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Infinitely Many Rotationally Symmetric Solutions to a Class of Semilinear Laplace-Beltrami Equations on the Unit Sphere

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Abstract

In this thesis, I show that the semilinear Laplace-Beltrami Equation has infinitely many solutions on the unit sphere which are symmetric with respect to rotations around some axis. This equation corresponds to a singular ordinary differential equation, which we solve using energy analysis. We obtain a Pohozaev-type identity to prove that the energy is continuously increasing with the initial condition and then use phase plane analysis to prove the existence of infinitely many solutions.

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Chapter 1

Introduction

The Laplace operator, or Laplacian, arises in many applications, including mathematical biology, physics, and economics. It is defined in Euclidean space as the divergence of the gradient of a function. In cartesian coordinates, the Laplacian is the sum of the second partial derivatives taken with respect to each coordinate. For a function f in $\mathbb{R}^n$, the Laplacian is thus

$$\Delta f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

The Laplace-Beltrami operator is a generalization of the Laplacian to Riemannian manifolds. Spheres of any dimension are Riemannian manifolds, and we can therefore apply the Laplace-Beltrami operator to functions over spheres (Guillemin and Pollack (2010)).

For a differentiable function f defined on a Riemannian manifold M, the Laplace-Beltrami operator is defined as the Laplacian of the extension of f that is constant on normal directions to M (see Jost (2011)). We will later derive the Laplace-Beltrami operator for rotationally symmetric functions over the n-dimensional unit sphere, and prove that

$$\Delta_s u(x_1, x_2, \ldots, x_{n-1}, z) = (1 - z^2)u'' + (1 - n)zu' \tag{1.1}$$

where z is the position along the z-axis embedded in $\mathbb{R}^n$.

The goal of this paper is to give sufficient conditions for the semilinear Laplace-Beltrami equation

$$\Delta_s u + (1 - |z|)f(u) = 0 \tag{1.2}$$

to have infinitely many solutions. Our main result is the following theorem.

Theorem 1.1. *If $n \geq 3$ and $f(u) = u^p$ for $1 < p < \frac{n+5}{n-3}$, then the boundary value problem*

$$\begin{cases} (1-z^2)u'' + (1-n)zu' + (1-|z|)f(u) = 0, & z \in [-1,1] \\ u'(1) = u'(-1) = 0 \end{cases} \tag{1.3}$$

has infinitely many solutions.

We begin the report by deriving the semilinear Laplace-Beltrami equation for functions on the unit sphere which are rotationally symmetric about an axis. We define a boundary value problem and then study the corresponding initial value problem with initial conditions $u(-1) = d$ and $u'(-1) = 0$.

Using an argument involving the contraction mapping principle, we prove that a solution exists to the in the interval $[-1, -1+\epsilon]$ for some $0 < \epsilon < 1$ which depends continuously on the initial condition d. Then using an energy argument, we prove that the function u is bounded and therefore that we can extend the solution to be defined over the interval $[-1, 0]$. Later this solution can be reflected about the equator $z = 0$ so that it is defined over the entire interval $[-1, 1]$.

Next we construct a Pohozaev identity to prove that the energy over some interval extending from -1 increases without limit as initial condition d increases. This allows us to use phase plane analysis, in which we prove that the argument function increases without bound. From this we are able to prove that there are infinitely many solutions to the boundary value problem associated with (1.2).

Chapter 2

The Laplace-Beltrami Operator

In this chapter, we derive the Laplace-Beltrami Equation for rotationally symmetric solutions on the unit sphere. If z is the distance from the origin on the z-axis, then the boundary value problem is given in (2.1):

$$\begin{cases} (1-z^2)u'' + (1-n)zu' + (1-|z|)f(u) = 0 \\ u'(1) = u'(-1) = 0 \end{cases} \tag{2.1}$$

where $f(u) = u^p$, $1 < p < \frac{n+5}{n-3}$, $n \geq 3$. Notice that $f(u)$ is locally Lipschitz continuous and superlinear, that is,

$$\lim_{|u| \to \infty} f(u) = \infty.$$

2.1 Derivation in Cartesian Coordinates

First we derive the Laplace-Beltrami operator on the unit sphere S^{n-1} by projecting all points $x = (x_1, x_2, \ldots, x_{n-1}, z)$ in $\mathbb{R}^n$ to the sphere and applying the operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{\partial^2}{\partial z^2}$. Additionally, we consider only rotationally symmetric solutions, that is, solutions that depend only on the variable z. So we consider functions of the form

$$u(x_1, x_2, \ldots, x_{n-1}, z) = v\left(\frac{z}{||x||}\right)$$

where scaling by the norm of x, $||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + z^2}$, projects each point onto the unit sphere.

We begin by calculating the partial derivatives. For $i = 1, 2, 3, \ldots, n-1$,

$$\frac{\partial u}{\partial x_i} = v'\left(\frac{z}{||x||}\right)\left(-\frac{x_i z}{||x||^3}\right),$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i^2} &= v'\left(\frac{z}{\|x\|}\right)\left[-\frac{z}{\|x\|^3} + \frac{3x_i^2 z}{\|x\|^5}\right] + v''\left(\frac{z}{\|x\|}\right)\left(\frac{x_i^2 z^2}{\|x\|^6}\right)\\ &= v'(z)\left(3x_i^2 z - z\right) + v''(z)\left(x_i^2 z^2\right),\end{aligned}$$

where in the last step we have set $\|x\| = 1$, since all points lie on the unit sphere. We also have

$$\frac{\partial u}{\partial z} = v'\left(\frac{z}{\|x\|}\right)\left[\frac{1}{\|x\|} - \frac{z^2}{\|x\|^3}\right]$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial z^2} &= v'\left(\frac{z}{\|x\|}\right)\left[\frac{3z^3}{\|x\|^5} - \frac{3z}{\|x\|^3}\right] + v''\left(\frac{z}{\|x\|}\right)\left[\frac{1}{\|x\|} - \frac{z^2}{\|x\|^3}\right]^2\\ &= v'(z)(-3z)(1-z^2) + v''(z)(1-z^2)^2.\end{aligned}$$

Putting this all together in the Laplace-Beltrami operator, we have

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_{n-1}^2} + \frac{\partial^2 u}{\partial z^2}\\ &= v'(z)\left[3x_1^2 z + 3x_2^2 z + \cdots + 3x_{n-1}^2 z - (n-1)z\right] + v'(z)\left[-3z(1-z^2)\right]\\ &\quad + v''(z)\left[x_1^2 z^2 + x_2^2 z^2 + \cdots + x_{n-1}^2 z^2\right] + v''(z)\left[(1-z^2)^2\right].\end{aligned}$$

Since $\|x\| = 1$, we see that $x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = 1 - z^2$. Then

$$\begin{aligned}\Delta u &= v'(z)\left[3z(1-z^2) + (1-n)z - 3z(1-z^2)\right]\\ &\quad + v''(z)\left[z^2(1-z^2) + (1-z^2)(1-z^2)\right]\\ &= v'(z)[(1-n)z] + v''(z)[1-z^2].\end{aligned}$$

Therefore we have calculated that the Laplace-Beltrami operator applied to the rotationally symmetric function u is

$$\Delta u = (1-z^2)v''(z) + (1-n)zv'(z). \tag{2.2}$$

Then we form the semilinear Laplace-Beltrami equation as follows:

$$(1-z^2)u'' + (1-n)zu' + (1-|z|)f(u) = 0 \tag{2.3}$$

where $f(u)$ is superlinear, that is, $\lim_{|u|\to\infty} \frac{f(u)}{u} = \infty$. We also assume that $f(u)$ is locally Lipschitz continuous. In fact, we use $f(u) = u^p$, where $1 < p < \frac{n+5}{n-3}$. The coefficient $(1-|z|)$ appears on the f term to avoid singularities at the poles $z = \pm 1$.

The boundary conditions come from the fact that solutions are rotationally symmetric. Therefore the slope at both poles of the sphere, $z = 1$ and $z = -1$, must be 0. With these boundary conditions, we define the problem:

$$\begin{cases} (1-z^2)u'' + (1-n)zu' + (1-|z|)f(u) = 0 \\ u'(1) = u'(-1) = 0 \end{cases},$$

as given in (2.1).

2.2 Spherical Coordinates

Using a spherical coordinate system, where t is the arc length from the north pole, the Laplace-Beltrami operator on the unit sphere can also be written for function f as

$$\Delta f(t) = \sin^{2-n} t \frac{\partial}{\partial t}\left(\sin^{n-2} t \frac{\partial f}{\partial t}\right) = f''(t) + (n-2)\frac{\cos t}{\sin t} f'(t). \tag{2.4}$$

(See Jost (2011)). Setting $f(t) = g(z(t))$, we have

$$f'(t) = g'(z(t))z'(t)$$

and

$$f''(t) = g''(z)(z'(t))^2 + g'(z(t))z''(t).$$

We also use that $\cos t = z(t)$, $\sin t = \sqrt{1-(z(t))^2}$. So

$$z'(t) = -\sin t \qquad \text{and} \qquad z''(t) = -\cos t.$$

Plugging these values into (2.4), we see

$$\begin{aligned} f''(t) + (n-2)\frac{\cos t}{\sin t} f'(t) &= g''(z')^2 + g'z'' + (n-2)\frac{\cos t}{\sin t} g'z' \\ &= g'' \sin^2 t + g'(-\cos t) + (n-2)\left(\frac{\cos t}{\sin t}\right)(-\sin t)g' \\ &= (\sin^2 t)g'' + (1-n)(\cos t)g' \\ &= (1-z^2)g'' + (1-n)zg', \end{aligned}$$

and this final expression is equivalent to the cartesian coordiantes given in (2.2).

Chapter 3

Local Existence of Solutions for an Initial Value Problem

3.1 The Initial Value Problem

In this chapter, we prove the existence, uniqueness and continuous dependence of solutions on the initial conditions for $z \in [-1, -1+\epsilon]$ for the initial value problem

$$\begin{cases} (1-z^2)u'' + (1-n)zu' + (1+z)f(u) = 0 \\ u(-1) = d, u'(-1) = 0. \end{cases} \tag{3.1}$$

We write the f coefficient here as $(1+z)$ instead of $(1-|z|)$ because here we are only considering $z \in [-1, 0]$. We find solutions using the integrating factor.

