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Reed's Conjecture and Cycle-Power Graphs

Alexa Serrato
Harvey Mudd College

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Alexa Serrato

Nicholas Pippenger, Advisor

Mohamed Omar, Reader

HARVEY MUDD COLLEGE

Department of Mathematics

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Abstract

Reed's conjecture is a proposed upper bound for the chromatic number of a graph. Reed's conjecture has already been proven for several families of graphs. In this paper, I show how one of those families of graphs can be extended to include additional graphs and also show that Reed's conjecture holds for a family of graphs known as cycle-power graphs, and also for their complements.

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Chapter 1

Introduction

In a proper vertex coloring of a graph, each vertex is assigned a color such that if two distinct vertices share an edge then they must be assigned different colors. The chromatic number of a graph G, which is denoted by $\chi(G)$, is the minimum number of colors required to properly color that graph. A graph which has a chromatic number of k is said to be k-chromatic. In this paper we will consider only simple, nonempty, finite, and undirected graphs.

In 1998, Reed conjectured that for any graph G, $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta(G)+1}{2} \right\rceil$, Reed (1998) where $\omega(G)$ and $\Delta(G)$ denote the size of a largest clique in G, and the largest vertex degree in G respectively. Reed's conjecture has been shown to be true for

- graphs in which the complement is disconnected Rabern (2008).
- graphs satisfying $\chi > \lceil \frac{n}{2} \rceil$ Rabern (2008).
- graphs satisfying $\chi > \frac{n+3-\alpha}{2}$ Rabern (2008).
- graphs satisfying $\Delta \geq n + 2 - (\alpha + \sqrt{n+5-\alpha})$ Rabern (2008).
- graphs in which $\alpha = 2$ and $\Delta \geq n - \alpha - 4$ Kohl and Schiermeyer (2010).
- graphs that are triangle-free with $\Delta \geq \frac{8(n-\alpha)+118}{21}$ Kohl and Schiermeyer (2010).

Here, χ, ω, and Δ are as defined above, n denotes the number of vertices in the graph, and α denotes the size of a largest independent set in the graph.

A K_i-free graph is one which contains no clique of size i. In particular, a triangle-free graph is one which is K_3-free.

The final item in the list above was proven in 2010 by Kohl and Schiermeyer by using the following theorem:

Theorem 1. *If $\omega(G) = 2$ and $k \geq 2$ then $\chi(G) \leq (k-1)\left\lceil \frac{n-\alpha}{n(k)-1} \right\rceil + 1$ where $n(k)$ denotes the smallest possible number of vertices in a triangle-free graph that is k-chromatic.*

If the definition of $n(k)$ is modified slightly, then we can extend this theorem as follows:

Definition 2. *Let $n_i(k)$ denote the fewest number of vertices possible in a K_i-free graph that is k-chromatic where i is as small as possible.*

Theorem 3. *If G is a graph with clique number ω and $k \geq 2$, then $\chi(G) \leq 1 + (k-1)\left\lceil \frac{n-\alpha}{n_{\omega+1}(k)-1} \right\rceil$.*

Proof. Let $k \geq 2$. We can partition the vertex set by choosing a maximum independent set I in G and defining $G_1 = G - I$. We can now partition the vertex set of G_1 into ℓ subsets $V_1, ..., V_\ell$ where $\ell = \left\lceil \frac{|V(G_1)|}{n_{\omega+1}(k)-1} \right\rceil$ such that $|V_i| \leq n_{\omega+1}(k) - 1$ for $i = 1, ..., \ell$.

Since the vertices of I can all be colored with the same color, we therefore have that $\chi(G) \leq 1 + \sum_{i=1}^{\ell} \chi(G[V_i])$. Furthermore, since $|V_i| \leq n_{\omega+1}(k) - 1$, $G[V_i]$ must be $k-1$-colorable and so $\chi(G[V_i]) \leq k-1$ for $i = 1, ..., \ell$. Thus, we see that

$$\begin{aligned} \chi(G) &\leq 1 + (k-1)\ell \\ &= 1 + (k-1)\left\lceil \frac{n-\alpha}{n_{\omega+1}(k)-1} \right\rceil \end{aligned}$$

□

Note that in the case that $\omega = 2$, this theorem is equivalent to Theorem 1.

