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Report on Work at Brunel University

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Dear Professor White

Thank you for sending me a copy of Newsletters 1 and 2. I enclose with this letter a copy of the report of work that I undertook at Brunel University, London, whilst on sabbatical leave from the City University, London.

I think that the report deserves a wider audience than it has thus far had, and I would like you to consider it for publication within the auspices of the Humanistic Mathematics Network.

I look forward to any further material from you,

Yours sincerely

Dr Anthony Briginshaw

1. Some observations on teaching undergraduate mathematics.

Introduction.

This Report is an essay on current perspectives in mathematics education. It is largely concerned with undergraduate mathematics education, and focuses particularly on how mathematics is taught to first and second year engineering undergraduates. It is clear that, as always, such a narrow focus cannot fail to have ramifications in neighbouring areas, and I shall at least mention the following:

(i) the history and philosophy of mathematics,

(ii) the treatment of mathematics in schools,

(iii) the great success of mathematics as the servant of physics and engineering,

(iv) misconceptions of what mathematics is and what it seeks to achieve,

(v) behavioural phenomena in the lecture meeting, and

(vi) methods of assessment of lecturer performance.

The essay is a distillation from a period of intensive enquiry undertaken throughout the 1986/7 session, a session which was spent as study leave in the Department of Mathematics at Brunel University.

The net result of the enquiry is that many more questions were raised than I could answer, so that in this Report I re-iterate those questions and outline whatever I have achieved in the way of an answer.

2. Teaching mathematics: the mathematical language.

Teachers of mathematics, at all levels, are concerned with the transmission of learning. They use, as the medium of communication, a hybrid language, part English, part Mathematics . In one sense the mathematical language is not a natural language, in that it has not grown as a readily accessible medium for communication and for general discourse. In another sense the language is a natural one, for it has grown with and alongside physics and engineering and has come to describe scientific and engineering usage and procedure with unrivalled success. Indeed, we might say that it is a medium for discourse within certain communities, the various communities of science and engineering. However, the mathematical language is succinct and economical, and, indeed, it has been honed to economical perfection through centuries of selection (Cajori, 1928) . Thus, unlike English, or any similar natural language of general written or spoken discourse it has very low "redundancy". Yet high redundancy seems to be a quality which indicates that the language is a good vehicle for the transmission of information. Thus the mathematical language would seem *a priori* to be a poor vehicle for communication to learners! In spite of that, I suggest that, with care, this difficulty is subject to automatic alleviation in the currency of a course of mathematical instruction. Briefly, there is a continual need for conceptual scaffolding throughout the course, and this scaffolding is the means by which a quite small number of major concepts and techniques is acquired. When facility and confidence with these concepts and techniques is achieved, by accretion, and with a large degree of overkill and redundancy, the objectives of the course are attained. In a University mathematics course, some familiarity with the axiomatic method and with logical implication is also acquired, but not as much as we think; to most students, especially those for which mathematics is not a major subject, facility and confidence are quite enough .

3. Language and communication.

Claud Shannon is regarded as the founder of the modern theory of communication. Shannon's original work (1948) gave rise to the collaboration of Shannon and Weaver (1949) and that was followed by the publication of Bell's book (1953). Since then, the field has expanded vigorously and that expansion has been matched by prolific publication. We need only touch here on the very basic idea of the theory, and, in particular, introduce the concept of entropy, a measure of the quantity of information which is transmitted per symbol of language.

Colloquially, we envisage a distinction between "language" and "code". "Language" (literally "tongue") is perhaps a naturally occurring spoken or written system of discourse. On the other hand, "code" is a restricted, curtailed, economical or displaced version of such a language. More technically, the word "code" has also come to indicate strings of symbols with which we address a machine. However, if we look upon a code as a device used for protecting security, then a good code is one which transmits a message to insiders and which is impenetrable to outsiders. On the other hand, a poor code is one which transmits a message to all and is not impenetrable. Thus English is a good language but a poor code (for English speakers) and Mathematics is a poor language but a good code (for English speakers!).

4. Self information of an event; entropy of a set of events.

Shannon defined the self information, $I(E)$, of an event E as a function of the probability p of its occurrence.

It is $I(E) = \log\left(\frac{1}{p}\right)$ which defines the self information of E, or, in other words, the quantity of information which is transmitted when E

occurs. The convention is that base 2 logarithms are used for this measure and that $I(E)$ is measured in "bits". Notice that the quantity of information transmitted by E increases with its "surprise" value.