The second order equation in (3.1) is equivalent to

$$(1-z^2)\left[u'' + \frac{(1-n)z}{1-z^2}u'\right] = -(1+z)f(u).$$

Multiplying and dividing by the integrating factor $(1-z^2)^{(n-1)/2}$, we have

$$\begin{aligned} -(1+z)f(u) &= \frac{(1-z^2)}{(1-z^2)^{\frac{n-1}{2}}}\left[u''(1-z^2)^{\frac{n-1}{2}} + \frac{z(1-n)(1-z^2)^{\frac{n-1}{2}}}{1-z^2}u'\right] \\ &= (1-z^2)^{\frac{3-n}{2}}[(1-z^2)^{\frac{n-1}{2}}u']', \end{aligned}$$

or

$$\left((1-z^2)^{\frac{n-1}{2}}u'\right)' = -(1-z^2)^{\frac{n-3}{2}}(1+z)f(u).$$

Now integrating both sides, this becomes

$$(1-z^2)^{\frac{n-1}{2}} u' = -\int_{-1}^{z} (1-x^2)^{\frac{n-3}{2}} (1+x) f(u(x) dx.$$

Integrating again, after dividing by $(1-z^2)^{\frac{n-1}{2}}$, we have

$$u(z) = d - \int_{-1}^{z} (1-y^2)^{\frac{1-n}{2}} \int_{-1}^{y} (1-x^2)^{\frac{n-3}{2}} (1+x) f(u(x)) dx dy.$$

Thus finding solutions to (3.1) is equivalent to finding fixed points of the operator

$$\Gamma(u,d) = d - \int_{-1}^{z} w(y) \int_{-1}^{y} (1-x^2)^{\frac{n-3}{2}} (1+x) f(u(x)) dx dy$$

where $w(t) = (1-t^2)^{\frac{1-n}{2}}$.

In the following section we prove, using the contraction mapping principle, that such a fixed point exists.

3.2 The Contraction Mapping Principle

Let $I = [-1, -1+\epsilon]$ and $J = [d_0 - M, d_0 + M]$ for fixed d_0, some M, and some $0 < \epsilon < 1$. Then let $X = \mathcal{C}(I,J)$ be the space of continuous functions that map I to J. We will show in the remainder of this section that X is a complete metric space, that Γ maps from $X \times Y$ to X, where $Y = [d_0 - \epsilon, d_0 + \epsilon]$, and that Γ is a contraction.

Lemma 3.1. *Let $X = \mathcal{C}(I,J)$ be the space of continuous functions mapping I to J. Then X is a complete metric space with metric*

$$d(u,v) = \max_{x \in I} |u(x) - v(x)|.$$

Proof. Suppose that $\{u_n\}$ is a Cauchy sequence in X. Then, given $\delta > 0$, there is some N such that if $n,m \geq N$, then $d(u_n,u_m) < \delta$. Fix $x_0 \in I$. The sequence $\{u_n(x_0)\}$ is a sequence in J, which is itself a subset of $\mathbb{R}$. Now, for $n,m \geq N$,

$$\begin{aligned} d(u_n(x_0), u_m(x_0)) &= |u_n(x_0) - u_m(x_0)| \\ &\leq \max_{x \in I} |u_n(x) - u_m(x)| \\ &= d(u_n, u_m) \\ &< \delta. \end{aligned}$$

Therefore, the sequence $\{u_n(x_0)\}$ is a Cauchy sequence, so it must converge. Say $\{u_n(x_0)\} \to u^*(x_0)$. Similarly, for any $y \in I$, there is a $u^*(y)$ such that $\{u_n(y)\} \to u^*(y)$ and an M_y such that if $n \geq M_y$, then

$$|u_n(y) - u^*(y)| < \delta^*$$

where $\delta^* > 0$ is arbitrary.

Now let $M = \sup_{y \in I} M_y$. Then for $n \geq M$, we have

$$d(u_n, u^*) = \max_{x \in I} |u_n(x) - u^*(x)| < \delta^*.$$

Thus $\{u_n\}$ converges to u^*. Therefore every Cauchy sequence in X converges, and X is complete. □

It is not immediately clear that $\Gamma(u,d)$ maps into X, so we next prove that if $z \in I$ then $\Gamma(u(z), d) \in J = [d_0 - M, d_0 + M]$.

Lemma 3.2. *Let $(u,d) \in X \times Y$. Then $\Gamma(u,d) \in X$.*

Proof. We aim to show that $|\Gamma(u,d) - d_0| \leq M$. Since $u(z) \in J$ and J is compact, and since f is continuous, $f(u)$ restricted to $u \in X$ is bounded. Let K_1 be such that $|f(u)| \leq K_1$.

$$\begin{aligned}
|\Gamma(u,d) - d_0| &= \left| d - \int_{-1}^{z} w(y) \int_{-1}^{y} (1-x^2)^{\frac{n-3}{2}} (1+x) f(u(x)) dx dy - d_0 \right| \\
&\leq |d - d_0| + \int_{-1}^{z} w(y) \int_{-1}^{y} (1-x^2)^{\frac{n-3}{2}} (1+x) |f(u(x))| dx dy \\
&\leq \epsilon + K_1 \int_{-1}^{z} (1-y^2)^{-1} \int_{-1}^{y} \left(\frac{1-x^2}{1-y^2} \right)^{\frac{n-3}{2}} (1+x) dx dy \\
&\leq \epsilon + K_1 \epsilon \ln \left| \frac{2}{1-z} \right| \\
&\leq \epsilon \left(1 + K_1 \ln \left| \frac{2}{2-\epsilon} \right| \right).
\end{aligned}$$

We want $\epsilon \left(1 + K_1 \ln \left|\frac{2}{2-\epsilon}\right|\right) \leq M$, which is equivalent to

$$\epsilon \leq 2 - 2e^{\frac{1-M}{K_1}}.$$

So if ϵ is small enough, then Γ will stay within J. Thus $\Gamma(u,d) \in X$. □

Now we prove that $\Gamma(u,d)$ is a contraction and is continuous.

Lemma 3.3. *The function $\Gamma : X \times Y \to X$ is continuous and is a contraction in the first variable.*

Proof. We will see that there is a constant $\delta < 1$ such that

$$d(\Gamma(u,d),\Gamma(v,d)) = \max_{z\in I} |\Gamma(u(z),d) - \Gamma(v(z),d)| < \delta d(u,v)$$

for all $u, v \in X$, $d \in Y$.

We have

$$\begin{aligned}
|\Gamma(u(z),d)-&\Gamma(v(z),d)| \\
&= \left| \int_{-1}^{z} w(y) \int_{-1}^{y} (1-x^2)^{\frac{n-3}{2}} (1+x)[f(v(x)) - f(u(x))] dxdy \right| \\
&\leq \int_{-1}^{z} w(y) \int_{-1}^{y} (1-x^2)^{\frac{n-3}{2}} (1+x)|f(u(x)) - f(v(x))| dxdy \\
&\leq K_2 d(u,v) \int_{-1}^{z} w(y) \int_{-1}^{y} (1-x^2)^{\frac{n-3}{2}} (1+x) dxdy
\end{aligned}$$

where $K_2 < 1$ is a Lipschitz constant such that $d(f(u), f(v)) \leq K_2 d(u,v)$. Then

$$\begin{aligned}
|\Gamma(u(z),d)-&\Gamma(v(z),d)| \\
&\leq K_2 d(u,v) \int_{-1}^{z} (1-y^2)^{-1} \int_{-1}^{y} \left(\frac{1-x^2}{1-y^2} \right)^{\frac{n-3}{2}} (1+x) dxdy \\
&\leq K_2 d(u,v) \int_{-1}^{z} (1-y^2)^{-1} \int_{-1}^{y} (1)(\epsilon) dxdy \\
&= K_2 d(u,v) \epsilon \int_{-1}^{z} \frac{y+1}{1-y^2} dy \\
&= K_2 d(u,v) \epsilon \int_{-1}^{z} \frac{1}{1-y} dy \\
&= K_2 d(u,v) \epsilon \ln \left| \frac{2}{1-z} \right|.
\end{aligned}$$

So

$$\begin{aligned}
d(\Gamma(u,d),\Gamma(v,d)) &\leq \max_{z\in I} \left(K_2 d(u,v) \epsilon \ln \left| \frac{2}{1-z} \right| \right) \\
&= K_2 d(u,v) \epsilon \ln \left| \frac{2}{2-\epsilon} \right|
\end{aligned}$$

which we can make arbitrarily small. Particularly, given $\delta > 0$, we can choose ϵ such that $0 < \epsilon < 2 - 2e^{-\delta/K_2}$. Then

$$\begin{aligned} d(\Gamma(u,d),\Gamma(v,d)) &< K_2 d(u,v)\epsilon \ln\left|\frac{2}{2-(2-2e^{-\delta/K_2})}\right| \\ &< K_2 d(u,v)\ln|e^{\delta/K_2}| \qquad \text{assuming } \epsilon < 1 \\ &= \delta d(u,v). \end{aligned}$$

Taking $\delta < 1$, this proves that Γ is a contraction in the first variable.

Now we prove that Γ is continuous. Let $\{u_n\}$ and $\{d_n\}$ be sequences which converge to u and d, respectively, with $u_n \neq u$, $d_n \neq d$. We wish to take the limit of $\Gamma(u_n, d_n)$ as $n \to \infty$. In order to do so, first we need to prove that the integrand

$$I_1 = w(y)\int_{-1}^{y}(1-x^2)^{\frac{n-1}{2}}(1+x)f(u_n(x))dx \tag{3.2}$$

converges uniformly.