In order to apply this theorem to a wider class of graphs, it is necessary to know some values of $n_i(k)$. Table 1.1 shows some known values of $n_i(k)$.

First note that if $i > k$ then $n_i(k) = k$. This is so because for any graph G, $\omega(G) \leq \chi(G)$. Furthermore, equality holds only for a complete graph and a complete graph on k vertices will be both k-chromatic and K_i-free. Jensen

$n_3(1) = 1$	$n_4(1) = 1$	$n_5(1) = 1$
$n_3(2) = 2$	$n_4(2) = 2$	$n_5(2) = 2$
$n_3(3) = 5$ Jensen and Toft (1995)	$n_4(3) = 3$	$n_5(3) = 3$
$n_3(4) = 11$ Jensen and Toft (1995)	$n_4(4) = 6$	$n_5(4) = 4$
$n_3(5) = 22$ Jensen and Toft (1995)	$n_4(5) = 11$ Jensen and Royle (2995)	$n_5(5) = 7$

Table 1.1 A summary of known values of $n_i(k)$

and Toft (1995) and Jensen and Royle (2995) used an exhaustive computer search to find the values of $n_3(3)$, $n_3(4)$, $n_3(5)$, and $n_4(5)$. The values of the remaining entries in Table 1.1 are derived from Corollary 5:

Theorem 4. *Let a be the smallest integer such that a graph G is K_a-free. If G is a-chromatic then fewest number of vertices that G can have is $a+2$. That is to say, $n_a(a) \geq a+2$.*

Proof. An a-chromatic graph must have at least a vertices. First note that the only a-chromatic graph on a vertices is the complete graph on a vertices, but this graph is not K_a-free.

Consider now a graph G on $a+1$ vertices which is a-chromatic and K_a-free, but does contain K_{a-1}. Let u and v be the two vertices in G which are not a part of the subgraph K_{a-1}.

We first consider the case in which u and v are not adjacent. Then we have that each of u and v can be adjacent to no more that $a-2$ of the vertices in K_{a-1} because otherwise K_a will be a subgraph of G. K_{a-1} requires $a-1$ colors in order to be properly colored. The vertices u and v can each receive the color of a vertex that they are not adajacent to, and such a vertex must exist. Thus, we see that such a graph only requires $a-1$ colors for a proper coloring.

Now we consider the case in which u and v are adjacent. Let the subgraph K_{a-1} be colored with the colors $1, 2, ..., a-1$. We can first assign u the color i where $i \in \{1, 2, ..., a-1\}$. This is possible since there must exist a vertex in K_{a-1} which u is not adjacent to since otherwise we would have K_a as a subgraph. If A denotes the neighborhood of u and B denotes the neighborhood of v then we see that $|A \cap B| \leq a-3$ because otherwise we would have a K_a as a subgraph. Thus, we have that there are at most $a-3$ colors of the $a-1$ colors used to color K_{a-1} and u which v can not receive. Since u and v are adjacent but $u, v \notin A \cap B$, there is possibly an additional color which v may not receive, namely the color of u. Then we have that there are at most $a-2$ colors which v may not receive. Thus, there exists $j \in \{1, 2, ..., a-1\}$ such that v may receive the color j and G will be properly colored. We now have that in each case, G can be properly colored with $a-1$ colors and so is not a-chromatic. This is a contradiction and so we can now conclude that there is no such graph on $a+1$ vertices which is K_a-free and a-chromatic. Thus, the fewest number of vertices in an a-chromatic graph that is K_a-free is at least $a+2$. □

We can show now that $n_a(a) = a+2$ by providing a construction for a graph on $a+2$ vertices which is a-chromatic and K_a-free:

Corollary 5. *The fewest number of vertices in an a-chromatic graph that is K_a-free is $a+2$. That is to say, $n_a(a) = a+2$.*