For a set of events $\{E_i\}$ with respective probabilities of occurrence $\{p_i\}$ we define the <u>entropy</u>, $H\{E_i\}$ by

$$H\{E_i\} = \sum p_i \log\left(\frac{1}{p_i}\right) .$$

We can look upon entropy as the average information transmitted per event for a sequence of events.

5. <u>Redundancy of a code or language</u>.

If a code has n symbols with actual probabilities of occurrence $\{p_i\}$, then we may calculate two numbers, the actual entropy and the equiprobable entropy (that which would occur if all symbols were of equiprobable occurrence).

The quantity $R = 1 - \dfrac{\text{actual entropy}}{\text{equiprobable entropy}}$ (see Usher, 1984)

is then referred to as the <u>redundancy</u> of the code. Notice that the equiprobable entropy is the one which corresponds to the case of <u>maximal</u> information transfer per symbol. Thus, if the actual entropy is low, then the code is a relatively inefficient information transfer mechanism. It is, correspondingly, a good vehicle for communication, for, even if the receiver misreads or ignores some of the symbols, the receiver may still get the import of the message.

Both Shannon and Weaver (1949) and Bell (1953) estimated the redundancy of written English at around 80%, basing their calculations on a <u>conditional entropy</u>, assuming intersymbol influence.Thus, a conditional entropy

$$H(j/i) = -\sum_i \sum_j p(i,j) \log p(j/i)$$

takes the place of the cruder measure H of paragraph 4 above. Here, in particular, $H(j/i)$ assumes influence only in adjacent symbols, and $p(i,j)$, $p(j/i)$ are, respectively, joint and conditional probabilities; there may be more complicated intersymbol influence. With this means of estimation, I suspect a very low value for redundancy of the mathematical language, which implies that it is a good code, but a poor vehicle for communication (except to those who are already "fluent").

6. Tactical surprise in undergraduate mathematics courses.

If we accept that the mathematical language has low redundancy then that fact must affect teaching style. Thus in an undergraduate mathematics course, especially one where mathematics is not the major study, so that motivation may be low, the lecturer must seek to exploit whatever tricks and strategies he can command to achieve two ends. Firstly, to optimize receiver tuning (student attention) for a given information flow, a matter which is not wholly affected by the nature of the subject matter. Secondly, to maximize the information flow. Thus, in order to catch and hold student attention those tricks may encompass changes of pace and vocal tone and appropriate use of humour and anecdote and may stretch to limited histrionics. To maximize the information flow, on the other hand, tactical surprise might be used in one of two ways, either at the motivational stage of a new section or within the development of a set of theorems by a judicious selection of pattern and proof. It might occur by appeal to any facility for pattern recognition that students may already have acquired whether it be a recognition of analogue in structure or in usage. It might occur in a particularly neat or succinct set of implications which justify a technique or procedure.

Whatever attempts are made at tactical surprise, however, it is difficult to achieve, and, worse still, it is often post facto. The limits of tactical surprise, indeed, must be set alongside the realization that mathematics is mostly formal, methodical and economical, in other words, it is intrinsically not surprising.

This is allied, consciously or unconsciously, to the attention profile which is associated with a target student group, which roughly indicates a variable attention span, with a lack of receptivity, both at 17-20 minutes and at 34-37 minutes. These occur willynilly in a fifty minute lecture presentation. There are, as I have said, two types of surprise. Motivational surprise, by which we hope to capture attention by pointing out the utility of a prospective technique before outlining the details, or by pointing out the structural beauty of the mathematics in its own right; that is, respectively, motivation either at the modelling level or at the aesthetic level of cognition. It is facile to assume that the former should be reserved for undergraduate engineers and the latter for students of pure mathematics, I have not found such restrictions to be an effective aid to good communication for either group. The second type of surprise is dramatic surprise and it is a phenomenon, or collection of phenomena that is very difficult to describe adequately; let us say that it has something to do with the lecturer as performer, and his or her ability to engage in limited histrionics or to inject appropriate humour or expertly to pace the flow of information to suit the target group.

7. Strategic redundancy in undergraduate mathematics courses.

If the mathematical language is such a fundamentally poor medium for communication, yet the whole point of it is to communicate that which cannot be communicated in English, how is it that generations of scientists and engineers come through unscathed? In my view, because of

the strategic redundancy inherent in the undergraduate mathematics course. Much of the time and effort spent in, for example, a first year methods course for engineering students is scaffolding. It is essential for building the edifice of mathematical knowledge with which a student proceeds to year two, but it is redundant at the end of year one. On my estimation, perhaps 90% of the work and effort expended in a first year course is scaffolding. It is that part by which the course is motivated topic by topic and by which each topic development is given plausibility and each resulting technique is given facility. It is essential for the successful execution of the course, but it is redundant when the course is over.