Since $\lim_{n\to\infty} u_n = u$ and f is continuous, we have $\lim_{n\to\infty} f(u_n) = f(u)$. Let $\delta > 0$. Then there is an N such that if $n \geq N$, then

$$d(f(u_n), f(u)) < \delta/\epsilon^2.$$

Now consider the difference in absolute value between

$$I_2 = w(y)\int_{-1}^{y}(1-x^2)^{\frac{n-1}{2}}(1+x)f(u(x))dx$$

and I_1.

$$\begin{aligned} |I_1 - I_2| &= (1-y^2)^{\frac{1-n}{2}}\int_{-1}^{y}(1-x^2)^{\frac{n-1}{2}}(1+x)\,|f(u_n(x)) - f(u(x))|\,dx \\ &\leq (2\epsilon-\epsilon^2)^{\frac{1-n}{2}}\int_{-1}^{y}(2\epsilon-\epsilon^2)^{\frac{n-1}{2}}(\epsilon)d(f(u_n),f(u))dx \\ &\leq \epsilon(y+1)d(f(u_n),f(u)) \\ &\leq \epsilon^2 d(f(u_n),f(u)) \\ &< \delta \end{aligned}$$

for all $y \in [-1, -1+\epsilon]$. Therefore the integrand converges uniformly. Now we can say

$$\begin{aligned} \lim_{n\to\infty}\Gamma(u_n,d_n) &= \lim_{n\to\infty}\left(d_n - \int_{-1}^{z} w(y)\int_{-1}^{y}(1-x^2)^{\frac{n-1}{2}}(1+x)f(u_n(x))dxdy\right) \\ &= d - \int_{-1}^{z} w(y)\int_{-1}^{y}(1-x^2)^{\frac{n-1}{2}}(1+x)f(u(x))dxdy \\ &= \Gamma(u,d). \end{aligned}$$

Therefore Γ is continuous in both variables. □

Finally, we apply the contraction mapping principle and prove the following theorem:

Theorem 3.1. *The function Γ has a unique fixed point $u(z)$ which is a solution to (3.1) for $z \in I$ and depends continuously on the initial condition d.*

Proof. By the previous lemmas, we have a function $\Gamma : X \times Y \to X$, where X is a complete metric space and Γ is a contraction in the first variable. Thus we can apply the Contraction Mapping Principle to find its unique fixed point:

$$u(z) = d - \int_{-1}^{z} w(y) \int_{-1}^{y} (1 - x^2)^{\frac{n-3}{2}} (1 + x) f(u(x)) dx dy.$$

Furthermore, this function $u \in X$ is a solution to the initial value problem (3.1) and depends continuously on d. □

We have shown that a solution exists to the initial value problem in the interval $[-1, -1 + \epsilon]$. In the next chapter we show that we can extend this solution to the entire interval $[-1, 0]$.

Chapter 4

Global Existence of Solutions for an Initial Value Problem

4.1 Energy Analysis

Now that we have proved the existence of a solution to (3.1) over some interval $[-1, -1+\epsilon]$, we next need to show that the solution can be extended up to $z = 0$. To do this, we must first prove that the solution stays bounded in the interval $[-1, 0]$. We will do this using an energy argument, with energy defined as

$$E(z) = \frac{(u'(z))^2}{2} + \frac{1}{1-z} F(u(z)) \tag{4.1}$$

where $F(t) = \int_0^t f(s)ds$.

Lemma 4.1. *Energy is bounded in the interval* $[-1, 0]$. *In particular,*

$$E(z) \leq \frac{e}{2} F(d),$$

and $|u'(z)|$ *is bounded for* $z \in [-1, 0]$.

Proof. Differentiating equation (4.1), we see

$$\begin{aligned}
E'(z) &= u'u'' + \frac{1}{1-z} f(u)u' + \frac{1}{(1-z)^2} F(u) \\
&= u'\left(\frac{z(n-1)}{1-z^2} u'\right) + \frac{1}{(1-z)^2} F(u) \\
&\leq \frac{1}{(1-z)^2} F(u) \\
&\leq \frac{1}{1-z} E(z).
\end{aligned}$$

Note that $E(-1) = \frac{1}{2}F(d)$ and $\frac{1}{1-z} \leq 1$ for $z \in [-1, 0]$, so

$$E(z) \leq E(-1)e^{z+1} \leq \frac{e}{2} F(d).$$

Therefore the energy is bounded from above. This gives us

$$E(z) = \frac{(u')^2}{2} + \frac{1}{1-z} F(u) \leq \frac{e}{2} F(d),$$

and since $F(u)$ is bounded below, we can say $F(u) \geq K$ for some K. Then

$$\frac{(u')^2}{2} + \frac{1}{2} K \leq \frac{(u')^2}{2} + \frac{1}{1-z} F(u) \leq \frac{e}{2} F(d),$$

so

$$(u')^2 \leq eF(d) - K.$$

This implies that $|u'|$ is bounded above. □

This result will allow us to extend the solution to values of z higher than $z = -1 + \epsilon$.

4.2 Extension to $z = 0$

We have proved that a solution exists from -1 to $-1 + \epsilon$ for some $0 < \epsilon < 1$. To show that we can in fact extend this solution all the way up to $z = 0$, we use the following lemma:

Lemma 4.2. *Define a as*

$$a = \sup\{t : u(z) \text{ defined for } z \in [-1, t)\}.$$

Then $\lim_{z \to a^-} u(z)$ *and* $\lim_{z \to a^-} u'(z)$ *exist.*

Proof. By definition, $a \geq -1 + \epsilon$. Suppose that $a \leq 0$. We know from the previous section that $|u'|$ is bounded above, so let K be such that $|u'(z)| < K$ for all z. Let $\{z_k\}$ be a Cauchy sequence converging to a, $z_k \in [-1, a)$. Given $\delta > 0$, let k, j be large enough that $|z_k - z_j| < \delta/K$. Then by the Mean Value Theorem,

$$|u(z_k) - u(z_j)| \leq |z_k - z_j||u'(\xi)|$$

for some $\xi \in [z_k, z_j]$. Then

$$|u(z_k) - u(z_j)| < (\delta/K)K = \delta,$$

which shows that the sequence $\{u(z_k)\}$ is Cauchy. Therefore it must converge and the limit exists. We define $u(a) = \lim_{z\to a-} u(z)$.

We make a similar argument for $\lim_{z\to a-} u'(z)$. Again by the Mean Value Theorem,

$$|u'(z_k) - u'(z_j)| \leq |z_k - z_j||u''(\xi)|$$

for some $\xi \in [z_k, z_j]$. Since $u''(z) = \frac{z(n-1)}{1-z^2}u'(z) + \frac{1}{1-z}f(u(z))$ and both $|u'|$ and $|f(u)|$ are bounded, we see that $|u''|$ is bounded. It follows that the desired limit exists. Then we define $u'(a) = \lim_{z\to a-} u'(z)$. $\square$

Theorem 4.1. *A solution $u(z)$ to the initial value problem (3.1) exists for $z \in [-1, 0]$.*

Proof. We conclude from the previous lemmas that u is defined over $[-1, a]$ for some $-1 < a \leq 0$. Let $u(a) = d$ and $u'(a) = m$. Now we can solve the initial value problem

$$\begin{cases} (1-z^2)u'' + (1-n)zu' + (1+z)f(u) = 0 \\ u(a) = d, u'(a) = m \end{cases} . \tag{4.2}$$

The first equation in 4.2 is equivalent to

$$(1-z^2)^{\frac{3-n}{2}} \left((1-z^2)^{\frac{n-1}{2}} u'\right)' = -(1+z)f(u).$$

Integrating this equation from a to z, we have

$$(1-z^2)^{\frac{n-1}{2}} u'(z) - (1-a^2)^{\frac{n-1}{2}} u'(a) = -\int_a^z (1-x^2)^{\frac{n-3}{2}} (1+x)f(u(x))dx$$

or

$$u'(z) = (1-z^2)^{\frac{1-n}{2}} \left[(1-a^2)^{\frac{n-1}{2}} m - \int_a^z (1-x^2)^{\frac{n-3}{2}} (1+x)f(u(x))dx\right].$$

Integrating again from a to z, this becomes

$$u(z) = d + \frac{m}{w(a)} \int_a^z w(y)dy - \int_a^z w(y) \int_a^z (1-x^2)^{\frac{n-3}{2}} (1+x) f(u(x)) dx dy,$$

where $w(t) = (1-t^2)^{\frac{1-n}{2}}$, as before. But notice that since $a > -1$, the integrals are finite. Therefore $u(z)$ exists for any z, $a < z \leq 0$. Thus we have the solution $u(z)$ defined for $z \in [-1, 0]$. $\square$

Chapter 5

Pohozaev Identity

5.1 Derivation

Now we derive a Pohozaev identity for the initial value problelm (3.1). Pohozaev identities are useful in estimating solutions of nonlinear partial differential equations (see Pohozaev (1965)). Such identities result from multiplying the differential equation by suitable quantities that, after integration, produce relations that do not depend on the derivatives of highest order. We define

$$p(z) = (1 - z^2)^{\frac{n-3}{2}}. \tag{5.1}$$

and

$$q(z) = 2(1 - z^2)^{\frac{n-1}{2}} h(z), \tag{5.2}$$

where $h(z) = (1 - z^2)^{\frac{n-3}{2}} \int_z^0 (1 - y^2)^{\frac{1-n}{2}} dy$. Multiplying (3.1) by pu and qu' and then integrating, we obtain a Pohozaev identity.

The result is the following theorem:

Theorem 5.1. *Let $n > 3$, $f : \mathbb{R} \to \mathbb{R}$, and $F(u) = \int_0^u f(s)ds$. Then the following identity holds for $z \in [-1, 0]$:*

$$\begin{aligned}
&(1 - z^2)^{\frac{n+1}{2}} h(z)(u')^2 + (1 - z^2)^{\frac{n-1}{2}} uu' + 2(1 - z^2)^{\frac{n-1}{2}} (1 + z)h(z)F(u) \\
&\quad = \int_{-1}^{z} \frac{(1 - y^2)^{\frac{n-1}{2}}}{1 - y} \left[F(u)\left(h(y)(6y - 4ny + 2) - 2\right) - uf(u)\right] dy,
\end{aligned} \tag{5.3}$$

where $h(z) = (1 - z^2)^{\frac{n-3}{2}} \int_z^0 (1 - y^2)^{\frac{1-n}{2}} dy$.