Proof. Consider joining C_5 to K_{a-3} for $a \geq 3$. This means that each vertex of C_5 will be made adjacent to every vertex in K_{a-3}. Then we have that the resulting graph is a-chromatic because we need $a-3$ colors to properly color K_{a-3} and C_5 is an odd cycle and so requires 3 colors. These three colors for C_5 can't be the same as any of the colors used in K_{a-3} because every vertex of C_5 is made adjacent to every vertex of K_{a-3} in the join of these two graphs. This indicates that $n_a(a) \leq a+2$. Since in Theorem 4 we were able to show that $n_a(a) \geq a+2$, we can now conclude that $n_a(a) = a+2$. □

Using the fact that $n_a(a) = a+2$, we can now generalize Theorem 3 as follows:

Corollary 6. *If G is a K_a-free then $\chi(G) \leq 1 + (a-1)\left\lceil \frac{n-\alpha}{a+1} \right\rceil$.*

Using Corollary 6, we can now show that Reed's Conjecture is satisfied for a slightly wider class of graphs:

Theorem 7. *If G is K_a-free and $\Delta \geq \frac{2(a-1)(n-\alpha+a)}{a+1} + 1 - a$ then $\chi(G) \leq \left\lceil \frac{\Delta+\omega+1}{2} \right\rceil$ and so Reed's Conjecture is satisfied.*

Proof. First note that $\Delta + \omega + 1 = \Delta + a \geq \frac{2(a-1)(n-\alpha+a)}{a+1} + 1$.

We then have that $\left\lceil \frac{\Delta+\omega+1}{2} \right\rceil = \left\lceil \frac{\Delta+a}{2} \right\rceil \geq (a-1)\left\lceil \frac{n-\alpha}{a+1} \right\rceil + 1 \geq \chi(G)$ where the final inequality in the line above comes from Corollary 6. □

We now consider some properties of graphs on $n_i(k)$ vertices which are K_i-free and k-chromatic.

Proposition 8. *A graph G on $n_i(k)$ vertices that is K_i-free and k-chromatic is vertex critical.*

Proof. Suppose to the contrary that if we remove a vertex from the graph G, the chromatic number is unchanged. Then we have a graph which is K_i-free and k-chromatic though with fewer than $n_i(k)$ vertices, contradicting the minimality of $n_i(k)$. □

Conjecture 9. *In a graph G on $n_i(k)$ vertices that is K_i-free and k-chromatic, there exists at least 1 color which only occurs once in a proper k-coloring of G.*

Chapter 2

Cycle-Power Graphs

The Strong Perfect Graph Theorem Chudnovsky et al. (2006) states that if G is a graph in which no induced subgraph is an odd hole or odd antihole, then $\chi(G) = \omega(G)$.

Definition 10. *An odd hole is a cycle with an odd number of vertices greater than or equal to 5 which contains no chords.*

Definition 11. *An odd antihole is the complement of an odd hole.*

Applying the fact that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$ for any graph G, we therefore see that if G has no odd holes or odd antiholes, then

$$\chi(G) = \omega(G) \leq \left\lceil \frac{\omega(G) + \omega(G)}{2} \right\rceil \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil$$

and so Reed's conjecture is satisfied for such a graph. Thus, in order to find a counterexample to Reed's conjecture, one might consider a graph which does contain either an odd hole or an odd antihole.

The following result regarding possible counterexamples to Reed's conjecture was shown by Rabern (2008):

Theorem 12. *If G is an even-order counterexample to Reed's conjecture, then $\overline{G}$ has a perfect matching.*

A perfect matching in a graph on n vertices is a set of $\frac{n}{2}$ edges in which no two edges share a common vertex. Figure 2.1 shows an example of a graph with a perfect matching where the edges of the perfect matching have been bolded.