Concerning the bedrock knowledge of mathematics that a student actually needs, that is, those basic ideas of definition, notation and technique which occur over and over again in physics and engineering, we might conclude that if they could be implanted in memory banks and logic circuits direct, it could be done in one tenth of the normal course length.

It is in the provision of scaffolding that the intrinsic information transfer redundancy occurs for the mathematics lecture course.

8. Mastery of the mathematical language.

In speaking of English as a language we recognise such terms as essay, article, novel, poem, description, reporting, etc., and we distinguish between creative writing and criticism; equally we may refer to written material as being classic, modern, mainstream or avant garde. To a quite marked degree these concepts have their analogues in mathematics insofar as we treat mathematics as a language in its own right. The analogies go some way toward explaining some of the snobberies that arise between

teachers of undergraduates must come from the ranks of the "research mathematicians". Unfortunately, this is as misleading as saying that every poet can be a reporter, essayist or playwright, in other words, it is an empty assertion, usually made without any attempt to analyse what is going on, whether in research or in teaching. Another snobbery attempts to devalue "teaching" vis-a-vis "research", this is rather naive, too, for in order to validate research the researcher must communicate it, and the process of communication of new ideas even to peers is still "teaching". Naturally "new writing" is for aficianados i.e. academic mathematicians, who often are themselves, "writers". There are certainly "critics" of "new writing", necessarily themselves peers, and often playing the role of assessors, before publication. Too often there is the danger that, in the exciting world of avant garde "mathematical writing", critics and aficianados alike will too readily place new writing before mathematically immature minds. That danger is perennial in mathematics, as in music, or art, or literature.

As I have already noted in paragraph 5, the mathematical language is likely to have a low redundancy. Does this imply that it is inherently a poor vehicle for SOCIAL (i.e. educational or classroom) communication? That, of course, does not deny the fact that once the language is mastered, and fluency is gained, the language can be used with confidence as the outstandingly good vehicle for SCIENTIFIC communication that it is.

9. Taxonomies.

A taxonomy is a categorisation of a discipline by way of various traits and qualities, those of Bloom and Piaget, for example, are decided by "depth of cognisance". Following Jolliffe and Ponsford (1986), I propose the following taxonomy as being the most sharply focused as a means of cognitive analysis regarding mathematics learning at post 16 ages. In

particular, it is valid for the first and second years of an undergraduate mathematics or mathematics-related degree course. One would expect that incursions would be made into levels 0,1,2, in varying measures at post 16 ages and the implication of the classification is self evidently that the sequence 0,1,2,3 indicates an increasing depth of cognisance.

LEVEL 0	NUMBER	SPATIAL AWARENESS	FUNCTION
LEVEL 1 (Skill acquisition)	LANGUAGE	NOTATION	TECHNIQUE
LEVEL 2a (Abstraction)	ABSTRACTION	FORMALITY	STRUCTURE
LEVEL 2b (Logical implication)	PROOF	RIGOUR	AXIOMATICS
LEVEL 2c (Mathematical utility)	MODELLING	MATHEMATICS/PHYSICS INTERFACE	
LEVEL 2d (Analogy)	SPECIALIZATION	GENERALIZATION	ANALOGY
LEVEL 3 (Invention)	ADVANCED INTUITION	INVENTION	

We might ask which of these qualities is to be regarded as being of primary importance in the communication of mathematics as a service discipline, and particularly in the communication of mathematics to engineers, whether they are already qualified or in training. My use of the word "training" of course provokes an immediate objection from any self respecting educator. An honours degree programme in engineering is not just a training course, it should stretch the intellect and enable the aspiring engineer both to understand current practice and to venture beyond it. We need to encourage people to THINK and provide them with the intellectual equipment firstly to be able to respond to extreme and anomolous behaviour, and finally to be capable of engineering invention, however modest, on their own behalf.

Of course mathematics lecturers should, ideally, themselves be well aware of the interplay between mathematics, physics and engineering, not only as it

stands now, but as it has developed over the centuries. That is asking a great deal of the mathematics lecturer, with the implication that he or she should have some knowledge of both the history and philosophy of mathematics, as well as an overview of many areas of current mathematical practice, including some insights into how mathematics fulfils its modelling role. For an overview of what mathematics is, see Temple (1981), Howson (1972), and Roman (1975). For a discussion of the extent to which such polymath qualities are feasible in the modern world, at the same time being compatible with the demands put on the research mathematician, see Kline (1977, 1980).

