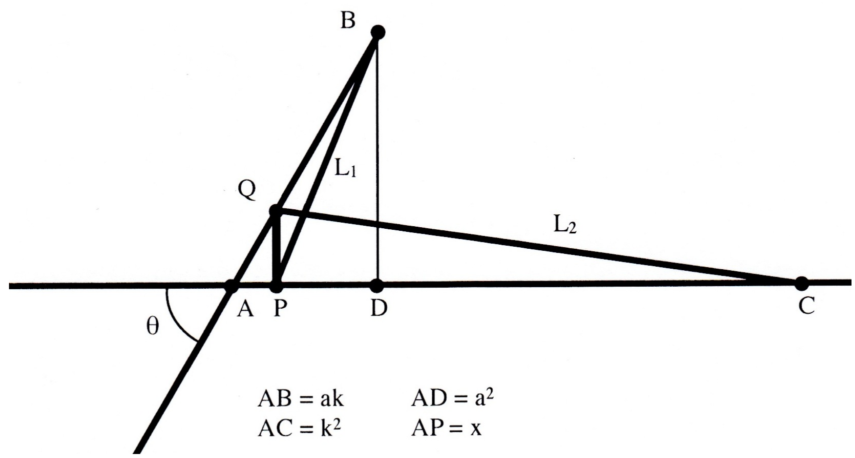

Figure 5: Simplified and generalized proof of Fichtner's theorem: $(k/a)L_1 = L_2$.

proportional ($AB/AD = ak/a^2 = k/a = k^2/(ak) = AC/AB$), Euclid VI.6 implies that triangle ABC is also similar to ADB (and APQ). In particular, $\angle ABC$ is also a right angle. Also, consequently, $BC/DB = AB/AD = k/a$, so $BC = (k/a)DB$.

Since triangle ABC is similar to APQ, $AQ/AP = AC/AB = k/a$, or $AQ = (k/a)AP$. It then follows that $AQ/AB = (k/a)AP/(ak) = AP/a^2 = AP/AD$. The similarity of ABC to ADB therefore maps Q to P, hence L_2 to L_1, implying Fichtner's theorem $L_2 = (k/a)L_1$.

Euclid VI.8 shows that the perpendicular from the right angle to the base splits triangle ABC to two similar subtriangles BDC and ADB. Fichtner's theorem then reflects a very simple relationship: Right triangle ADB is similar to ABC, the lengths of their sides being in the ratio $AB/AD = ak/a^2 = k/a$; point P on AD then corresponds to point Q on AB, and the line $L_1 = PB$ corresponds to the line $L_2 = QC$, again in the ratio k/a. The relation $k/a = \sec\theta$, however, does limit "nice" geometric values to the original $k/a = 2, \theta = 60°$ (or possibly $k/a = \sqrt{2}$ at $\theta = 45°$).

One can also prove these results, much less perspicuously, by using multiple applications of the Pythagorean theorem to write L_1 and L_2 in terms of a, k, and x. First $(BD)^2 = (ak)^2 - (a^2)^2$ and $(PD)^2 = (a^2 - x)^2$, so

$$L_1^2 = (BD)^2 + (PD)^2 = a^2k^2 - 2a^2x + x^2$$

By the similarity of ABD and AQP, $QP = (x/a^2)BD$, and $PC = k^2 - x$, so

$$\begin{aligned} L_2^2 = (QP)^2 + (PC)^2 &= (x/a^2)^2[a^2k^2 - a^4] + (k^2 - x)^2 \\ &= (x^2k^2/a^2) - x^2 + k^4 - 2xk^2 + x^2 \\ &= (k/a)^2[x^2 + a^2k^2 - 2xa^2] = (k/a)^2 L_1^2. \end{aligned}$$

The computation $(AB)^2 + (BC)^2 = (AB)^2 + (BD)^2 + (DC)^2 = 2a^2k^2 - a^4 + (k^2 - a^2)^2 = k^4 = (AC)^2$ implies by the law of cosines that $\angle ABC$ is a right angle. The three triangles ABC, BDC, and ADB, each containing a right angle, and each pair also sharing one acute angle, are consequently all similar, with sides in length ratios of $a : \sqrt{k^2 - a^2} : k$. Each individual triangle also has its sides in the same length ratios. One deduces $\angle BCQ = \angle DBP$ (so $\angle ACQ = \angle ABP$) from the tangents, $\tan \angle BCQ = QB/BC$ and $\tan \angle DBP = PD/DB$. These are equal because $PD/QB = AD/AB = \cos\theta = DB/BC$. This then implies $L_2/(BC) = \sec \angle BCQ = \sec \angle DBP = L_1/(DB)$, whence $L_2 = (k/a)L_1$, an alternate proof of the generalized Fichtner theorem.

Figure 4 actually shows just one of the four possible cases. Point P on the line AC could lie to the left of A, between A and D (Figure 5), between D and C, or to the right of C. Similar results and proofs hold in each case, for any a and k satisfying $0 < a < k$.

