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Arithmetical Graphs, Riemann-Roch Structure for Lattices, and the Frobenius Number Problem

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Abstract

If $R = (r_1, \ldots, r_n) \in \mathbb{Z}^n_{>0}$ with $\gcd(r_1, \ldots, r_n) = 1$, calculating the Frobenius number of R is in general NP-hard. Dino Lorenzini defines the arithmetical graph, which naturally arises in arithmetic geometry, and a notion of genus, the g-number, that in specific cases coincides with the Frobenius number of R. A result of Dino Lorenzini's gives a method for quickly calculating upper bounds for the g-number of arithmetical graphs. We discuss the arithmetic geometry related to arithmetical graphs and present an example of an arithmetical graph that arises in this context. We also discuss the construction for Lorenzini's Riemann-Roch structure and how it relates to the Riemann-Roch theorem for finite graphs shown by Matthew Baker and Serguei Norine.

We then focus on the connection between the Frobenius number and arithmetical graphs. Using the Laplacian of an arithmetical graph and a formulation of chip-firing on the vertices of an arithmetical graph, we show results that can be used to find arithmetical graphs whose g-numbers correspond to the Frobenius number of R. We describe how this can be used to quickly calculate upper bounds for the Frobenius number of R.

Contents

List of Figures

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Chapter 1

Introduction and Background

1.1 Introduction

If $R = (r_1, \dots, r_n) \in \mathbb{Z}^n_{>0}$ with $\gcd(r_1, \dots, r_n) = 1$, calculating the Frobenius number of R is in general NP-hard. Dino Lorenzini defines the arithmetical graph, which naturally arises in arithmetic geometry, and a notion of genus, the g-number, that in specific cases coincides with the Frobenius number of R. Characterizing in which cases these notions coincide is of interest due to its implications for when an arithmetical graph has a Riemann-Roch structure and due to its relationship to a classic NP-hard problem. In this thesis, we discuss the connection between arithmetical graphs and the Frobenius number as well as show results that can be used to construct arithmetical graphs whose g-number corresponds to the Frobenius number of R.

In this chapter we discuss arithmetical graphs, how they arise in arithmetic geometry, and how they can be used to quickly calculate an upper bound for the Frobenius number of R.

1.2 Arithmetical Graphs

We give the definition of an arithmetical graph as introduced in Lorenzini (1989). Let G be a connected undirected multigraph with no loop edges on n vertices, $v_1, \dots, v_n$. Let A be the adjacency matrix of G, i.e. A is the matrix in which the $(i, j)^{\text{th}}$ entry is the number of edges between v_i and v_j. Let $\operatorname{diag}(\delta_1, \dots, \delta_n)$ be a diagonal matrix with $\delta_i \in \mathbb{Z}_{\geq 0}$, and define the Laplacian

$$M = \operatorname{diag}(\delta_1, \dots, \delta_n) - A.$$

Let $R = (r_1, \ldots, r_n) \in \mathbb{Z}_{>0}^n$ such that $\gcd(r_1, \ldots, r_n) = 1$.

Definition 1.1. The data (G, M, R) is an *arithmetical graph* if $MR = 0$.

The following example shows a way of specifying an arithmetical graph structure for any graph G.

Example 1.1. Let G be a connected undirected multigraph with no loop edges on n vertices, $v_1, \ldots, v_n$, and adjacency matrix A. Let d_i be the degree of v_i and

$$M = \operatorname{diag}(d_1, \ldots, d_n) - A.$$

Let $R = (1, \ldots, 1)$. Then (G, M, R) is an arithmetical graph. Note that in this case M is the usual graph Laplacian, so we see that the Laplacian of an arithmetical graph is a generalization of the usual graph Laplacian. We say that an arithmetical graph (G, M, R) with $R = (1, \ldots, 1)$ is *simple*.

We also define the linear rank of an arithmetical graph.

Definition 1.2. Let (G, M, R) be an arithmetical graph with

$$M = \operatorname{diag}(\delta_1, \ldots, \delta_n) - A,$$

where A is the adjacency matrix of G. The *linear rank* $g_0(M)$ of (G, M, R) is defined by

$$2g_0(M) - 2 = \sum_{i=1}^{n} r_i(\delta_i - 2).$$

Suppose that d_i is the degree of vertex v_i in G and a_{ij} is the number of edges between v_i and v_j. Then because $MR = 0$, $r_i\delta_i = \sum_{j \neq i} r_j a_{ji}$. And because $r_i d_i = \sum_{j \neq i} r_i a_{ij}$,

$$\sum_{i=1}^{n} r_i d_i = \sum_{i=1}^{n} r_i \delta_i.$$

Therefore we also have

$$2g_0(M) - 2 = \sum_{i=1}^{n} r_i(d_i - 2).$$

By writing g_0 in this form, we see that we can think of the linear rank as a generalization of the first Betti number of a graph.