Proof. To derive the Pohozaev identity for this equation, we multiply the Laplace-Beltrami equation (2.3) by $p(z)u$ and $q(z)u'$ and integrate. Multiplying by $p(z)u$, we have

$$(1-z^2)^{\frac{n-1}{2}}uu'' + (1-n)(1-z^2)^{\frac{n-3}{2}}zuu' + (1-z^2)^{\frac{n-3}{2}}(1+z)uf(u) = 0.$$

Integrating by parts, this becomes

$$\begin{aligned}(1-z^2)^{\frac{n-1}{2}}uu' - \int_{-1}^{z} u'\left[(1-y^2)^{\frac{n-1}{2}}u' - (n-1)(1-y^2)^{\frac{n-3}{2}}yu\right]dy \\ + \int_{-1}^{z}(1-n)(1-y^2)^{\frac{n-3}{2}}yuu'dy + \int_{-1}^{z}(1-y^2)^{\frac{n-3}{2}}(1+y)uf(u)dy = 0,\end{aligned}$$

which simplifies to

$$\begin{aligned}(1-z^2)^{\frac{n-1}{2}}uu' - \int_{-1}^{z}(1-y^2)^{\frac{n-1}{2}}(u')^2dy \\ = -\int_{-1}^{z}\frac{(1-y^2)^{\frac{n-1}{2}}}{1-y}uf(u)dy. \quad (5.4)\end{aligned}$$

Next we multiply the Laplace-Beltrami equation by $q(z)u'$ to get

$$\begin{aligned}2(1-z^2)^{\frac{n+1}{2}}h(z)u'u'' + 2(1-n)(1-z^2)^{\frac{n-1}{2}}h(z)z(u')^2 \\ +2(1-z^2)^{\frac{n-1}{2}}(1+z)h(z)u'f(u) = 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}(1-z^2)^{\frac{n+1}{2}}\left(\frac{(u')^2}{2}\right)' + (1-n)(1-z^2)^{\frac{n-1}{2}}h(z)z(u')^2 \\ +(1-z^2)^{\frac{n-1}{2}}(1+z)h(z)\,(F(u))' = 0.\end{aligned}$$

Integrating by parts, this becomes

$$\begin{aligned}(1-z^2)^{\frac{n+1}{2}}h(z)\left(\frac{(u')^2}{2}\right) + (1-z^2)^{\frac{n-1}{2}}(1+z)h(z)F(u) \\ -\int_{-1}^{z}\frac{(u')^2}{2}\left[(1-y^2)^{\frac{n+1}{2}}h'(y) - (n+1)(1-y^2)^{\frac{n-1}{2}}yh(y)\right]dy \\ + \int_{-1}^{z}(1-n)(1-y^2)^{\frac{n-1}{2}}h(y)y(u')^2dy \\ -\int_{-1}^{z}F(u)\left[(1-y^2)^{\frac{n-1}{2}}(1+y)h'(z) + (1-y^2)^{\frac{n-1}{2}}h(y)\right. \\ \left. - (n-1)(1-y^2)^{\frac{n-3}{2}}(1+y)yh(y)\right]dy = 0.\end{aligned}$$

A simple calculation shows that

$$h'(z) = -(1-z^2)^{-1} - (n-3)(1-z^2)^{\frac{n-5}{2}} z \int_{-1}^{z} (1-y^2)^{\frac{1-n}{2}} dy,$$

so

$$(1-z^2)h'(z) = -1 - (n-3)zh(z).$$

Substituting this into the integrated equation and simplifying, we have

$$\begin{aligned}\frac{1}{2}(1-z^2)^{\frac{n+1}{2}} h(z)(u')^2 + \frac{1}{2}\int_{-1}^{z}(1-y^2)^{\frac{n-1}{2}}(u')^2 dy \\ + (1-z^2)^{\frac{n-1}{2}}(1+z)h(z)F(u) \\ = \int_{-1}^{z} \frac{(1-y^2)^{\frac{n-1}{2}}}{1-y} F(u)\left[h(y)\left(3y-2ny+1\right)-1\right] dy. \quad (5.5)\end{aligned}$$

Now we multiply (5.5) by 2 and add it to (5.4) to get

$$\begin{aligned}(1-z^2)^{\frac{n-1}{2}} uu' + (1-z^2)^{\frac{n+1}{2}} h(z)(u')^2 - \int_{-1}^{z}(1-y^2)^{\frac{n-1}{2}}(u')^2 dy \\ + \int_{-1}^{z}(1-y^2)^{\frac{n-1}{2}}(u')^2 dy + 2(1-z^2)^{\frac{n-1}{2}}(1+z)h(z)F(u) \\ = 2\int_{-1}^{z} \frac{(1-y^2)^{\frac{n-1}{2}}}{1-y} F(u)\left[h(y)\left(3y-2ny+1\right)-1\right] \\ - \int_{-1}^{z} \frac{(1-y^2)^{\frac{n-1}{2}}}{1-y} uf(u) dy.\end{aligned}$$

This simplifies to

$$\begin{aligned}(1-z^2)^{\frac{n+1}{2}} h(z)(u')^2 + (1-z^2)^{\frac{n-1}{2}} uu' + 2(1-z^2)^{\frac{n-1}{2}}(1+z)h(z)F(u) \\ = \int_{-1}^{z} \frac{(1-y^2)^{\frac{n-1}{2}}}{1-y}\left[F(u)\left(h(y)\left(6y-4ny+2\right)-2\right) - uf(u)\right] dy,\end{aligned}$$

as stated in (5.3). □

Now that we have derived the Pohozaev identity, we consider what this means for functions f of the form $f(u) = u^p$. But first, we consider the behavior of $h(z)$ near -1. Notice that the integrand blows up as y approaches -1. However, we can take the limit of $h(z)$ as $z \to -1$ using L'Hopital's rule to find:

$$\begin{aligned}
\lim_{z\to -1} h(z) &= \lim_{z\to -1} \frac{\int_z^0 (1-y^2)^{\frac{1-n}{2}}\,dy}{(1-z^2)^{\frac{3-n}{2}}} \\
&= \lim_{z\to -1} \frac{0-(1-z^2)^{\frac{1-n}{2}}}{\frac{3-n}{2}(1-z^2)^{\frac{1-n}{2}}(-2z)} \\
&= \lim_{z\to -1} \frac{1}{(3-n)z} \\
&= \frac{1}{n-3}.
\end{aligned}$$

So we can say that for $z \approx -1$,

$$h(z) \approx \frac{1}{n-3}.$$

Now suppose $f(u) = u^p$, $F(u) = \frac{1}{p+1}u^{p+1}$. The Pohozaev identity is now

$$\begin{aligned}
&(1-z^2)^{\frac{n+1}{2}} h(z)(u')^2 + (1-z^2)^{\frac{n-1}{2}} uu' + \frac{2}{p+1}(1-z^2)^{\frac{n-1}{2}}(1+z)h(z)u^{p+1} \\
&\quad = \int_{-1}^{z} \frac{(1-y^2)^{\frac{n-1}{2}}}{1-y}\left[\frac{u^{p+1}}{p+1}\left(h(y)\left(6y-4ny+2\right)-2\right)-u^{p+1}\right] dy.
\end{aligned}$$

We want to find conditions on which the Pohozaev energy is positive, so we constrain the integrand of the right hand side to be greater than zero. Notice that $\frac{(1-y^2)^{\frac{n-1}{2}}}{1-y}u^{p+1}$ is always positive, so we simply need

$$\frac{1}{p+1}\left[h(y)\left(6y-4ny+2\right)-2\right]-1>0.$$

Taking this expression near $y=-1$, where $h(y) \approx \frac{1}{n-3}$, gives

$$\frac{1}{p+1}\left[\frac{1}{n-3}(-6+4n+2)-\frac{2(n-3)}{n-3}\right]-1>0$$

or

$$\frac{1}{p+1}\left[\frac{2n+2}{n-3}\right]>1.$$

There for we must have

$$p+1<\frac{2(n+1)}{n-3} \qquad \text{or} \qquad p<\frac{n+5}{n-3}. \tag{5.6}$$

So if $f(u) = u^p$ where $p < \frac{n+5}{n-3}$, then the energy defined by the Pohozaev identity is positive. We will use this fact in the next section when doing energy analysis for $n > 3$.

Before moving on to the energy analysis, we discuss how p and q were chosen as stated in (5.1) and (5.2). We began by symbolically multiplying the Laplace-Beltrami equation (2.3) by $p(z)u$ and $q(z)u'$. Multiplying by $p(z)u$ gives

$$p(z)(1-z^2)uu'' + p(z)(1-n)zuu' + p(z)(1+z)uf(u) = 0.$$

Integrating by parts, this becomes

$$\begin{aligned} p(z)(1-z^2)uu' - \int_{-1}^{z} u' \left[p(z)(1-z^2)u' + p'(z)(1-z^2)u + p(z)(-2z)u\right] dz \\ + \int_{-1}^{z} uu'p(z)(1-n)zdz + \int_{-1}^{z} p(z)(1+z)uf(u) = 0, \end{aligned}$$

which simplifies to

$$\begin{aligned} p(z)(1-z^2)uu' + \int_{-1}^{z} uu' \left[-p'(z)(1-z^2) + (3-n)zp(z)\right] dz \\ - \int_{-1}^{z} (u')^2 p(z)(1-z^2)dz + \int_{-1}^{z} p(z)(1+z)uf(u) = 0. \end{aligned}$$

A Pohozaev identity should not depend on derivatives of highest order, so we want to the integral term containing uu' to be zero. Therefore we set

$$p'(z)(1-z^2) = (3-n)zp(z).$$

Separating, we have

$$\frac{p'}{p} = \frac{(3-n)z}{1-z^2},$$

and integrating with respect to z,

$$\ln p\big|_z^0 = \frac{n-3}{2}\ln(1-z^2)\Big|_z^0.$$

This becomes

$$\ln\frac{p(0)}{p(z)} = \frac{n-3}{2}\ln\frac{1-0}{1-z^2},$$

or

$$\frac{p(0)}{p(z)} = \left(\frac{1}{1-z^2}\right)^{\frac{n-3}{2}}.$$

Finally, we set $p(0) = 1$ and rearrange to get
$$p(z) = (1 - z^2)^{\frac{n-3}{2}},$$
as in (5.1).