6. Alternatives

Fichtner's theorem (Figure 4) is clearly true: Figure 5 gives a proof and an algebraic generalization. But there are still several causes for doubt whether this result is "Raphael's theorem." It requires that the star, contrary to appearances (Figures 2 and 3), be symmetric. The critical chord L_2 doesn't occur at all in Euclid's diagram (Figure 2). It remains unexplained what Euclid is doing in the picture with his compass. Is he drawing a circle? Bisecting some line or angle? Constructing a perpendicular? And now my Figure 5 proof has uncovered the fact that the result doesn't even have much to do with the full hexagram.

These circumstances perhaps justify glancing at some alternative theories–each with its own weaknesses–about Raphael's figure. Joost-Gaugier (who does not mention Fichtner) suggests a different mathematical explanation–a construction rather than a theorem. From weaker hypotheses–two similar triangles–this draws a weaker conclusion: one can form a hexagram.[16] I write here in more detail:

Proposition. *Any two similar triangles, with a scale factor between* $1:1$ *and* $2:1$*, can be superimposed in a hexagram as on Euclid's slate.*

Proof. Let triangle ABC have angles α, β, and γ. It is easy to see, first of all, that drawing the lines joining the midpoints of its sides divides ABC into four congruent subtriangles: By Euclid VI.2 [2] the inner lines each parallel outer ones, hence (I.29) cut off equal angles, proving the three corner subtriangles have angles α, β, and γ. Then $\alpha+\beta+\gamma=\pi$ (I.32) implies the same in the central subtriangle. The four subtriangles, being similar to ABC and sharing sides, are therefore congruent; since the scale factor is $2:1$, this makes the figure a "degenerate" hexagram.

An α-β-γ triangle $A'B'C'$ similar to ABC with a scale factor between $1:1$ and $2:1$ is at least as big as these subtriangles, and so can be placed upon ABC in "hexagram" positions containing those three edge midpoints of ABC. Let $B'C'$ so placed be parallel to BC (see Figure 6); one checks that $A'C'$ is then parallel to AC: Setting notation as in Figure 6, we see that the assumption that BC is parallel to $B'C'$ implies that $\angle AF'E' = \angle ACB = \gamma = \angle B'C'A'$, proving in turn that AC is parallel to $A'C'$. Similarly, $A'B'$ is parallel to AB because $\angle AE'B' = \pi-\alpha-\gamma=\beta=\angle C'B'A'$. Thus $A'B'C'$ may be superimposed on ABC to form a hexagram with corresponding sides parallel.

[16] Joost-Gaugier in [9, pages 82 and 206] uses the construction as evidence that the geometer in Raphael's painting is Euclid, and not, as had been alternatively suggested, Archimedes:

> The form appears to be a demonstration of parallelism, the single biggest contribution made by Euclid. [On similarity see Euclid, *The Elements*, VI.22.I, also I.12 and XI.6-9].

(The second Euclid reference seems mistaken.) But the argument from parallelism does not seem historically fully justified: While parallel lines are important for proving major Euclidean theorems (e.g., that the angles of a triangle add up to 180°), the great present-day importance of Euclid's fifth (parallel) postulate stems from its independence, so that denying it permits the construction of non-Euclidean geometries. But those steps—parallel postulate equivalents, Clavius, 1574; its denial, Saccheri, 1733; full non-Euclidean geometries, Gauss, Lobachevski, Bolyai, 1800s—mostly happened centuries after Raphael [25, pages 80–109].

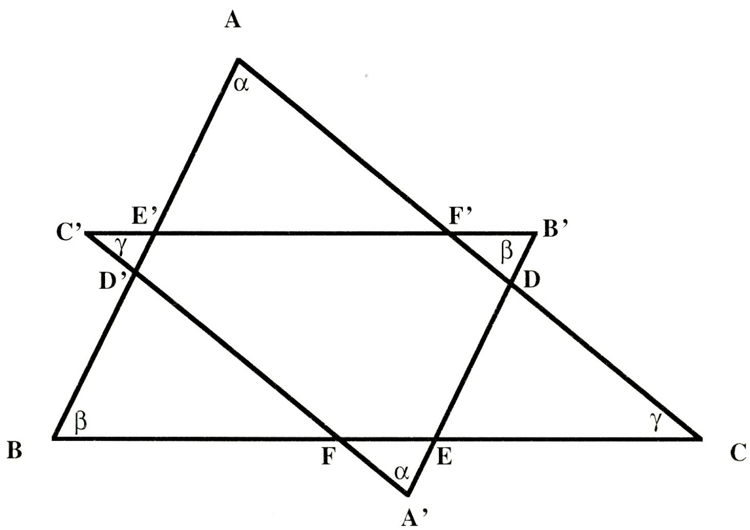

Figure 6: Aligning similar triangles to form a hexagram with corresponding sides parallel.