1.3 Curves and Regular Models

We now introduce a setting in which arithmetical graphs naturally occur. Let $\mathcal{O}_K$ be a complete discrete valuation ring with an algebraically closed residue field k. Let K be the field of fractions of $\mathcal{O}_K$. Then define

$$S = \operatorname{Spec} \mathcal{O}_K, \qquad \eta = \operatorname{Spec} K, \qquad s = \operatorname{Spec} k.$$

Definition 1.3. Let X be a smooth proper geometrically connected curve over η. Then a proper flat morphism $\psi : \mathcal{X} \to S$ is a *regular model* of X if $\mathcal{X}$ is connected and regular and the generic fiber $\mathcal{X}_\eta$ of $\mathcal{X}$ is isomorphic to X as η schemes.

Note that the generic fiber is the fiber over the generic point $(0) \in S$. As a subscheme of S, we consider the point (0) as the spectrum of its residue field. The stalk of the sheaf of regular functions of $S = \operatorname{Spec} \mathcal{O}_K$ at (0) is

$$\mathcal{O}_{S,(0)} = (\mathcal{O}_K)_{(0)} = K.$$

Thus the residue field at (0) is K so the point as a subscheme of S is isomorphic to $\eta = \operatorname{Spec} K$. Thus the generic fiber is defined by the fiber product $\mathcal{X}_\eta = \mathcal{X} \times_S \eta$.

$$\begin{array}{ccc} \mathcal{X}_\eta & \longrightarrow & \eta \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\psi} & S \end{array}$$

Therefore when we consider $\mathcal{X}_\eta$ to be a scheme over η, the structure map is the fiber product projection morphism.

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_K$. Then we can consider the point $\mathfrak{m}$ as $\operatorname{Spec} \mathcal{O}_K/\mathfrak{m} = \operatorname{Spec} k = s$, the closed subscheme of S. Thus we get a special fiber defined by $\mathcal{X}_s = \mathcal{X} \times_S s$.

$$\begin{array}{ccc} \mathcal{X}_s & \longrightarrow & s \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\psi} & S \end{array}$$

As an effective Cartier divisor on $\mathcal{X}$, we can write

$$\mathcal{X}_s = \sum_{i=1}^{n} r_i C_i$$

where the C_i are the irreducible components of $\mathcal{X}$ and the r_i are the multiplicity of C_i in $\mathcal{X}_s$.

Consider the intersection matrix M where the $(i, j)^{\text{th}}$ entry is $(C_i \cdot C_j)$, the intersection number of C_i and C_j. Let G be the graph with vertices $v_1, \ldots, v_n$ such that the number of edges between v_i and v_j, $i \neq j$ is $(C_i \cdot C_j)$. It can be shown that when $i \neq j$, $(C_i \cdot C_j) \geq 0$, so this defines a multigraph. It can also be shown that if $R = (r_1, \ldots, r_n)$ then $MR = 0$. Because the generic fiber of $\mathcal{X}$ is geometrically connected, $\mathcal{X}_s$ is connected. Thus G is connected. Therefore the data $(G, -M, \gcd(r_1, \ldots, r_n)^{-1}R)$ is an arithmetical graph. Note that a type, as introduced in Artin and Winters (1971), is the data

$$(n, M, R, (p(C_1), \ldots, p(C_n)),$$

where $p(C_i)$ is the arithmetical genus of C_i. In Winters (1974), he characterizes which types appear as special fibers of regular models of curves, and in particular his results show the following fact described in the proof of (Lorenzini, 2012: 4.2).

Fact 1.1 (Winters)**.** Let (G, M, R) be an arithmetical graph. There exists a complete discrete valuation ring $\mathcal{O}_K$ with algebraically closed residue field k and field of fractions K and a smooth proper geometrically connected curve X of genus $g_0(M)$ over η with a regular model $\mathcal{X}$ over S such that the special fiber $\mathcal{X}_s$ has irreducible components $C_1, \ldots, C_n$ such that the dual graph is G, the intersection matrix is M, and the multiplicity of C_i in $\mathcal{X}_s$ is r_i.