A cycle-power graph is a member of a family of graphs in which some graphs and their complements may contain a perfect matching and/or an

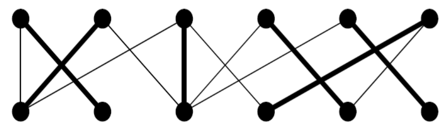

Figure 2.1 A graph with a perfect matching

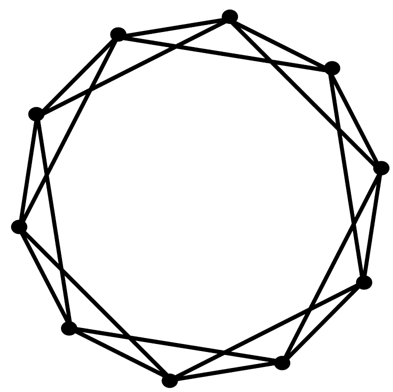

Figure 2.2 The cycle-power graph C_{10}^2.

odd hole. Since having an odd hole or a perfect matching can possibly cause a graph to hold as a counterexample to Reed's conjecture, we will now study the colorability of cycle-power graphs and their complements.

Definition 13. *A cycle-power graph, denoted by C_n^d is one which can be constructed by placing n vertices around a circle and then making each vertex adjacent to the d nearest vertices in each direction along the circle. That is to say that each vertex v is made adjacent to the $2d$ other vertices that are nearest v.*

Let's first begin by noting some properties of C_n^d. In order for C_n^d to have meaning as a graph, given a value for d, we must have that $n > 2d+1$. This condition ensures that there will be enough vertices in the graph for each vertex to be made adjacent with a total of $2d$ other distinct vertices. Note that the graph C_{2d+1}^d is equivalent to the complete graph on $2d+1$ vertices. We will not consider a complete graph to be a cycle-power graph and so we will require that a proper cycle-power graph must satisfy $n > 2d+1$. It can then be observed that for all proper cycle-power graphs, $\Delta(C_n^d) = 2d$ since $2d$ is the degree of every vertex.

Definition 14. *Suppose that vertices $v_1, v_2, v_3..., v_i$ are vertices in a cycle-power graph such that if we travel in one direction along the largest cycle beginning at vertex v_1, the next vertex that we encounter is v_2, then v_3,..., then v_i. Then we say that $v_1, v_2, v_3..., v_i$ are consecutive vertices.*

Proposition 15. *Any $d+1$ consecutive vertices in a cycle-power graph form a maximum clique.*

Proof. Let those $d+1$ consecutive vertices be denoted by $v_1, v_2, ..., v_{d+1}$. Vertex v_1 will be made adjacent to vertices $v_2, ..., v_{d+1}$ because those are the d nearest vertices to v_1 in one direction. For $i, j \in \{2, 3, ..., d+1\}$, v_i and v_j will be adjacent because they will be less and a distance of d apart. Since this is true for any $d+1$ consecutive vertices and no other vertex in a cycle-power can be made adjacent to all $d+1$ of these vertices, we therefore have that any $d+1$ consecutive vertices in a cycle-power graph form a maximal clique.

Note that the size of a maximum clique in a cycle-power graph is the same as the size of a maximal clique. Consider any set of $d+2$ vertices in a cycle-power graph. Let the distance between two vertices be given by the fewest number of edges along the largest cycle between those two vertices. Then we have that there must exist a pair of vertices which are a distance of $d+1$ apart. However, since a cycle power graph contains at least $2d+2$ vertices and each vertex can only be made adjacent to vertices which are no more than a distance of d away from it, we therefore have that these two vertices in our set of $d+2$ must not be adjacent. Thus, we see that the size of a maximum clique in our cycle-power graph is given by $d+1$. □

Now that we can describe the quantities $\omega(C_n^d)$ and $\Delta(C_n^d)$, we can determine the upper bound that Reed's conjecture suggests for a cycle-power graph and show that this bound holds for the chromatic number of a cycle-power graph.