Next we want to find q, so we multiply the Laplace-Beltrami equation by $q(z)u'$ to get
$$q(z)(1 - z^2)\left(\frac{(u')^2}{2}\right)' + q(z)(1 - n)z(u')^2 + q(z)(1 + z)u'f(u) = 0.$$
Integrating by parts,
$$q(z)(1 - z^2)\left(\frac{(u')^2}{2}\right)\Bigg|_{-1}^{z} - \int_{-1}^{z}\left(\frac{(u')^2}{2}\right)\left[q'(z)(1 - z^2) + q(z)(-2z)\right] dz$$
$$+ \int_{-1}^{z}(u')^2 q(z)(1 - n)z dz + \int_{-1}^{z} q(z)(1 + z)\,(F(u))'\,dz = 0.$$
Then
$$q(z)(1 - z^2)\left(\frac{(u')^2}{2}\right) + \int_{-1}^{z}(u')^2\left[-\frac{1}{2}q'(z)(1 - z^2) + q(z)(2 - n)z\right] dz$$
$$+F(u)q(z)(1 + z) - \int_{-1}^{z} F(u)\left[q'(z)(1 + z) + q(z)\right] dz = 0.$$

In summary, we have the following equations involving p and q, respectively:
$$(1 - z^2)^{\frac{n-1}{2}} uu' - \int_{-1}^{z}(u')^2(1 - z^2)^{\frac{n-1}{2}} dz + \int_{-1}^{z}(1 - z^2)^{\frac{n-3}{2}}(1 + z)uf(u)dz = 0 \quad (5.7)$$

$$q(z)(1 - z^2)\left(\frac{(u')^2}{2}\right) + \int_{-1}^{z}(u')^2\left[-\frac{1}{2}q'(z)(1 - z^2) + q(z)(2 - n)z\right] dz + F(u)q(z)(1 + z) - \int_{-1}^{z} F(u)\left[q'(z)(1 + z) + q(z)\right] dz = 0 \quad (5.8)$$

Summing the two equations, we have
$$\frac{1}{2}q(z)(1 - z^2)(u')^2 + (1 - z^2)^{\frac{n-1}{2}} uu' + F(u)q(z)(1 + z)$$
$$+ \int_{-1}^{z}(u')^2\left[-\frac{1}{2}q'(y)(1 - y^2) + q(y)(2 - n)y - (1 - y^2)^{\frac{n-1}{2}}\right] dy$$
$$= \int_{-1}^{z}\Big(F(u)\left[q'(y)(1 + y) + q(y)\right] - (1 - y^2)^{\frac{n-3}{2}}(1 + y)uf(u)\Big) dy \quad (5.9)$$

To get a Pohozaev identity, the integral term involving $(u')^2$ should be zero. This gives us the following ordinary differential equation for q:

$$-\frac{1}{2}q'(z)(1-z^2)+q(z)(2-n)z-(1-z^2)^{\frac{n-1}{2}}=0$$

or

$$q'(z)-\frac{2(2-n)z}{1-z^2}q(z)=-\frac{2(1-z^2)^{\frac{n-1}{2}}}{1-z^2}. \tag{5.10}$$

Multiplying through by the integrating factor $(1-y^2)^{2-n}$, we have

$$\left((1-z^2)^{2-n}q(z)\right)'=-2(1-z^2)^{\frac{n-3}{2}}(1-z^2)^{2-n}.$$

Then integrating from z to 0,

$$q(0)-(1-z^2)^{2-n}q(z)=-2\int_z^0(1-y^2)^{\frac{1-n}{2}}dy$$

and setting $q(0)=0$, we have

$$q(z)=2(1-z^2)^{n-2}\int_z^0(1-y^2)^{\frac{1-n}{2}}dy$$

or

$$q(z)=2(1-z^2)^{\frac{n-1}{2}}h(z),$$

where $h(z)=(1-z^2)^{\frac{n-3}{2}}\int_z^0(1-y^2)^{\frac{1-n}{2}}dy$. Therefore we have found the functions p and q that, when multiplying the Laplace-Beltrami equation by pu and qu', results in the Pohozaev identity (5.3).

In the next section, we use the Pohozaev identity to show that the energy stays large when the initial condition is large.

5.2 Energy Analysis

We use the Pohozaev identity to prove that energy stays high for high initial condition d.

Theorem 5.2. *Let $n\geq 3$. If $f(u)=u^p$, $1<p<\frac{n+5}{n-3}$, and*

$$E(z,d)=\frac{(u'(z))^2}{2}+\frac{1}{1-z}F(u(z)),$$

then

$$\lim_{d\to\infty}E(z,d)=\infty$$

for $z\in[-1,0]$.

Proof. Let $k \in [0,1]$ be fixed, and let t_0 be such that $u(t_0) = kd$ and $d \geq u(t) \geq kd$ for $t \in [-1, t_0]$. Multiplying (2.3) through by integrating factor $(1-z^2)^{\frac{n-1}{2}}$, we have

$$\left((1-z^2)^{\frac{n-1}{2}} u'\right)' = -(1-z^2)^{\frac{n-1}{2}}(1+z)f(u).$$

Then integrating from -1 to t, this becomes

$$(1-t^2)^{\frac{n-1}{2}} u' = -\int_{-1}^{t} (1-z^2)^{\frac{n-1}{2}}(1+z)f(u(z))dz.$$

Since $f(u) = u^p$ and $u(z) \leq d$ for $z \in [-1, t_0]$, we have

$$(1-t^2)^{\frac{n-1}{2}} u' \geq -\int_{-1}^{t} (1-z^2)^{\frac{n-1}{2}}(1+z)d^p dz.$$

Factoring $1-z^2$ and $1-t^2$ and noticing that $1-z \leq 2$ for $z \in [-1, 0]$,

$$\begin{aligned}
(1+t)^{\frac{n-1}{2}}(1-t)^{\frac{n-1}{2}} u' &\geq -d^p \int_{-1}^{t} (1-z)^{\frac{n-1}{2}}(1+z)^{\frac{n+1}{2}} dz \\
&\geq -d^p 2^{\frac{n-1}{2}} \int_{-1}^{t} (1+z)^{\frac{n+1}{2}} dz \\
&= -d^p 2^{\frac{n-1}{2}} (1+t)^{\frac{n+3}{2}} \left(\frac{2}{n+3}\right).
\end{aligned}$$

Solving the inequality for u' and noticing that $\frac{1}{1-t} \leq 1$ results in

$$\begin{aligned}
u' &\geq -\frac{d^p 2^{\frac{n+1}{2}}}{n+3} \frac{(1+t)^{\frac{n+3}{2}-\frac{n-1}{2}}}{(1-t)^{\frac{n-1}{2}}} \\
&\geq -\frac{d^p 2^{\frac{n+1}{2}}}{n+3}(1+t)^2.
\end{aligned}$$

Now we integrate this from -1 to t_0. Then

$$\begin{aligned}
u(t_0) - u(-1) &\geq -\frac{d^p 2^{\frac{n+1}{2}}}{n+3} \frac{(1+t_0)^3}{3} \\
kd - d &\geq -\frac{d^p 2^{\frac{n+1}{2}}}{3(n+3)}(1+t_0)^3.
\end{aligned}$$

Rearranging for $(1+t_0)$, we have

$$\begin{aligned}(1+t_0)^3 &\geq -\frac{3(n+3)(k-1)}{d^{p-1}2^{\frac{n+1}{2}}}\\ &= \frac{3(n+3)(1-k)}{2^{\frac{n+1}{2}}}d^{1-p}\end{aligned}$$

This gives

$$1+t_0 \geq Kd^{\frac{1-p}{3}}, \tag{5.11}$$

where $K = \left(\frac{3(n+3)(1-k)}{2^{\frac{n+1}{2}}}\right)^{\frac{1}{3}}$.

This lower bound on $1+t_0$ will allow us to bound for the energy. Let

$$\begin{aligned}P(z,d) = (1-z^2)^{\frac{n+1}{2}}h(z)(u')^2 + (1-z^2)^{\frac{n-1}{2}}uu'\\ + 2(1-z^2)^{\frac{n-1}{2}}(1+z)h(z)F(u)\end{aligned}$$

be the left hand side of the Pohozaev identity given in (5.3).

Let $n > 3$ and $g(y) = \frac{(1-y^2)^{\frac{n-1}{2}}}{1-y}$. Then using (5.3),

$$\begin{aligned}P(t_0,d) &= \int_{-1}^{t_0} u^{p+1}g(y)\left[\frac{1}{p+1}\left(h(y)(6y-4ny+2)-2\right)-1\right]dy\\ &\geq \int_{-1}^{t_0}(kd)^{p+1}g(y)\left[\frac{1}{p+1}\left(\frac{1}{n-3}(-4+4n)-2\right)-1\right]dy\end{aligned}$$

since $u \geq kd$ and $y \geq -1$. Then

$$\begin{aligned}P(t_0,d) &\geq (kd)^{p+1}\int_{-1}^{t_0} g(y)\left[\frac{1}{p+1}\left(\frac{(4n-4)}{n-3}-2\right)-1\right]dy\\ &= (kd)^{p+1}\left[\left(\frac{2n+2}{(p+1)(n-3)}\right)-1\right]\int_{-1}^{t_0}(1-y)^{\frac{n-3}{2}}(1+y)^{\frac{n-1}{2}}dy\\ &\geq (kd)^{p+1}\left[\left(\frac{2n+2}{(p+1)(n-3)}\right)-1\right]\int_{-1}^{t_0}(1)^{\frac{n-3}{2}}(1+y)^{\frac{n-1}{2}}dy\\ &= (kd)^{p+1}\left[\left(\frac{2n+2}{(p+1)(n-3)}\right)-1\right]\left(\frac{2}{n+1}\right)(1+t_0)^{\frac{n+1}{2}}\\ &= (kd)^{p+1}\left[\left(\frac{4}{(p+1)(n-3)}\right)-\frac{2}{n+1}\right](1+t_0)^{\frac{n+1}{2}}\\ &\geq (kd)^{p+1}\left[\left(\frac{4}{(p+1)(n-3)}\right)-\frac{2}{n+1}\right](Kd^{\frac{1-p}{3}})^{\frac{n+1}{2}}\\ &= Cd^{(p+1)+\left(\frac{1-p}{3}\right)\left(\frac{n+1}{2}\right)}\end{aligned}$$

where $C = k^{p+1}K^{\frac{n+1}{2}}\left[\left(\frac{4}{(p+1)(n-3)}\right) - \frac{2}{n+1}\right]$ is positive, since $k > 0$, $K > 0$, and $\frac{4}{(p+1)(n-3)} - \frac{2}{n+1} > 0$, since $p+1 < \frac{2(n+1)}{n-3}$.