Therefore all arithmetical graphs arise from this construction.

1.4 Riemann-Roch Structure for Integer Lattices

In Lorenzini (2012), a notion of a Riemann-Roch structure is defined for rank $n-1$ sublattices of $\mathbb{Z}^n$ that are perpendicular to a specified $R \in \mathbb{Z}^n_{>0}$. This notion of Riemann-Roch structure generalizes the Riemann-Roch structure for graphs defined in Baker and Norine (2007), and in particular is a natural setting to study Riemann-Roch properties of the arithmetical graphs defined in Lorenzini (1989). We begin by defining the setting for this Riemann-Roch structure for integer lattices.

We consider $\mathbb{Z}^n$ to be a group of divisors. The Riemann-Roch structure we define in this section will show why we want to think of $\mathbb{Z}^n$ as a group of divisors. In the case when our lattice is defined by a graph, as in Example 1.2, we have the graph divisors proposed in Baker and Norine (2007). In this case we can think of these divisors as the free abelian group on the vertices of the graph.

We say a divisor $D \in \mathbb{Z}^n$ is *effective* if its entries are all nonnegative. Let $R = (r_1, \ldots, r_n) \in \mathbb{Z}^n_{>0}$ be such that $\gcd(r_1, \ldots, r_n) = 1$. For any divisor $D \in \mathbb{Z}^n$, we define its degree with respect to R to be

$$\deg_R(D) = D \cdot R.$$

We define the rank $n-1$ lattice Λ_R to be

$$\Lambda_R = \{D \in \mathbb{Z}^n \mid \deg_R(D) = 0\}.$$

Let $\Lambda \subset \Lambda_R$ be an integer lattice of rank $n-1$. We use Λ to define divisor classes.

Definition 1.4. The *Picard group* with respect to Λ is

$$\mathrm{Pic}(\Lambda) = \mathbb{Z}^n/\Lambda.$$

We define $[D] \in \mathrm{Pic}(\Lambda)$, the divisor class of $D \in \mathbb{Z}^n$, to be the image of D in $\mathrm{Pic}(\Lambda)$.

Example 1.2. Let G be a connected undirected multigraph with no loop edges on n vertices, $v_1, \ldots, v_n$. Let M be the Laplacian of G,

$$M = \mathrm{diag}(d_1, \ldots, d_n) - A,$$

where A is the adjacency matrix of G and d_i is the degree of vertex v_i. Then we define the rank $n-1$ lattice associated to this graph to be

$$\Lambda_G = \{D \in \mathbb{Z}^n \mid \text{there exists } D' \in \mathbb{Z}^n \text{ such that } D = MD'\}.$$

In other words Λ_G is the image of M acting on $\mathbb{Z}^n$ by left multiplication. Let $R = (1, \ldots, 1)$. Then because $MR = 0$ and M is symmetric

$$\Lambda_G \subset \Lambda_R.$$

We use $\mathrm{Pic}(\Lambda)$ to define the g-number, a notion of genus, of the lattice Λ. This definition of the g-number is such that if Λ has a Riemann-Roch structure, the Riemann-Roch theorem for the lattice can be stated analogously to the classical Riemann-Roch theorem.

Definition 1.5. The *g-number* of Λ, denoted $g(\Lambda)$, is the least integer such that if $D \in \mathbb{Z}^n$ with $\deg_R(D) \geq g(\Lambda)$, there exists an effective divisor $E \in \mathbb{Z}^n$ such that

$$[D] = [E] \in \mathrm{Pic}(\Lambda).$$

It is shown in (Lorenzini, 2012: 2.5) that $g(\Lambda)$ exists for any $\Lambda \subset \Lambda_R$ of rank $n-1$. Note that because R has strictly positive entries, if E is an effective divisor, then

$$\deg_R(E) \geq 0.$$

Also because $\Lambda \subset \Lambda_R$, all divisors in the same divisor class have the same degree. Therefore

$$g(\Lambda) \geq 0.$$

The next example, given in (Lorenzini, 2012: 2.4), shows that we can think of the g-number of Λ as a generalization of the Frobenius number.