Theorem 16. *All cycle-power graphs satisfy Reed's conjecture.*

Proof. Since for C_n^d, we have that $\omega(C_n^d) = d+1$ and $\Delta(C_n^d) = 2d$. In order for Reed's Conjecture to be satisfied for C_n^d, we must have that

$$\chi(C_n^d) \leq \left\lceil \frac{\omega(C_n^d) + \Delta(C_n^d) + 1}{2} \right\rceil = \left\lceil \frac{d+1+2d+1}{2} \right\rceil = \left\lceil \frac{d}{2} \right\rceil + d + 1.$$

Suppose that $n \bmod (d+1) = x$. Then we can color consecutive vertices with the colors, $1, 2, ..., d+1+\lceil \frac{x}{2} \rceil$. Then the next set of consecutive vertices along the largest cycle can be colored with the colors $1, 2, ..., d + 1 + \lfloor \frac{x}{2} \rfloor$. Any remaining number of vertices will be divisible by $d + 1$ and those vertices can be partitioned and properly colored by repeating the colors $1, 2, ...d + 1$ until we run out of vertices to color. Since, $0 \leq x \leq d$, at maximum, we will use $d + 1 + \left\lceil \frac{d}{2} \right\rceil$ colors, which is the maximum allowable number of colors that can be used in order for Reed's Conjecture to hold. This coloring scheme shows that it is possible to color any cycle-power graph such that Reed's conjecture will be satisfied.

□

Although Reed's conjecture gives an upper bound for the chromatic number of a graph, many cycle-power graphs can actually be colored with far fewer colors than the bound which Reed's conjecture suggests. As previously noted, any $d + 1$ consecutive vertices must each receive a unique color. To work towards establishing a chromatic number for a cycle-power graph, we can begin by establishing a general coloring scheme that can be used. If the set $\{1, 2, 3, ..., d + 1 + \left\lceil \frac{d}{2} \right\rceil\}$ denotes the set of colors that can be used to color our cycle-power graph, then we can let the colors $1, 2, ..., i$ denote a *chain* of colors. In order to color a cycle-power graph by covering the vertices of the graph with chains, we must use chains of length $d + 1$ or greater. This is so because any consecutive $d + 1$ vertices must each receive a different color since any $d + 1$ vertices form a clique. Such a coloring will be referred to as a *chain coloring* and the *chain chromatic number*, or the size of the largest chain required to properly chain-color our graph, will be denoted by χ^*.

Proposition 17. *Any cycle-power graph can be properly chain colored by using chains of only 2 distinct, consecutive lengths.*

Proof. Suppose that a cycle-power graph has been properly chain colored. Note that this is possible and a construction of this is provided in the proof Theorem 16. Then if we consider the length of a longest chain ℓ and the length of a shortest chain s, then if $\ell + s$ is even, we can consider using the vertices which these two chains cover to instead create two chains of length $\frac{\ell+s}{2}$. Otherwise, if $\ell + s$ is odd, then we can consider using the vertices which these two chains cover to instead create one chain of length $\frac{\ell+s+1}{2}$ and one of length $\frac{\ell+s-1}{2}$. We can continue repeating this process for the longest and shortest chains that are used. If $\ell \geq s + 2$, then each time we do this, the number of chains of the longest length will either decrease or remain the same. This process can be repeated until only two consecutive chain lengths are used. Note that it is possible, to only use one chain length to color some cycle-power graphs, but two distinct chain lengths is the maximum that will be needed. □

Since only two chain lengths, specifically two consecutive chain lengths, are needed, a proper chain-coloring of a cycle-power graph can be realized by finding a nonnegative linear combination of the the two chain lengths which which is equal to the number of vertices in the cycle-power graph. In other words, let j and $j+1$ be the lengths of the two chain lengths used. Note that $d+1 \leq j < j+1 \leq d+1+\lceil \frac{d}{2} \rceil$. We can now look for the smallest value of j such that we can write $n = aj + b(j+1)$ for some $a, b \in \{0, 1, 2, 3, ...\}$.