So we have

$$P(t_0, d) \geq Cd^{(p+1)+\left(\frac{1-p}{3}\right)\left(\frac{n+1}{2}\right)}. \tag{5.12}$$

Note that if

$$(p+1) + \left(\frac{1-p}{3}\right)\left(\frac{n+1}{2}\right) > 0$$

then we must have $p < \frac{n+7}{n-5}$. But from (5.6), we have that $p < \frac{n+5}{n-3} < \frac{n+7}{n-5}$. Therefore the exponent of d is greater than 0.

Now, notice

$$\begin{aligned}
P(t_0, d) &= (1-t_0^2)^{\frac{n+1}{2}} h(t_0)\left(u'(t_0)\right)^2 + (1-t_0^2)^{\frac{n-1}{2}} u(t_0)u'(t_0) \\
&\qquad + 2(1-t_0^2)^{\frac{n-1}{2}}(1+t_0)h(t_0)F(u(t_0)) \\
&= 2h(t_0)(1-t_0^2)^{\frac{n+1}{2}}\left[\frac{(u'(t_0))^2}{2} + \frac{1}{1-t_0}F(u(t_0))\right] \\
&\qquad + (1-t_0^2)^{\frac{n-1}{2}} u(t_0)u'(t_0) \\
&= 2h(t_0)(1-t_0^2)^{\frac{n+1}{2}} E(t_0, d) + (1-t_0^2)^{\frac{n-1}{2}} u(t_0)u'(t_0).
\end{aligned}$$

Using this fact along with (5.12), we see that

$$2h(t_0)(1-t_0^2)^{\frac{n+1}{2}} E(t_0, d) \geq Cd^{(p+1)+\left(\frac{1-p}{3}\right)\left(\frac{n+1}{2}\right)} - kd(1-t_0^2)^{\frac{n-1}{2}} u'(t_0).$$

Since the coefficient of $E(t_0, d)$ is a positive constant with respect to d, as is $-k(1-t_0^2)^{\frac{n-1}{2}} u'(t_0)$, and since we have a positive exponent on d, we can now say that

$$\lim_{d\to\infty} E(z, d) = \infty$$

for $z \in [-1, 0]$, thus proving the theorem for $n > 3$.

For $n = 3$, the Laplace-Beltrami equation is $(1-z^2)u'' - 2zu' + (1+z)f(u) = 0$. Multiplying through by $(1+z)u'$, this becomes

$$\left(\frac{(u')^2}{2}\right)'(1-z)(1+z)^2 - 2z(1+z)(u')^2 + (1+z)^2\left(F(u)\right)' = 0.$$

Integrating by parts, we have

$$\begin{aligned}
\left(\frac{(u')^2}{2}\right)&(1-z)(1+z)^2 + F(u)(1+z)^2 \\
&= \int_{-1}^{z} \frac{(u')^2}{2}\left[2(1-t)(1+t) - (1+t)^2\right] dt + 2\int_{-1}^{z} t(1+t)(u')^2 dt \\
&\qquad + 2\int_{-1}^{z}(1+t)F(u)dt \\
&= \frac{1}{2}\int_{-1}^{z}(u')^2(1+t)^2 dt + 2\int_{-1}^{z}(1+t)F(u)dt.
\end{aligned}$$

Then

$$\begin{aligned}
E(t_0,d) &= \frac{(u'(t_0))^2}{2} + \frac{1}{1-t_0}F(u(t_0)) \\
&= \frac{1}{(1-t_0)(1+t_0)^2}\left[\frac{1}{2}\int_{-1}^{t_0}(u')^2(1+t)^2 dt + 2\int_{-1}^{t_0}(1+t)F(u)dt\right] \\
&\geq \frac{1}{(2)(1)^2}\left[\frac{1}{2}\int_{-1}^{t_0}(u')^2(1+t)^2 dt + 2\int_{-1}^{t_0}(1+t)\frac{(kd)^{P+1}}{P+1}dt\right] \\
&= \frac{1}{2}\left[\frac{1}{2}\int_{-1}^{t_0}(u')^2(1+t)^2 dt + \frac{2(kd)^{P+1}}{P+1}\left(t_0 + \frac{1}{2}t_0^2 + \frac{1}{2}\right)\right]
\end{aligned}$$

Since $(u'(t))^2(1+t)^2 > 0$ for $t \in (-1, t_0]$ and $\frac{1}{2}t_0^2 + t_0 + \frac{1}{2} > 0$, there exist constants K_1, K_2 such that

$$E(t_0,d) \geq K_1 + K_2 d^{P+1}.$$

Therefore we can say that

$$\lim_{d\to\infty} E(z,d) = \infty$$

for $z \in [-1, 0]$. This completes the proof. $\square$

Chapter 6

Phase Plane Analysis

At this point, we have found a solution $u(z,d)$ defined and bounded over $z \in [-1,0]$. Now we use the shooting method so that we can satisfy the boundary conditions. We do this using phase plane analysis, defining functions $\rho(z,d)$ and $\theta(z,d)$ that map out the curve defined by $(u(z,d), u'(z,d))$, and then proving that the argument function θ increases without bound.

6.1 Defining ρ and θ

Define $x(z) = u(z)$ and $y(z) = u'(z)$. This results in the following system:

$$\begin{cases} x' = y \\ y' = \frac{(n-1)z}{1-z^2}y - \frac{f(x)}{1+|z|} \\ x(-1) = d, y(-1) = 0 \end{cases} . \tag{6.1}$$

An initial goal is to prove the following lemma, which will be useful when we later define $\theta(z)$.

Lemma 6.1. *The ordered pair (x,y) never intersects the origin. That is, $(x,y) \neq (0,0)$ for all $z \in [-1,0]$.*

Proof. We use the energy as defined previously:

$$E(z,d) = \frac{y(z)^2}{2} + \frac{F(x(z))}{1+|z|}.$$

Notice that

$$E(-1,d) = \frac{y(-1)^2}{2} + \frac{F(x(-1))}{1+|-1|} = \frac{F(d)}{2} > 0,$$

so the initial energy is nonzero. We previously showed that $E(z,d) \to \infty$ as $|d| \to \infty$, so there exists a D such that if $|d| > D$, then $E(z,d) \geq 1$ for all $z \in [-1,0]$. In particular, suppose that $z \in [-1,0]$. Assuming $d > D$, then we can say that

$$E(z,d) = \frac{y(z)^2}{2} + \frac{F(x(z))}{1+|z|} \geq 1.$$

If $x(z) = 0$, then $F(x(z)) = \int_0^{x(z)} f(s)ds = 0$, so $y(z) \neq 0$. Conversely, if $y(z) = 0$, then this inequality tells us that $x(z) \neq 0$. Therefore $(x,y) \neq (0,0)$ for any value of $z \in [-1,0]$. □

Now define

$$\rho(z,d) = \sqrt{x(z,d)^2 + y(z,d)^2}$$

and notice that $\rho(z,d) > 0$ for all $z \in [-1,0]$, since $(x,y) \neq (0,0)$.

Next we want to find a continuous function $\theta(z)$ which satisfies

$$\begin{cases} \theta(-1,d) = 0 \\ x(z,d) = \rho(z,d)\cos(\theta(z,d)) \\ y(z,d) = -\rho(z,d)\sin(\theta(z,d)) \end{cases} \tag{6.2}$$

for all $z \in [-1,0]$. In fact, such a θ is given by $\theta(z,d) = \tan^{-1}\left(-\frac{y(z,d)}{x(z,d)}\right)$, but we will need to prove that this is well defined.

We begin by defining a local θ_0 near $z = -1$, and then we will extend this function over the entire interval. We know that $x(-1) = d \neq 0$. Without loss of generality, suppose that $x(-1) > 0$. Then there exists an interval $[-1, -1+\epsilon)$, for some $\epsilon > 0$, over which $x(z) > 0$.

For $z \in [-1, -1+\epsilon)$, we define $\theta_0(z) = \tan^{-1}\left(-\frac{y(z)}{x(z)}\right)$, which is well-defined because $x(z) \neq 0$ for any value in this interval. This function satisfies the conditions for θ given in (6.2).

Let

$$S = \{z \in (-1,0]; \theta : [-1,z] \to \mathbb{R} \text{ exists, satisfies (6.2), and coincides with } \theta_0 \text{ over } [-1,-1+\epsilon)\}.$$

Then put $a = \sup S$. We know that $a \geq -1+\epsilon$, and we wish to prove that $a = 0$. Suppose that $a < 0$. If $x(a) \neq 0$, then $\rho(a) = \sqrt{x(a)^2 + y(a)^2} > y(a)$, so

$$\frac{y(a)}{\rho(a)} < 1.$$

Also, by continuity of $y(z)/\rho(z)$, and since $(1 - |y(a)/\rho(a)|)/2 > 0$, there exists $\delta > 0$ such that if $|z - a| < \delta$ then $|y(z)/\rho(z) - y(a)/\rho(a)| < (1 - |y(a)/\rho(a)|)/2$. Rearranging, this implies that

$$\left|\frac{y(z)}{\rho(z)}\right| \leq \frac{1 + |y(a)/\rho(a)|}{2} < 1.$$

But $y(z)/\rho(z) = -\sin(\theta(z))$, and sine is invertible over $(-1, 1)$. Let k be an odd, positive integer such that $\theta(z) \in (-\frac{k\pi}{2}, \frac{k\pi}{2})$ for all $z \in (a - \delta, a + \delta)$ and let $\sin_k^{-1}(s)$ be defined, for $|s| < 1$, as the inverse sine map to the interval $(-\frac{k\pi}{2}, \frac{k\pi}{2})$. Then put

$$\theta(z) = \sin_k^{-1}\left(-\frac{y(z)}{\rho(z)}\right)$$

for $z \in (a - \delta, a + \delta)$.