Example 1.3 (Lorenzini)**.** The Frobenius number of $r_1, \ldots, r_n \in \mathbb{Z}$ with $\gcd(r_1, \ldots, r_n) = 1$ is the largest integer that cannot be written in the form $\sum_{i=1}^n x_i r_i$ for any $x_i \geq 0$. Let $g(r_1, \ldots, r_n)$ be one more than their Frobenius number. Then

$$g(r_1, \ldots, r_n) = g(\Lambda_R).$$

Proof. Let $D \in \mathbb{Z}^n$ with $\deg_R(D) \geq g(r_1, \ldots, r_n)$. Then there exists effective $E = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that

$$\deg_R(D) = \sum_{i=1}^n x_i r_i = \deg_R(E).$$

Thus $\deg_R(D-E) = 0$ so $D - E \in \Lambda_R$. Thus $[D] = [E] \in \operatorname{Pic}(\Lambda_R)$ so

$$g(r_1, \ldots, r_n) \geq g(\Lambda_R).$$

Also by definition, $\deg_R(E) \neq g(r_1, \ldots, r_n) - 1$ for any effective divisor E. Because divisors in the same class have the same degree, this implies that any divisor D with $\deg_R(D) = g(r_1, \ldots, r_n) - 1$ is not equivalent to any effective divisor. Note also that because $\gcd(r_1, \ldots, r_n) = 1$, there exists a divisor D with $\deg_R(D) = g(r_1, \ldots, r_n) - 1$. Thus

$$g(r_1, \ldots, r_n) = g(\Lambda_R).$$

$\square$

We can now define the Riemann-Roch structure on the lattice Λ proposed in Lorenzini (2012).

Definition 1.6. A *Riemann-Roch* structure on Λ is a function $h : \mathbb{Z}^n \to \mathbb{Z}_{\geq 0}$ that is constant on divisor classes and satisfies the following.

1. There exists a divisor K such that for all divisors D

$$h(D) - h(K - D) = \deg_R(D) + 1 - g(\Lambda).$$

2. If D is a divisor with $\deg_R(D) \leq 0$ then

$$h(D) = \begin{cases} 1, & [D] = [0] \in \mathrm{Pic}(\Lambda) \\ 0, & \text{otherwise} \end{cases}.$$

3. For any divisor D, there exists an effective divisor E such that $[D] = [E] \in \mathrm{Pic}(\Lambda)$ if and only if $h(D) \geq 1$.

The next example shows that this Riemann-Roch structure is a generalization of the setting for the Riemann-Roch theorem for graphs shown in Baker and Norine (2007).

Example 1.4 (Lorenzini)**.** Let G, M, R be defined as in Example 1.2. For $D \in \mathbb{Z}^n$, let $h(D)$ be the least nonnegative integer for which there exists an effective divisor E such that $\deg_R(E) = h(D)$ and $D - E$ is not equivalent to any effective divisors in $\mathrm{Pic}(\Lambda_G)$. Then

1. if m is the number of edges in G then

$$g(\Lambda_G) = m - n + 1,$$

2. and the function h is a Riemann-Roch structure on Λ_G.

Proof. We will show that this is just a restatement of the Riemann-Roch theorem for graphs shown in Baker and Norine (2007).

1. Let d_i be the degree of vertex v_i in G. Define the divisor K to be

$$K = (d_1 - 2, \dots, d_n - 2).$$

Thus K is equivalent to the canonical divisor defined in (Baker and Norine, 2007: 1.6). If r is the dimension function defined in (Baker and Norine, 2007: 1.6), then $h(D) = r(D) + 1$. Thus by the Riemann-Roch theorem for graphs, for any divisor D

$$h(D) - h(K - D) = \deg_R(D) + 1 - (m - n + 1).$$

If $\deg_R(D) \geq m - n + 1$

$$h(D) \geq h(K - D) + 1 \geq 1.$$

Thus by definition of h, $D = D - 0$ is equivalent to an effective divisor in $\text{Pic}(\Lambda_G)$, so
$$g(\Lambda_G) \leq m - n + 1.$$
Also by Corollary 3.4 in (Baker and Norine, 2007: 3.2), there exists a divisor degree $m - n$ that is not equivalent to any effective divisors in $\text{Pic}(\Lambda_G)$. Thus
$$g(\Lambda_G) = m - n + 1.$$

2. From the Riemann-Roch theorem for graphs, it follows that h satisfies properties (1) and (3) of a Riemann-Roch structure. Because $R = (1, \ldots, 1)$, the only effective divisor of degree 0 is $0 \in \mathbb{Z}^n$. Thus if $\deg_R(D) \leq 0$, either $h(D) = 0$ or $[D] = [0]$, in which case $h(D) = h(0) = 1$. Thus h is a Riemann-Roch structure for Λ_G.