10. Pedagogical perspectives.

Clearly the traditional approach to the teaching of undergraduate mathematics will increasingly become subject to modification as the result of the availability of computer aids. Structured learning is already available as a means of self-paced instruction under the general title of "Keller Plan". There are several modifications of this plan in current use, and the scheme has proved to be an effective means of communicating service mathematics material when it is allied to back-up facilities in the form of video taped lectures and typed hand-outs. It has certain drawbacks in the proliferation of new problem sheets which are required to ensure that real progress takes place in successive years. It also suffers, in my view, from the fact that it seems not to take into account a general behavioural trait, namely that learning is not, at all levels of the taxonomy (or even at any of them) an instantaneous process. At what level of skills is the Keller Plan working, for example? Are other aspects of knowledge and understanding accumulating with various time delays with the Keller Plan as with the traditional lecture method?

It has become standard practice in many institutions to give lectures in mathematics to quite large groups and provide tutorial back-up in smaller groups. That tutorial back-up usually consists of working of problem sheets together with individual attention. It would seem that micro-computer networks have an increasingly large role to play in this area, for there seems to be no reason why they should not be useful in providing illustrations especially using the graphics facilities that they possess, or in the demonstration of model solutions. What they cannot do, in my view, is to teach concept, nor can they motivate, nor be capable of the surprise, humour or timing that is the hallmark of the expert human communicator.

We are reluctant to accept that knowledge transfer is a multiply fuzzy process, that is that the knowledge is inevitably fuzzy in conception prior to transmission, then it is fuzzily transmitted and finally fuzzily received. In the course of a lecture programme we witness several learning phenomena at work. Depending on the cognitive level there are several indices of delay; there are also cross-disciplinary effects of great sublety by which just the flavour of one lecturer's approach in one discipline will pay dividends in some unforseen way elsewhere.

In addition, there is enormous redundancy and overkill built into the traditional lecture programme method of tuition, and that possibly accounts for what seems to be its continuing success, or at least acceptability.

11. History of reforms in mathematics teaching.

The year 1871 saw one of the great causes célèbres of mathematical education in full flow. The dispute arose between the ranks of the teachers of school mathematics on the one hand and the scions of the University of Cambridge on the other. In particular, on the school side, were members of the College of

Preceptors, and, particularly, a formation called the Association for the Improvement of Geometrical Teaching; on the University side such famous names as Todhunter, Kelland and Dodgson.

What, then, was the argument about? It was about the balance between the formality and rigour that University mathematicians demand in their version of the mathematical disciplines, and the hands-on experience and plausibility that school teachers judge to be essential in their role.

It is a dispute that recurs constantly at all levels of mathematics teaching, from primary school course to undergraduate course, and it is one whose intensity was to peak again in the era of the "modern mathematics" controversy.

In 1871, as in 1969, University mathematicians somewhat arrogantly assumed that they, and only they, knew what mathematics was really about, and they tried to use their position as masters of the mathematical high ground to deny others a say in the educational process. The Universities did not have the argument all their own way, however, opposition was strong, not least at the various thriving technical schools, both in London, and elsewhere. That opposition would almost certainly have been reinforced by opinion in the nation's premier military academies, for outstanding mathematics teachers and text book writers were to be found at the Royal Military Academy (Woolwich), the Royal Naval College (Greenwich), and the Royal Military College (Sandhurst). These institutions were at the leading edge when it came to instruction in the techniques and procedures of applicable mathematics. So, too, in London, were the Cowper Street School, off City Road, and the Borough Road school, and, outside the capital, the various northern Mechanical Institutes, including the Manchester Institute.

In 1969, the titles of "reformer" and "traditionalist" were interchanged; the proposers of reform were in the Universities, its opponents, such as they were, in the schools. The issue, however, was the same, how to reconcile the conflicting needs of rigorous mathematics and vocational mathematics.

"Mathematics is formal, logical.and wonderful" would say the reformers; unable to deny this, prospective critics of the reform were muted in their opposition. Indeed, the educational difficulty that "modern mathematics" provokes is only manifest when the reformers bring forth their next implication: "therefore we must teach it formally and logically in our schools and universities."