Since $\sin_k^{-1}$ is continuous, $\lim_{z \to a} \theta(z) = \theta(a)$. So θ is defined at a, and satisfies the criteria in (6.2). That is, $a \in S$. Since θ is continuous in $[-1, a]$ and $x(a) \neq 0$, we can extend θ to an interval of the form $[-1, a + \epsilon]$.

Similarly, if $y(a) \neq 0$, then $\rho(a) > x(a)$, so $\frac{x(a)}{\rho(a)} < 1$, and we can define

$$\theta(z) = \cos_k^{-1}\left(\frac{x(z)}{\rho(z)}\right),$$

where k is such that $\theta(z) \in (k\pi, (k+1)\pi)$ for all z in an interval $(a - \delta, a + \delta)$. By the same reasoning as above, we see that θ is continuous in $[-1, a]$, with $y(a) \neq 0$, and we can also extend θ to an interval of the form $[-1, a + \epsilon]$.

We can continue in this manner, extending the interval over which θ is defined, whenever $a < 0$. Therefore θ is in fact defined in $[-1, 0]$. In summary, we have now defined and proven the existence of $\rho(z, d) = \sqrt{x(z, d)^2 + y(z, d)^2}$ and $\theta(z, d)$ satisfying (6.2).

6.2 Analysis of the argument θ

The goal of this section is to prove the following theorem:

Theorem 6.1. *The θ function depends continuously on d, and*

$$\lim_{d \to \infty} \theta(z, d) = \infty.$$

We will prove this in the following way: given a positive integer J, there exists a d_0 such that if $d \geq d_0$, then $\theta(0, d) > J\pi$.

Proving this will mean that (u, u') crosses the x axis infinitely many times, i.e., $\theta = 2k\pi$ for infinitely many integers k. This means that there are infinitely many starting positions that yield $u'(0, d) = 0$. This gives us infinitely many solutions to our boundary value problem.

We outline the proof as follows: first we prove that $\theta(z, d)$ is always increasing for $z > -\frac{3}{4}$, by finding the derivative and showing that it is positive. Then we divide the image of θ to belong either in an interval of the form $[\frac{j\pi}{2} - \delta, \frac{j\pi}{2} + \delta]$ or of the form $[\frac{j\pi}{2} + \delta, \frac{(j+2)\pi}{2} - \delta]$, for a nonnegative odd integer j and δ sufficiently small. We will show that, no matter which of these intervals contains $\theta(z)$, there is a larger z' where $\theta(z')$ leaves this interval. Then we can show that θ increases in a way such that it eventually leaves every such interval, and in this way, it continues to increase so that $\theta(0, d) > J\pi$ for some sufficiently large d.

But before stating the proof, we must calculate θ' and prove a lemma that will be used in the eventual proof.

6.2.1 Calculating $\theta'(z)$

We implicitly differentiate $x(z) = \rho(z)\cos(\theta(z))$ to get

$$x'(z) = \rho'(z)\cos(\theta(z)) - \rho(z)\sin(\theta(z))\theta'(z).$$

Rearranging, we see

$$\theta'(z)\rho(z)\sin(\theta(z)) = \rho'(z)\cos(\theta(z)) - x'(z)\rho(z)\sin(\theta(z)).$$

Now we calculate $\rho'(z)$ from the definition $\rho(z) = \sqrt{(x(z))^2 + (x'(z))^2}$. Recalling $x' = y$,

$$\rho'(z) = \frac{2xx' + 2x'x''}{2\sqrt{x^2 + (x')^2}} = \frac{x'(x + x'')}{\rho}.$$

Substituting $x' = y = -\rho\sin(\theta)$,

$$\rho'(z) = \frac{-\rho\sin(\theta)(x + x'')}{\rho} = -(x + x'')\sin(\theta).$$

Now we plug this into our equation for θ',

$$\theta'(z)\rho\sin(\theta) = -(x + x'')\sin(\theta)\cos(\theta) - x'.$$

Cancelling $\sin\theta$ in the left term, and recalling that $x' = -\rho\sin\theta$,

$$\theta'(z) = \frac{-(x+x'')\cos\theta}{\rho} + 1.$$

From the original differential equation, we have

$$x'' = \frac{(n-1)z}{1-z^2}x' - \frac{f(x)}{1-z},$$

so

$$\theta'(z) = 1 - \frac{x\cos\theta}{\rho} - \left(\frac{(n-1)z}{1-z^2}x' - \frac{f(x)}{1-z}\right)\frac{\cos\theta}{\rho}.$$

Since $x = \rho\cos\theta$, we can say

$$\theta'(z) = 1 - \frac{\rho\cos^2\theta}{\rho} - \left(\frac{(n-1)z}{1-z^2}x' - \frac{f(x)}{1-z}\right)\frac{\cos\theta}{\rho},$$

so

$$\theta'(z) = \sin^2\theta - \left(\frac{(n-1)z}{1-z^2}x' - \frac{f(x)}{1-z}\right)\frac{\cos\theta}{\rho}, \tag{6.3}$$

and (6.3) is the expression for $\theta'(z)$ that we will use.

Lemma 6.2. *If $\theta(\hat{z}) = \frac{k\pi}{2}$ for a nonnegative integer k and some $\hat{z} \in [-1, 0]$, then $\theta(z, d) > \frac{k\pi}{2}$ for all $z > \hat{z}$.*

Proof. Suppose $z_1 \in [-1, 0]$ is the smallest such number where $\theta(z_1) = \frac{k\pi}{2}$. Let $z_2 \in [-1, 0]$ be the smallest number such that $z_2 > z_1$ and $\theta(z_2) = \frac{k\pi}{2}$. Since $\theta(-1) = 0$ and $\theta(z_1) \geq 0$, and θ is continuous, we know that $\theta'(z_1) \geq 0$. Similarly, we know $\theta'(z_2) \leq 0$. We will contradict this statement by conditioning on k.

Suppose k is even. From (6.3) we have

$$\theta'(z_2) = \sin^2(k\pi/2) + \frac{f(x)}{1-z_2}\frac{\cos(k\pi/2)}{\rho} - \left(\frac{(n-1)z_2}{1-z_2^2}x'\right)\frac{\cos(k\pi/2)}{\rho}.$$

But k is even, so $\sin^2(k\pi/2) = 0$. Also, notice $x' = -\rho\sin(k\pi/2) = 0$, so

$$\theta'(z_2) = 0 + \frac{f(x)}{1-z_2}\frac{\cos(k\pi/2)}{\rho} - 0.$$

Next we multiply this expression by $\frac{\rho\cos\theta}{x} = 1$.

$$\theta'(z_2) = \frac{1}{1-z_2}\frac{f(x)}{x}\cos^2(k\pi/2).$$

It is clear that $\frac{1}{1-z_2} > 0$ and $\cos^2(k\pi/2) > 0$. Since $f(x)/x \to \infty$, we can choose a large enough d such that $\frac{f(x(z,d))}{x(z,d)} > 0$. Therefore

$$\theta'(z_2) > 0$$

for k even.

Now suppose k is odd. We see that

$$\theta'(z_2) = \sin^2(k\pi/2) + \left(\frac{f(x)}{1-z_2} - \frac{(n-1)zx'}{1-z_2^2}\right)\frac{\cos(k\pi/2)}{\rho}.$$

But in this case, $\cos(k\pi/2) = 0$, so

$$\theta'(z_2) = \sin^2(k\pi/2) > 0.$$

In either case, we have found that $\theta'(z_2) > 0$. Therefore $\theta(z) \neq \frac{k\pi}{2}$ for any $z > z_1$. Since θ is continuous, it follows that

$$\theta(z) > \frac{k\pi}{2}$$

for all $z > z_1$. □

Corollary 1. *Consider $k = 0$. Since $\theta(-1,d) = 0$, we see that $\theta(z,d) > 0$ for all $z > -1$.*

In particular, as we will use in the proof in the next section, picking $z \geq -\frac{3}{4}$ ensures that $\theta(z,d) > 0$.

Lemma 6.3. *The following limit holds:*

$$\lim_{|d|\to\infty} \rho(z,d) = \infty.$$

Proof. We know from a previous chapter that $\lim_{|d|\to\infty} E(z,d) = \infty$. Let $\{d_n\}$ be a sequence converging to infinity, and let $z \in [-1,0]$. Then $\lim_{n\to\infty} E(z,d_n) = \infty$. Now suppose to the contrary that ρ is bounded for any d, that is,

$$\rho(z,d_n) \leq M$$

for all n. But notice that

$$\rho(z,d_n) = \sqrt{(x(z,d_n))^2 + (x'(z,d_n))^2} \geq |x(z,d_n)|,$$

so

$$|x(z,d_n)| \le M.$$

Similarly, $\rho(z,d_n) \ge |x'(z,d_n)|$, so

$$|x'(z,d_n)| \le M$$

for all n. This gives us that

$$\begin{aligned} E(z,d_n) &= \frac{(x'(z,d_n))^2}{2} + \frac{F(x(z,d_n))}{1-z} \\ &\le \frac{M^2}{2} + \frac{F(M)}{2}. \end{aligned}$$

Since $F(M)$ is finite, this shows that $E(z,d_n)$ is bounded for all n, which is a contradiciton. Therefore ρ increases without bound as d increases. That is,

$$\lim_{|d|\to\infty} \rho(z,d) = \infty.$$

□

6.2.2 Proof of Theorem 6.1

Proof of Theorem 6.1. Let J be given and $z \ge -\frac{3}{4}$. Let

$$0 < \delta \le \min\left\{\frac{\pi}{6}, \frac{1}{16Jn}\right\}.$$

Since $\lim_{|x|\to\infty} \frac{f(x)}{x} = \infty$, and since δ has been fixed, there is an X_J such that if $|x| \ge X_J$, then

$$\frac{f(x)}{x} \ge \frac{16J^2\pi + 3(n-1)}{2\cos^2(\frac{\pi}{2}-\delta)}.$$

From Lemma 6.3, there exists a d_0 such that if $|d| \ge d_0$, then

$$\rho(z,d) \ge \frac{X_J}{\cos(\frac{\pi}{2}-\delta)}.$$

Let $|d| \ge d_0$. Then $X_J \le \rho(z,d)\cos\theta = x$. So $|x| \ge X_J$. Now, from (6.3),

$$\theta'(z) = \sin^2\theta - \frac{(n-1)z}{1-z^2}\frac{x'\cos\theta}{\rho} + \frac{f(x)}{1-z}\frac{\cos\theta}{\rho}.$$