$\square$

1.5 Arithmetical Graphs and Riemann-Roch for Lattices

We have shown that we can associate to a graph G a lattice Λ_G such that the Riemann-Roch structure for Λ_G is a restatement of the Riemann-Roch theorem for graphs. We can similarly define an integer lattice associated to an arithmetical graph, but unlike in the case of graphs we are not guaranteed that the associated lattice has a Riemann-Roch structure.

Let (G, M, R) be an arithmetical graph. Then we define the rank $n - 1$ lattice associated to this arithmetical graph to be

$$\Lambda_M = \text{im}\, M = \{D \in \mathbb{Z}^n \mid \text{there exists } D' \in \mathbb{Z}^n \text{ such that } D = MD'\}.$$

Then because $MR = 0$ and M is symmetric

$$\Lambda_M \subset \Lambda_R.$$

Note that when $R = (1, \ldots, 1)$ and $M = \text{diag}(d_1, \ldots, d_n) - A$, d_i is the degree of vertex v_i, we get that $\Lambda_M = \Lambda_G$.

A natural question to ask is if there is some relationship between $g(\Lambda_M)$ and $g_0(M)$. The following fact is shown in (Lorenzini, 2012: 4.2).

Fact 1.2 (Lorenzini)**.** Let (G, M, R) be an arithmetical graph. Then $g(\Lambda_M) \leq g_0(M)$. Also if $g(\Lambda_M) = g_0(M)$, Λ_M has a Riemann-Roch structure.

Lorenzini proves this fact by using Fact 1.1, which guarantees the existence of a curve X of genus $g_0(M)$ with regular model $\mathcal{X}$ whose special fiber defines (G, M, R). He uses the Riemann-Roch theorem for curves on X and the fact that X has genus $g_0(M)$ to guarantee the existence of an effective divisor $E \in \mathbb{Z}^n$ that is equivalent to $D \in \mathbb{Z}^n$ in $\mathrm{Pic}(\Lambda_M)$ for any D with degree at least $g_0(M)$.

Because of this relationship between $g_0(M)$ and $g(\Lambda_M)$, we might think that g_0 may be determined by the lattice Λ_M. The following example shows that there are arithmetical graphs $(G, M, R), (G', M', R)$ with $R \neq (1, \ldots, 1)$ such that $\Lambda_M = \Lambda_{M'}$ and $g_0(M) \neq g_0(M')$. Thus the linear rank does not only depend on the lattice induced by the arithmetical graph.

Example 1.5. Let $R = (1, 2, 5)$ and G, G' be graphs on the vertices v_1, v_2, v_3 defined by the figure below.

Figure 1.1 Graphs G and G' labeled with edge multiplicities

Define M and M' by

$$M = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 26 & -10 \\ 0 & -10 & 4 \end{pmatrix}, \qquad M' = \begin{pmatrix} 58 & -4 & -10 \\ -4 & 2 & 0 \\ -10 & 0 & 2 \end{pmatrix}.$$

Then (G, M, R) and (G', M', R) are arithmetical graphs with

$$\Lambda_M = \Lambda_{M'}, \qquad g_0(M) \neq g_0(M').$$

Proof. We see that $(G, M, R), (G', M', R)$ satisfy the definition of arithmetical graphs. Also define

$$P = \begin{pmatrix} 11 & -1 & -2 \\ -7 & 0 & 1 \\ -20 & 0 & 3 \end{pmatrix}.$$

Because $\det(P) = -1$, $P \in \mathrm{GL}_n(\mathbb{Z})$. Thus because $M' = MP$

$$\Lambda_M = \operatorname{im} M = \operatorname{im} M' = \Lambda_{M'}.$$

Also

$$g_0(M) = 31 \neq 29 = g_0(M').$$

□

An interesting problem is to classify which arithmetical graphs satisfy $g(\Lambda_M) = g_0(M)$. As any arithmetical graph (G, M, R) with $g(\Lambda_M) = g_0(M)$ is guaranteed to have a Riemann-Roch structure, such a classification would help us understand which integer lattices have a Riemann-Roch structure.

