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At The Gate Of Discovery

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Synopsis

This is the story of how a mathematical problem was discovered. Although it was never solved, it gave great joy to the discoverer.

Some time back I had the pleasure of following my girlfriend to her job. This is the gate she entered.

I guess many things can come to mind when you face a gate like this, some even mathematical. I noticed, being also a teacher of information technology, that 128 is a power of 2. Or, to put it a way my students might understand, you get it from 1 by constantly doubling.

Another thing struck me. All the digits, the 1, 2 and 8, are also powers of 2. "Hmmm, that's nice!", I thought.

One might say that I had found a pearl. But, as with pearls, its value would increase if it was unique. So, leaving the gate slowly, looking down at the ground to concentrate while walking, I started with 1 and doubled, searching for more pearls, hoping to find none.

My mind managed to double to $4,096$, but it then had problems going on without messing things up. I therefore waited till I had pencil and paper at hand, and remembered something I had read about Lipman Bers, a famous mathematician:

> As a committed social democrat, he was somewhat self-conscious of having devoted his life to the mathematics from which he derived so much pleasure. He said, "Here I am, a grownup man, worrying about whether the limit set of a Kleinian group has positive measure and willing to invest a great deal of effort to find the answer." [1, page 18]

Why would I care if 128 was the only power of 2 with all its digits also powers of 2? I could think of only two reasons. I had discovered the problem and that made me proud. I couldn't leave my discovery without following its path. The second reason was that I thought my discovery would impress my girlfriend. Especially if I could come up with a solution. I was obviously hoping for more than Lipman Bers:

> Mathematicians work for the grudging admiration of a few close friends [1, page 17].

A third reason came a bit later. I had no idea if the problem was hard or trivial. How creative could I be trying to crack it? And, if I couldn't crack it, how far could I go? These were some of my thoughts, along with a pedagogical one, what makes a student study problems just for the heck of it? How do we create an environment to stimulate it?

Armed with paper I was able to double fifty times without finding another pearl. Suddenly, I had a brilliant idea! I would concentrate on the last two digits and throw away anything to the left of them. That gave me the sequence

$$01\ 02\ 04\ 08\ 16\ 32\ 64\ 28\ 56\ \ldots$$

My hope was that the numbers would soon repeat. I would then throw away all numbers that had other digits than 1, 2, 4 and 8, and study the remaining numbers.

It repeated rather soon:

01 02 04 08 16 32 64 28 56 12 24 48 96 92 84 68 36 72 44 88 76 52 04

It left me with these survivors:

28(7) 12(9) 24(10) 48(11) 84(14) 44(18) 88(19)

An explanation may be in order. 2^7 ends in 28 and is therefore a candidate. The sequence repeats itself after 20 multiplications by 2, so $2^{20} \times 2^7$ is also a candidate. In general, 2^{20k+7} is a candidate where $k = 0, 1, 2, \cdots$. There are seven of these candidate groups.

(A pedagogical side note added a year later than the writing of the original piece: I had to read the preceding paragraphs a few times till I understood what I had tried to convey. This is in my mind not a bad thing. If one takes the idea of students constructing their mathematical knowledge seriously, vagueness in exposition is to be applauded. Textbook writers often go to the other extreme. They drown their students in words and explanations, robbing the reader of any opportunity of actual thinking. György Pólya expressed it in his ninth commandment for teachers: "...let them find out for themselves as much as feasible" [3].)

My hope was that if I cast my net a bit wider, using the last three digits, the number of groups would go down. My dream was that by looking at enough digits only one candidate group would remain.

Was the dream justified? I thought. The thought was interrupted by another thought. "A dream is a dream. A starting point. To ask if the dream is justifiable takes time and energy away from finding out where the dream leads you!" I had these two voices fighting in my head. I followed the second voice as I was more concerned with the joy a mathematical journey might provide, and not if it led to a safe harbor.

With three digits I found the sequence to repeat after 100 steps and, sad to say, with as many as 13 candidate groups:

128(7) 144(18) 288(19) 824(30) 184(34) 888(39) 248(51) 488(59) 424(70)
848(71) 112(89) 224(90) 448(91)

Possible candidates were now on the form 2^{100k+7}, etc.

After this setback I changed my approach, remembering Piet Hein's grook: "Problems worthy of attack, prove their worth by hitting back" [2].

Every power of 2 of any interest is divisible by 4. That means that the last two digits have to be a multiple of 4. In other words a candidate has to end with

00, 04, 08, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 88, 92 or 96

Throwing out unworthy ones, I was left with seven:

12, 24, 28, 44, 48, 84 and 88

Every power of interest is also divisible by 8, which means that the last three digits have to be divisible by 8. I listed them all and threw out the ones that used other digits than 1, 2, 4 and 8. There were 13:

112, 128, 144, 184, 224, 248, 288, 424, 448, 488, 824, 848 and 888

Again, I was disappointed. The number of candidates increased instead of decreasing. It struck me that I got seven and thirteen groups, just as above. Hardly a coincidence I thought, as I rushed to attack at a different front.