Suppose $\theta(z) \in [\frac{j\pi}{2} - \delta, \frac{j\pi}{2} + \delta]$, where j is a nonnegative odd integer. As in the proof of the previous section, we can multiply the rightmost term by $\frac{\rho \cos\theta}{x} = 1$ to see that

$$\frac{f(x)}{1-z}\frac{\cos\theta}{\rho} = \frac{1}{1-z}\frac{f(x)}{x}\cos^2\theta > 0.$$

Therefore

$$\theta'(z) \geq \sin^2\theta - \frac{(n-1)z}{1-z^2}\frac{x'\cos\theta}{\rho} + 0.$$

Next, since $-\frac{3}{4} \leq z \leq 0$, we see that $-\frac{12}{7} \leq \frac{z}{1-z^2} \leq -\frac{3}{4}$. So

$$\theta'(z) \geq \sin^2\theta + \frac{3(n-1)}{4}\frac{x'\cos\theta}{\rho}.$$

And since $\rho = \sqrt{x^2 + (x')^2} \geq |x'|$, we see that $\left|\frac{x'}{\rho}\right| \leq 1$. Further, when θ is in this interval, $|\cos\theta| < \delta$, and $\delta \leq \frac{1}{16Jn} \leq \frac{1}{3(n-1)}$. Now we can say

$$\begin{aligned}
\theta'(z) &\geq \sin^2\theta - \frac{3(n-1)}{4}\delta \\
&\geq \sin^2\theta - \frac{3(n-1)}{4}\left(\frac{1}{3(n-1)}\right) \\
&\geq \sin^2\theta - \frac{1}{4}
\end{aligned}$$

Also, in this interval, $\sin^2\theta \geq \sin^2\left(\frac{j\pi}{2} - \delta\right) = \cos^2(\delta) \geq \cos^2(\pi/6) = \frac{3}{4}$. So

$$\theta'(z) \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$$

So $\theta'(z) \geq \frac{1}{2}$ for $\theta(z) \in [\frac{j\pi}{2} - \delta, \frac{j\pi}{2} + \delta]$.

For $\theta(z) \in [\frac{j\pi}{2} + \delta, \frac{(j+2)\pi}{2} - \delta]$,

$$\begin{aligned}
\theta'(z) &= \sin^2\theta + \frac{f(x)}{1-z}\frac{\cos\theta}{\rho} - \frac{(n-1)z}{1-z^2}\frac{x'\cos\theta}{\rho} \\
&\geq 0 + \frac{1}{1-z}\frac{f(x)}{x}\cos^2\theta + \frac{3(n-1)}{4}\frac{xx'}{\rho^2}.
\end{aligned}$$

Similarly to before, we see that $\left|\frac{x}{\rho}\right| \leq 1$, as is $\left|\frac{x'}{\rho}\right| \leq 1$. So this gives us

$$\theta'(z) \geq \frac{1}{1-z}\frac{f(x)}{x}\cos^2\theta - \frac{3(n-1)}{4}.$$

Now, since $|x| \geq X_J$, and $\frac{1}{1-z} \geq \frac{1}{2}$, we have

$$\begin{aligned}\theta'(z) &\geq \frac{1}{2}\left(\frac{16J^2\pi + 3(n-1)}{2\cos^2\theta}\right)\cos^2\theta - \frac{3(n-1)}{4} \\ &= 4J^2\pi.\end{aligned}$$

So $\theta'(z,d) \geq 4J^2\pi$ for $\theta(z,d) \in [\frac{j\pi}{2}+\delta, \frac{(j+2)\pi}{2}-\delta]$.

Thus $\theta(z,d)$ is increasing for z in the interval $[-3/4, 0]$. Let us assume that $\theta(-3/4,d) \in [\frac{j\pi}{2}-\delta, \frac{j\pi}{2}+\delta]$ for some odd integer j. Since $\theta'(-3/4,d) \geq \frac{1}{2}$, $\theta(z,d)$ cannot remain in this interval for longer than 4δ. So for some $z \in (-3/4, -3/4+4\delta)$, it is true that $\theta(z) \in [\frac{j\pi}{2}+\delta, \frac{(j+2)\pi}{2}-\delta]$. Specifically, using the Intermediate Value Theorem, we can say there exists a $z_1 \in (-3/4, -3/4+4\delta)$ such that

$$\theta(z_1,d) = \frac{j\pi}{2} + \delta.$$

Now suppose $z > z_1$ and $\theta(z,d) \in [\frac{j\pi}{2}+\delta, \frac{(j+2)\pi}{2}-\delta]$. Here, $\theta'(z,d) \geq 4J^2\pi$, so θ cannot remain in this interval for longer than $\frac{\pi-2\delta}{J^2\pi}$. Again using the Intermediate Value Theorem, there is a $z_2 \in (z_1, z_1 + \frac{\pi-2\delta}{4J^2\pi})$ where

$$\theta(z_2,d) = \frac{(j+2)\pi}{2} - \delta.$$

By similar argumentation, there exists a $z_3 \in (z_2, z_2+4\delta)$ where $\theta(z_3,d) = \frac{(j+2)\pi}{2} + \delta$. Now, notice that

$$\theta(z_3,d) - \theta(-3/4,d) \geq \pi,$$

so, specifically, $\theta(z_3,d) \geq \pi$, since $\theta(-3/4,d) \geq 0$. Also,

$$z_3 \leq -3/4 + (4\delta + \frac{\pi-2\delta}{4J^2\pi} + 4\delta).$$

Repeating this argument J times, we can find a $\hat{z}$ such that

$$\theta(\hat{z},d) \geq J\pi$$

and $\hat{z} \in (-3/4, -3/4 + J(8\delta + \frac{\pi - 2\delta}{4J^2\pi}))$. So

$$\begin{aligned}\hat{z} &\leq -\frac{3}{4} + 8J\delta + \frac{1}{4J} - \frac{\delta}{2J\pi} \\ &\leq -\frac{3}{4} + 8J\left(\frac{1}{16Jn}\right) + \frac{1}{4J} \\ &\leq -\frac{3}{4} + \frac{1}{2n} + \frac{1}{4J} \\ &\leq 0,\end{aligned}$$

assuming $n \geq 1$ and $J \geq 1$.

This tells us that, if $\theta(-3/4)$ is in an interval of the form $[\frac{j\pi}{2} - \delta, \frac{j\pi}{2} + \delta]$, then $\theta(z)$ reaches $J\pi$ at some $\hat{z}$, $-3/4 \leq \hat{z} \leq 0$.

Now suppose $\theta(-3/4) \in [\frac{j\pi}{2} + \delta, \frac{(j+2)\pi}{2} - \delta]$ for some odd j. By similar argumentation, there is a $\hat{z} \in (-3/4, -3/4 + J(4\delta + \frac{2(\pi-2\delta)}{4J^2\pi}))$ where

$$\theta(\hat{z}, d) \geq J\pi.$$

And notice that

$$\begin{aligned}\hat{z} &\leq -\frac{3}{4} + J(4\delta + 2\left(\frac{\pi - 2\delta}{4J^2\pi}\right)) \\ &\leq -\frac{3}{4} + 4J\delta + \frac{1}{2J} - \frac{\delta}{J\pi}) \\ &\leq -\frac{3}{4} + 4J\left(\frac{1}{16Jn}\right) + \frac{1}{2J} \\ &\leq -\frac{3}{4} + \frac{1}{4n} + \frac{1}{2J} \\ &\leq 0.\end{aligned}$$

since $n \geq 1$ and $J \geq 1$.

Therefore, since $\hat{z} \leq 0$, and θ is increasing in z, we have that

$$\theta(0, d) \geq J\pi$$

for $|d| \geq d_0$. In other words, we have proved that

$$\lim_{|d| \to \infty} \theta(0, d) = \infty.$$

□

Finally, we prove our main result.

6.2.3 Proof of Theorem 1.1

Proof of Theorem 1.1. We have seen that a solution $u(z,d)$ exists to the initial value problem (3.1) for $z \in [-1,0]$. The previous theorem states that $\lim_{|d|\to\infty} \theta(0,d) = \infty$. Hence by the Intermediate Value Theorem, there exists an integer N such that if $k \geq N$ then there exists $d_k > 1$ such that $\theta(0,d_k) = 2\pi k$. Theorefore $y(0,d) = u'(0,d) = 0$ for infinitely many d.

Thus we can reflect the solution about $z = 0$, defining $u(z,d) = u(-z,d)$ for $z \in [0,1]$. Hence we have obtained a symmetric solution over $[-1,1]$ for each d. Therefore we have proven that the Laplace-Beltrami equation has infinitely many rotationally symmetric solutions over $[-1,1]$. $\square$

Chapter 7

Conclusion and Future Work

We began by finding a solution to the initial value problem (3.1) near the south pole, over the interval $[-1, -1+\epsilon]$. This solution was obtained by using the integrating factor method to turn the initial value problem into a fixed point equation. Then we utilized the contraction mapping principle to prove that a unique fixed point exists for this equation; thus such a solution u does exist. Then, because the energy is bounded in the interval $[-1, 0]$, we were able to extend the solution up to the equator, $z = 0$.

Next a Pohozaev identity was constructed. This allowed us to show that the energy is positive at all points on the interval $[-1, 0]$. Since energy stays high when the initial condition is high, we were able to use phase plane analysis to show that the argument function continuously increases without bound. From there we were able to prove Theorem 1.1, concluding that there are infinitely many rotationally symmetric solutions to the Laplace-Beltrami equation on the unit sphere.

At this point, we have made several assumptions and restrictions that we hope to relax in future study. We have assumed that the function $f(u)$ is of the form u^p for some $p < \frac{n+5}{n-3}$. We may consider other types of functions in future work. Additionally, we have up to this point been considering non-singular solutions that are symmetric in z. In the future we may study non-symmetric solutions as well as possible singular solutions. Another interesting topic would be to consider solutions over manifolds other than the sphere.

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