This problem of classifying arithmetical graphs satisfying $g(\Lambda_M) = g_0(M)$ also has interesting applications in the study of algorithms. Consider the problem of finding the Frobenius number for $R = (r_1, \ldots, r_n)$. In general, this problem is NP-hard. We can find an upper bound for this problem because for any arithmetical graph (G, M, R),

$$g(r_1, \ldots, r_n) = g(\Lambda_R) \leq g(\Lambda_M) \leq g_0(M),$$

and from the definition, $g_0(M)$ can be computed with a polynomial number of arithmetic operations. It is shown in (Lorenzini, 2012: 4.3) that in general $g_0(M) - g(\Lambda_M)$ can be arbitrarily large.

If in polynomial time we can find an arithmetical graph (G, M, R) such that $g(\Lambda_M) = g(\Lambda_R)$, such as when $\Lambda_M = \Lambda_R$, and $g(\Lambda_M) = g_0(M)$, then in polynomial time we can find the Frobenius number, which in this case would be $g_0(M) - 1$. Thus the problem of finding families of arithmetical graphs that satisfy $g(\Lambda_M) = g_0(M)$ can lead to algorithms that solve the Frobenius number problem in special cases in polynomial time.

We can also use this connection to the Frobenius number problem to find examples when $g(\Lambda_M) < g_0(M)$. For example, if we have a collection $\mathcal{G}$ of arithmetical graphs such that for every $R = (r_1, \ldots, r_n)$, we can find in polynomial time a $(G, M, R) \in \mathcal{G}$ such that $g(\Lambda_R) = g(\Lambda_M) = g_0(M)$, then the existence of $\mathcal{G}$ would imply the existence of a polynomial time algorithm for the Frobenius number problem. Thus such a collection $\mathcal{G}$ is unlikely to exist. In fact if any NP-hard subproblem of the Frobenius number problem can be reduced to finding $g(\Lambda_M)$ for a collection $\mathcal{G}$ of arithmetical graphs, then we are likely to find examples of arithmetical graphs in $\mathcal{G}$ for which $g(\Lambda_M) < g_0(M)$.

In (Lorenzini, 2012: 4.5), it is shown that when $a, b > 1$, there is always an arithmetical graph (G, M, R) such that $\Lambda_M = \Lambda_R$ and $g(a, b) = g(\Lambda_R) = g(\Lambda_M) = g_0(M)$.

Note that given R, in general we know of no algorithm for finding an arithmetical graph (G, M, R) for which $\Lambda_M = \Lambda_R$. In Chapter 3, we discuss results that reduce this problem to finding a certain change of basis matrix.

Chapter 2

Arithmetic Geometry

In this chapter, we discuss topics in arithmetic geometry that have motivated the material in this thesis. We begin by describing an example of a curve and model and the associated arithmetical graph. We then conclude this chapter with an introduction to meromorphic functions and divisors on a scheme. As a reference for the algebraic and arithmetic geometry discussed in this section, we recommend Hartshorne (1977) and Liu (2002)

2.1 Example of a Model

In this section, we show an example of an arithmetical graph constructed as in Section 1.3. In particular, we show an example in which the special fiber has non-reduced components, and therefore the associated arithmetical graph has nontrivial vertex multiplicities. We use the notation from Section 1.3, so let $\mathcal{O}_K$ be a discrete valuation ring with field of fractions K and algebraically closed residue field k, and let $S = \operatorname{Spec} \mathcal{O}_K, \eta = \operatorname{Spec} K, s = \operatorname{Spec} k$.

Let $\mathcal{O}_K = \mathbb{C}[[t]]$, the ring of formal power series in t. Then $K = \mathbb{C}((t))$, the field of formal Laurent series, and $k = \mathbb{C}$. Note also that the algebraic closure of K is $\overline{K} = \mathbb{C}\{\{t\}\}$, the field of Puiseux series. Let $F \in \mathcal{O}_K[u, v, w]$ be the homogeneous polynomial

$$F = u^3 - u^2 w + tv^2 w + tvw^2.$$

Considering F as an element of $K[u, v, w]$, let $X \to \eta$ be the plane curve

$$X = \operatorname{Proj} K[u, v, w]/(F).$$

Thus $X \to \eta$ is projective and therefore proper.