I had said to myself earlier that I was in it for the mathematical journey, not for the end result. But this was not fun! The waves were too high and too frequent. I was not demanding smooth sailing, but some success would have been appreciated. I repeated Piet Hein's grook over and over, before I pressed on.

2^{50} equals $1,125,899,906,842,624$. When this number is multiplied by 2, what happens? Let's take each digit in turn.

When we start with the digit 0, we get 0 or 1, depending on if there was a carry or not. Assume the chance for a carry is 50%. 0 is not a power of 2, but 1 is, so the chance for getting a power of 2 with the digit 0 is 50%.

When we do the same calculation with the other digits we find this:

0	1	2	3	4	5	6	7	8	9
0/1	2/3	4/5	6/7	8/9	0/1	2/3	4/5	6/7	8/9
50%	50%	50%	0%	50%	50%	50%	50%	0%	50%

Since 2^{50} has 16 digits the chance for 2^{51} to be a pearl is therefore about $(50\%)^{16} = 0.000015$. (We ignore the fact that rightmost digit will not have a carry to worry about and that the result may have a seventeenth digit at the left.) As the number of digits increases as we continue to multiply by 2, the likelihood for another pearl to appear will drop towards zero.

I wrote a computer program in `Visual Basic` that doubled 1 ten thousand times and counted the number of "ugly" digits for each number. $2^{1,000}$ has $3,011$ digits and $1,785$ of them are ugly, i.e., not 1, 2, 4 or 8. By the way, no pearls, other than 128, were found. The likelihood that $2^{1,001}$ is a pearl is in the range of $0.5^{1,785}$ which happens every time you throw heads $1,785$ times in a row. Please email me when this happens!

After this playing around, I looked 128 up in Penguin's *Dictionary of Interesting Numbers* [4]. The book states the problem, so I was not the first to discover it, but it says that no one knows if 128 is the only pearl.

A computer will not create numbers with one thousand digits if you don't find a creative way to manipulate them. Here then is $2^{10,000}$ as a final attempt to get some admiration for my futile attempt at the gate of discovery:

19950631168807583848837421626835850838234968318861924548520089498529438
83022194663191996168403619459789933112942320912427155649134941378111759
37859320963239578557300467937945267652465512660598955205500869181933115
42508608460618104685509074866089624888090489894838009253941633257850621
56830947390255691238806522509664387444104675987162698545322286853816169
43157756296407628368807607322285350916414761839563814589694638994108409
60536267821064621427333394036525565649530603142680234969400335934316651
45929777327966577560617258203140799419817960737824568376228003730288548
72519008344645814546505579296014148339216157345881392570953797691192778
00826957735674444123062018757836325502728323789270710373802866393031428
13324140162419567169057406141965434232463880124885614730520743199225961
17962501309928602417083408076059323201612684922884962558413128440615367
38951487114256315111089745514203313820202931640957596464756010405845841
56607204496286701651506192063100418642227590867090057460641785695191145
60550682512504060075198422618980592371180544447880729063952425483392219
82707404473162376760846613033778706039803413197133493654622700563169937
45550824178097281098329131440357187752476850985727693792643322159939987
68866608083688378380276432827751722736575727447841122943897338108616074
23253291974813120197604178281965697475898164531258434135959862784130128
18540628347664908869052104758088261582396198577012240704433058307586903
93196046034049731565832086721059133009037528234155397453943977152574552
90510212310947321610753474825740775273986348298498340756937955646638621
87456949927901657210370136443313581721431179139822298384584733444027096

4182851005072927748364550578634501100852987812389473928699540834346158807043959118985815145779177143619698728131459483783202081474982171858011389071228250905826817436220577475921417653715687725614904582904992461028630081535583308130101987675856234343538955409175623400844887526162643568648833519463720377293240094456246923254350400678027273837755376406726898636241037491410966718557050759098100246789880178271925953381282421954028302759408448955014676668389697996886241636313376393903373455801407636741877711055384225739499110186468219696581651485130494222369947714763069155468217682876200362777257723781365331611196811280792669481887201298643660768551639860534602297871557517947385246369446923087894265948217008051120322365496288169035739121368338393591756418733850510970271613915439590991598154654417336311656936031122249937969999226781732358023111862644575299135758175008199839236284615249881088960232244362173771618086357015468484058622329792853875623486556440536962622018963571028812361567512543338303270029097668650568557157505516727518899194129711337690149916181315171544007728650573189557450920330185304847113818315407324053319038462084036421763703911550639789000742853672196280903477974533320468368795868580237952218629120080742819551317948157624448298518461509704888027274721574688131594750409732115080498190455803416826949787141316063210686391511681774304792596709376

Postscript: Please don't tell my girlfriend about WolframAlpha. It finds all the digits of $2^{10,000}$ in a snap.

Second postscript: My girlfriend is now my wife.

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