The following code (written in Python) is an algorithm that can be used to determine the chromatic number of C_n^d. The function chromatic takes in d and n, the two parameters of a cycle-power graph and return the number of colors that are required to chain color C_n^d either by using chains of one length of chains of two consecutive lengths. The value j will, in increasing order, take on the integer values from $d+1$ to $d+1+\lceil \frac{d}{2} \rceil$. We can then set i equal to n mod j. Then $n - i(j+1)$ will be divisible by j. If $n - i(j+1) \geq 0$ then we can write $n = aj + b(j+1)$ where a and b are nonnegative integers. If i was equal to 0, then we only need to used chains of length j and so $\chi^*(C_n^d) = j$. If $i \neq 0$, then we will need to use chains of lengths j and $j+1$ and so $\chi^*(C_n^d) = j+1$.

```
def chromatic(d,n):
   m=int(d+2+ceil(d/float(2)))
   j=d+1
   while j in range(d+1,m):
      i = n % j
      if n-i*(j+1) >= 0:
         if i==0:
            return ‘chromatic number:  ’  + str(j)
         else:
            return ‘chromatic number:  ’  + str(j+1)
      else:
         j+=1
```

Chapter 3

Complements of Cycle-Power Graphs

Like the cycle-power graph, the complement of an even-order cycle-power graph will contain a perfect matching. Additionally, if a particular cycle-power graph contains an odd hole, then the complement of that graph will contain an odd anti-hole. As mentioned in Chapter 3, a graph satisfying either of these criteria could possibly provide a counterexample to Reed's conjecture and for this reason, we will now turn our attention to the complements of cycle-power graphs.

Since in a cycle-power graph, any $d+1$ consecutive vertices form a clique, in the complement of a cycle-power graph, we must have that any $d+1$ consecutive vertices form an independent set. Given that $\Delta(C_n^d) = 2d$, we can see that $\Delta(\overline{C_n^d}) = n - 2d - 1$ We can also characterize the clique number of a cycle-power complement:

Proposition 18. *In the complement of a cycle-power graph on $k(d+1)+x$ vertices where $x \in \{0, 1, 2, ..., d\}$, the size of a largest clique is given by k.*

Proof. An independent set, or a set of vertices in which all vertices are pairwise nonadjacent, can be formed in a cycle power graph by partitioning the set of vertices into sets of $d+1$ consecutive vertices and possibly one additional set with d vertices or fewer. Then we can take the ith vertex from each each set of size $d+1$ to form our independent set. It isn't possible to include any additional vertices and still maintain an independent set since any $d+1$ consecutive vertices are all pairwise adjacent and all vertices in the one smaller set will be within a distance of less than $d+1$ from at least one vertex that is already in the set. Thus, to form the largest independent set possible, we can take one vertex, the ith vertex, from each of k sets of size $d+1$. This means then that $\omega(\overline{C^d_{k(d+1)+x}}) = \alpha(C^d_{k(d+1)+x}) = k$. □

Since in the complement of a cycle-power graph, any $d+1$ consecutive vertices form an independent set, we can color each of these vertices with the same color. Under this coloring scheme, $\overline{C^d_{k(d+1)+x}}$ will be colored with k colors if $x = 0$ and $k+1$ colors if $x \in \{1, 2, ..., d\}$. We can show now that these bounds will satisfy Reed's conjecture:

Theorem 19. *The complement of a cycle-power graph will satisfy Reed's conjecture.*

Proof. If the complement of a cycle-power graph is to satisfy Reed's conjecture then we must have that

$$\chi(C^d_{k(d+1)}) \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil = \left\lceil \frac{k + (n - 2d - 1) + 1}{2} \right\rceil = \left\lceil \frac{k+n}{2} \right\rceil - d$$

Let's first suppose that $x = 0$. Then making the appropriate substitution for n in the above expression, we see that if the complement of a cycle-power graph in which $(d+1)|n$ is to satisfy Reed's conjecture then we must have that

$$k \leq \left\lceil \frac{k + k(d+1)}{2} \right\rceil - d = \left\lceil \frac{k(d+2)}{2} \right\rceil - d.$$

Noting that a proper cycle-power graph on n vertices must satisfy $n \geq 2(d+1)$, we see that if Reed's conjecture is to hold, then the above inequality needs to hold for all $k \geq 2$ since the number of vertices in our cycle-power complement graph is being expressed as $n = k(d+1) + x$. We can analyze this inequality with the following two cases:

Case I: At least one of d and k is even. Then our inequality above can be simplified to $k \leq \frac{k(d+2)}{2} - d$. Further simplifying we see that this holds true whenever $k \geq 2$, as desired.