Also consider $X_{\overline{K}} \to \operatorname{Spec} \overline{K}$ defined by

$$X_{\overline{K}} = X \times_\eta \operatorname{Spec} \overline{K}.$$

Then

$$X_{\overline{K}} \cong \operatorname{Proj} \overline{K}[u, v, w]/(F),$$

where F is considered as an element of $\overline{K}[u, v, w]$. Because every projective plane curve is connected, $X_{\overline{K}}$ is connected, and therefore X is geometrically connected. To show that X satisfies the conditions specified in Section 1.3, we need to show that its structure morphism is smooth.

Proposition 2.1. The structure morphism $\varphi : X \to \eta$ is smooth.

Proof. For any point $p \in X$, let $\varphi_p^\# : K = \mathcal{O}_{\eta,\varphi(p)} \to \mathcal{O}_{X,p}$ be the induced ring homomorphism on the stalks. Because tensoring with a vector space preserves exactness, all ring homomorphisms from a field are flat. Therefore for all $p \in X$, $\varphi_p^\#$ is flat, so φ is flat. Finally because η has only one point, we only need to check that X is smooth. Thus we only need to show that $X_{\overline{K}}$ is regular.

To show that $X_{\overline{K}}$ is regular, we only need to show that it is regular at its closed points $p = (u : v : w) \in X_{\overline{K}} \subset \mathbb{P}^2_{\overline{K}}$. By definition, $X_{\overline{K}}$ is regular at p if $\mathcal{O}_{X_{\overline{K}},p}$ is regular. Thus regularity is a local property, and we can check that the closed points are regular in the standard affine patches of $X_{\overline{K}}$, which we denote

$$\begin{aligned} X_0 &= \operatorname{Spec} \overline{K}[x, y]/(f_0), \\ X_1 &= \operatorname{Spec} \overline{K}[x, z]/(f_1), \\ X_2 &= \operatorname{Spec} \overline{K}[y, z]/(f_2), \end{aligned}$$

where

$$\begin{aligned} f_0 &= x^3 - x^2 + ty^2 + ty, \\ f_1 &= x^3 - x^2 z + tz + tz^2, \\ f_2 &= 1 - z + ty^2 z + tyz^2. \end{aligned}$$

We can use the Jacobian criterion to check that points are regular in these closed subschemes of $\mathbb{A}^2_{\overline{K}}$.

First consider X_0. At any closed point $(x, y) \in \mathbb{A}^2_{\overline{K}}$, the Jacobian is

$$J_{(x,y)} = (3x^2 - 2x, 2ty + t).$$

Thus the Jacobian is zero only at the points $(0, 1/2)$ and $(2/3, 1/2)$. Because f_0 does not vanish at these points, they are not in X_0. Therefore all closed points of $X_{\overline{K}}$ contained in X_0 are regular.

Now consider X_1. At any closed point $(x, z) \in \mathbb{A}^2_{\overline{K}}$, the Jacobian is

$$J_{(x,z)} = (3x^2 - 2xz, -x^2 + t + 2tz).$$

Because all points $(u : v : w) \in X_{\overline{K}}$ with $w \neq 0$ are contained in X_0, we only need to check the points where $z = 0$. Then

$$J_{(x,0)} = (3x^2, -x^2 + t) \neq 0$$

for any $x \in \overline{K}$. Thus all closed points of $X_{\overline{K}}$ contained in X_1 are regular.

Finally we consider X_2. The only closed point of $\mathbb{P}^2_{\overline{K}}$ that remains to be checked is $(1 : 0 : 0)$, which corresponds to when $y = z = 0$ in X_2. Because $f_2(0, 0) = 1 \neq 0$, $(1 : 0 : 0) \notin X_{\overline{K}}$.

Therefore $X_{\overline{K}}$ is regular at all closed points and is therefore regular. Thus X is smooth, so $X \to \eta$ is smooth. $\square$

Consider $\mathcal{X} \to S$ defined by

$$\mathcal{X} = \operatorname{Proj} \mathcal{O}_K[u, v, w]/(F).$$

Then its generic fiber is

$$\mathcal{X}_\eta = \mathcal{X} \times_S \eta \cong \operatorname{Proj} K[u, v, w]/(F) = X.$$

Because $\mathcal{X}$ is a projective plane curve, it is connected. A computation of the Hilbert polynomials of the fibers of $\mathcal{X} \to S$ shows that it is flat. If $\mathcal{X}$ is regular, we can also show that the structure morphism is flat by the following argument.