To say that "mathematics is formal, logical and wonderful" is far less than half the story. It is also intuitive, inventive and pragmatic. Its acquisition is cumulative, but that accumulation is selective . Though it is successful, its success is not capable of being complete . Though it might have been thought once to be independent of experience, it seems now more likely to be quasi-empirical. That prospect will have the greatest possible effect on how it is taught, at all levels.

12. Quasi-empiricism: what is it?

Within the last twenty years, especially since the work of Lakatos, (1967, 1976, 1978) attention has been focused on mathematics AS IT IS not as axiomatists have conceived that IT OUGHT TO BE. In Tymoczko (1986), there is an extended critique of the platonist, logicist, formalist and intuitionist positions, and the philosophical basis of mathematics is re-examined. The extreme convolutions to which Hilbert, Russell and others were reduced in their search for the perfect axiom system, which were shown to be vain by

Godel, WOULD HAVE MADE mathematics dry, automatic and computer generatable. Fortunately, it is not dry, automatic and computer generatable. But what is it, then?

Mathematics is as it is practised. It develops now as mathematicians have always experienced its development, by generalization, specialization and analogy, by cross-fertilization with physics, by conjecture and refutation, by abstraction, and by intuition tempered with rigour.

These facts about mathematical reality are not observable without some appreciation of how mathematicians worked in the past, and this is nowhere better illustrated than by George Polya in his article in Tymoczko (1986) concerning $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Euler guessed at a result, but not wildly; he made just the right guess, the one IN THE IDIOM OF MATHEMATICS. But the guess, the conjecture, was not motivated from set theoretic foundations, nor with formal logic. It was the way that all mathematicians, at their most creative, work.

Tymoczko (1986) describes the field in which mathematicians work as "mathematics without foundations". That is to say, the search for firm foundations at a very deep level is vain. That did not stop Euler "doing mathematics". Should it stop us? Of course not.

My own conception of mathematical activity is as work on a mosaic of knowledge, never to be completed, and resting on somewhat spongey "foundations", but <u>fascinating</u>; and not only that, <u>useful</u>. Thus mathematics cannot be divorced from experience, whether foundationist philosophers like it or not. Lakatos defines "quasi-empiricism" as follows:

> the axioms or basic principles of the theory are the results of bold speculation that have survived the test of severe criticism, (Tymoczko, 1986).

Thus a theory stands UNTIL IT IS FALSIFIED.

Perhaps it is a little vague to talk of having "survived the test of severe criticism", but that seems to be the best that we can do.

We may perhaps look upon "severe criticism" as the operation of "rigour" in the following way. Bold speculation leads, and if it misleads into error by way of a proferred contradiction or counter example, then one of a succession of logical filters will indicate how and why. If the speculation is still unexplained, and if it escapes all existing filters then we have to rethink the position and design a new logical filter. The collection of logical filters is called "rigour".

13. <u>High ground v high peaks</u>.

What is the best preparation for a university lecturer in mathematics? It is tempting to assert that clearly, those with a good overview of what mathematics is will be the best teachers. It is then a short step to asserting that there are certainly such persons around, and that they are clearly research mathematicians, those who have the deepest knowledge of certain aspects of mathematical theory and practice. I dispute this last assertion. In two ways. Communication of some fund of knowledge requires two, at least, ingredients, a fund of knowledge to communicate and the ability to communicate it. The fund of knowledge is available to two categories of academics, those who occupy the high peaks in the mathematical landscape - the researchers, and those who occupy the high ground, what I shall call scholars. Constituents of neither group are guaranteed an innate ability to communicate well the material at their command, but the scholars are more likely to want to do it.

14. Assessment of teaching skills.

Following the foregoing discussions of what exactly is the practice of mathematics and what are the various theoretical problems of communicating it to others, there remains the problem of judging what is good communication practice, and who achieves it and how? That matter is the subject of Briginshaw and Newby (1987), which is submitted for publication elsewhere. Briefly, that paper observes that mathematics is particularly difficult to teach. The reasons for this are two-fold, and I have attempted to explain them more fully in this Report. Firstly, the mathematical language has low redundancy, and is not an easy vehicle for good communication; secondly, mathematics operates at so many levels of cognition that the multiply fuzzy ways in which mathematical knowledge is focused, transmitted and received are an order of magnitude more complex than those for a non-scientific discipline. Briginshaw and Newby (1987) conclude that a major input to the assessment of the teaching of mathematics to undergraduates must be by way of anonymous student questionnaire. They have therefore attempted to design a model questionnaire which is both skill specific (i.e. it judges ability to communicate) and subject specific (it focuses specifically on mathematics).

15. Acknowledgements

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