Case II: Both k and d are odd. Then our inequality above can be simplified to $k \leq \frac{k(d+2)+1}{2} - d$. Further simplifying, we see that this holds true whenever $k \geq 2 - \frac{1}{d}$, as desired.

Since this inequality is valid in either case, we can now see that Reed's conjecture has been proven for the complement of a cycle-power graph when $(d+1)|n$.

Let's now consider the case in which $(d+1) \nmid n$. Then in order for the complement of such a cycle-power graph to satisfy Reed's conjecture, we must have that

$$k+1 \leq \left\lceil \frac{k+k(d+1)+x}{2} \right\rceil - d = \left\lceil \frac{k(d+2)+x}{2} \right\rceil - d.$$

We can minimize the right side of the above inequality by taking $x = 1$ which simplifies the expression to

$$k+1 \leq \left\lceil \frac{k(d+2)+1}{2} \right\rceil - d.$$

Once again, we can get rid of the ceiling function by considering cases for the parity of k and d,

Case I: Both k and d are odd. Then our inequality above can be simplified to $k+1 \leq \frac{k(d+2)+1}{2}d$. Further simplifying, we see that this holds true whenever $k \geq 2 + \frac{1}{d}$. Since k is odd in this case, we only need for $k \geq 3$ and so Reed's conjecture is satisfied in this case.

Case II: At least one of k and d is even. Then our inquality above can be simplified to $k+1 \leq \frac{k(d+2)+2}{2} - d$. Further simplifying, we see that this inequaltiy holds for whenever $k \geq 2$, as desired.

Since Reed's conjecture holds in both of these cases for the complement of a cycle-power graph in which $(d+1) \nmid n$ and has also been shown to hold for the complement of a cycle-power graph when $(d+1)|n$, we can now conclude that Reed's conjecture holds for the complements of all cycle-power graphs. □

We can now show that the bounds that Reed's conjecture suggest for cycle-power complement graphs actually give the chromatic number for the complement of a cycle-power graph.

Theorem 20. *If* $n = k(d+1)$, *then* $\chi(\overline{C_n^d}) = k$. *If* $n = k(d+1) + x$ *where* $x \in \{1, 2, , ..., d\}$ *then* $\chi(\overline{C_n^d}) = k+1$.

Proof. First note that for a graph G on n vertices with and independent set of size α, $\frac{n}{\alpha} \leq \chi(G)$. This is so because each color class of an optimal coloring of G will be an independent set. Since the complement of a cycle-power graph on $n = k(d+1)$ vertices has an independence number of $d+1$, we therefore have that

$$k = \frac{n}{\alpha} \leq \chi(\overline{C_n^d}).$$

In theorem 19, we were also able to show that

$$\chi(\overline{C_n^d}) \leq k.$$

Combining these two inequalities, we therefore see that $\chi(\overline{C_n^d}) = k$.

Similarly, we can see that for a cycle-power complement graph on $n = k(d+1) + x$ vertices that

$$k + \frac{x}{d+1} = \frac{n}{\alpha} \leq \chi(\overline{C_n^d})$$

Since the chromatic number of a graph must be an integer and $x \in \{1, 2, ..., d\}$, we thus have that $k + 1 \leq \chi(\overline{C_n^d})$

In theorem 19, we were also able to show that

$$\chi(\overline{C_n^d}) \leq k + 1.$$

On combining these inequalities, we have that $\chi(\overline{C_n^d}) = k + 1$. $\square$

Chapter 4

Future Work

As we saw in Chapter 3, in 2008, Rabern was able to show that if a graph G is to be a counterexample to Reed's conjecture and has an even number of vertices, then $\overline{G}$ must have a perfect matching. Additionally, Reed's Conjecture is satisfied for all perfect graphs, but it is not known whether a graph which is not perfect will satisfy Reed's Conjecture. On combining these two conditions, one looking for a graph which holds as a counterexample to Reed's conjecture might consider studying another family of graphs which contains a perfect matching or an odd hole or antihole.

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