The construction on Euclid's slate is completed by constructing two line segments perpendicular to BC (hence parallel to each other) running up to $B'C'$, and drawing a diagonal in the resulting quadrilateral. □

A quite different line of explanation, offered by Valtieri [31], regards Euclid's hexagram as symbolic or schematic for the perspective and architecture in Raphael's entire painting.[17] Such hidden self-referentiality, while unusual to present-day scientific rationalism, could comport well with some aspects of Renaissance thought, such as the neoplatonic mysticism of Marsilio Ficino then current in Florence.[18] Taking (in Figure 1) the vanishing point between Plato and Aristotle as vertex V, one forms a downward-pointing isosceles triangle CDV, where C and D are, respectively, the tips of the left and right horizontal wall moldings at the base of the back arch of the vault above the two men. A hexagram then results by adjoining a congruent upward-pointing triangle ZLM, taking Z as the midpoint between A and B, the tips of the

[17]The arches in Raphael's painting are all half circles, and in her Figure 1, Valtieri [31] also works extensively to relate those circles and their centers to other salient points of the architecture or the hexagram.

[18]For entry to Marsilio Ficino and neoplatonic mysticism see [8], [9, especially pages 164–172], and [21].

wall moldings at the base of the front arch of the vault. This hexagram fits in a $5:8$ ($= 0.625$) rectangle. Lauenstein in [12] gives extensive arguments and constructions for a variant with golden ratio ≈ 0.618.[19] In a critique, Fichtner [3, pages 13–21] notes among weaknesses of Valtieri's theory that Z, L, and M are not themselves architecturally significant points.

A large study led by Mazzola ([16], conclusions also briefly summarized in [15, pages 186–187 and 201–202]) aimed to reconstruct by a computer the original three-dimensional geometry and ground plan of the scene. This reached the novel conclusion that, in the reconstruction, people themselves form the hexagram. One of its triangles (actually intersecting many people) is formed by the three people wearing yellow (one near Plato; the "Indian philosopher" in front of Pythagoras; and Ptolemy). The other triangle contains the three people with one hand on hip (one in armor facing Socrates; one to the right of Aristotle; and Euclid's youngest student kneeling in front), modulo some further adjustments made by reference to the people wearing white.

But, unfortunately, the reconstruction is not well-defined, since distances between people on the upper level (and Plato's step-length) come out excessively large. With so many figures in the scene (many already in vertical or horizontal lines), one may wonder about getting a hexagram simply by chance. Moreover, as already noted in [16, pages 31-32, 54], the reconstruction is not as satisfactory for figures at the back. Furthermore, the architecture itself is visibly distorted in the reconstruction (Figure 2 on page 67) compared to the original painting (Figure 1 on page 66): For instance, the floor pattern quadrilateral adjacent to the front step is shallower (measured relative to the step height) by about 20%, while the arches at the back are about 30% wider (both absolutely, and relative to the height of the Plato figure). Such problems in turn cast doubt on the far more subtle and complex analysis of the arrangement of the human figures. And, lastly, since in the painting itself the hexagram of people is so tilted it is rather hard even to visualize, this interesting and creative theory must posit that the

[19]Wittkower [33, pages 104–129, 150–154] insists at considerable length, though, that while irrational "geometrical" proportions like the golden ratio were important in the Middle Ages, the basis of Renaissance proportion was "arithmetical" whole number ratios: ". . . we may explode the old and continuously repeated myth of the predominant role of the Golden Section in the age of the Renaissance" (page 152).

manifest beauty and order of the painting stem from the viewer's own act of reconstructing the ground plan subconsciously.

I offer, finally, as a theory of my own, a "null hypothesis" (in both literal and statistical senses): that Euclid's figure may have no real mathematical meaning. The scene is a beautiful image of scholarship: the mathematicians of Athens would have been engrossed in some such geometric diagram. But, just as a Raphael "Madonna and Child" is an image of maternal tenderness, not an instructional diagram on how to hold one's baby, it might simply be misplaced ingenuity to seek an actual theorem on Euclid's slate.[20]

Raphael's *School of Athens* well deserves its fame as an image of an ideal world of intellectual life. Though the overall plan is clear, many details and identifications still remain undetermined [9]. Might Euclid's slate hold a new theorem? The present article has described some candidates; possibly a better one is still waiting to be found. In any case, the scene itself remains a magnificent image of an ideal life in mathematics.

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[20] In *The Madonna of the Chair*, "perhaps the most famous of all his Madonnas," for example, Raphael has compromised on clarity in Mary's pose to highlight the intimacy of the figures filling this circular (*tondo*) painting: "Mary's torso, like that of Christ, is set in profile in order to appear less crowded, while her legs are thrust up to provide a comfortable place for the child (although it is extremely difficult actually to reconstruct her pose)" [1, pages 112-113]. But the painting is simply too beautiful for that difficulty to matter!

Plato considered this issue of artistic vs. literal truth in his dialogue *Ion* [23]. Ion is a famous *rhapsode*—a performer / reciter of Homer—who is able to give an overwhelmingly powerful and "realistic" performance of, say, a great general. But as Socrates establishes by judicious questioning, in the event of a real war, one needs not Ion, but a real general!

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