Proposition 2.2. If $\mathcal{X}$ is regular, the structure morphism $\phi : \mathcal{X} \to S$ is flat.

Proof. Consider the standard affine cover of $\mathcal{X}$

$$\begin{aligned} \mathcal{X}_0 &= \operatorname{Spec} \mathcal{O}_K[x, y]/(f_0), \\ \mathcal{X}_1 &= \operatorname{Spec} \mathcal{O}_K[x, z]/(f_1), \\ \mathcal{X}_2 &= \operatorname{Spec} \mathcal{O}_K[y, z]/(f_2), \end{aligned}$$

where

$$\begin{aligned} f_0 &= x^3 - x^2 + ty^2 + ty, \\ f_1 &= x^3 - x^2z + tz + tz^2, \\ f_2 &= 1 - z + ty^2z + tyz^2. \end{aligned}$$

Because this is a cover by spectra of Noetherian rings, $\mathcal{X}$ is Noetherian. Thus because $\mathcal{X}$ is regular, connected, and Noetherian, it is integral. Therefore the coordinate rings $\mathcal{O}_{\mathcal{X}_i}(\mathcal{X}_i)$ are integral domains. Note that ϕ restricted to $\mathcal{X}_i$ is associated to the ring homomorphism $\mathcal{O}_K \to \mathcal{O}_{\mathcal{X}_i}(\mathcal{X}_i)$ given by the inclusion of $\mathcal{O}_K$ into the polynomial ring followed by the quotient map. Because the polynomials f_i have strictly positive degree, these ring homomorphisms are injective. Then because these are injective ring homomorphisms into integral domains, they give each $\mathcal{O}_{\mathcal{X}_i}(\mathcal{X}_i)$ a torsion-free module structure over $\mathcal{O}_K$. Then because $\mathcal{O}_K$ is a discrete valuation ring, these ring homomorphisms are flat. Therefore ϕ restricted to each $\mathcal{X}_i$ is flat, so $\phi : \mathcal{X} \to S$ is flat. $\square$

We now inspect the special fiber of $\mathcal{X}$ to get an arithmetical graph.

Let f be the image of F in $k[u, v, w]$. Then the special fiber of $\mathcal{X}$ is

$$\mathcal{X}_s = \mathcal{X} \times_S s \cong \operatorname{Proj} k[u, v, w]/(f),$$

and

$$f = u^3 - u^2 w = u^2(u - w).$$

Therefore $\mathcal{X}_s = C_1 + 2C_2$ is the union of a line C_1 and a double line $2C_2$ in the complex projective plane. Then by the Bezout identity we get that the intersection number is $C_1 \cdot 2C_2 = 2$. Thus the vertex multiplicities vector is $R = (1, 2)$, and the dual graph is G as shown below:

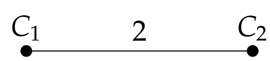

Figure 2.1 Dual graph G labeled with edge multiplicity

Thus the Laplacian is

$$M = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix},$$

and $g_0(M) = 1$, the genus of the cubic plane curve X. Using either the Smith normal form as discussed in Lorenzini (1989) or the chip-firing techniques discussed in Chapter 4, it can also be shown that for this arithmetical graph, $\Lambda_M = \Lambda_R$. Thus $g(\Lambda_M) = g(1, 2) = 0$.

We see in this example that although X is a reduced, nonsingular, and irreducible curve in a flat family of curves $\mathcal{X}$, the special fiber $\mathcal{X}_s$ is nonreduced, singular, and reducible. In particular, the polynomial F that defines X is irreducible in K, but its image f in the residue field, given by

$t \mapsto 0$, can be factored. Thus X, a smooth cubic curve, degenerates to $\mathcal{X}_s$, a union of lines.

2.2 Meromorphic Functions on Schemes

The theory of divisors on graphs and lattices parallels the theory of divisors on curves and schemes in general. In this section, to give context for the theory and terminology of divisors on graphs, we describe Cartier divisors on schemes. We first define the notion of meromorphic functions on a scheme. Meromorphic functions are defined locally by fractions of regular functions, and we will see that the sheaf of meromorphic functions generalizes the notion of rational functions. Note that in this section, all rings will be commutative rings with identity, and all ring homomorphisms will be morphisms of commutative rings with identity and therefore will preserve identity.

Definition 2.1. Let A be a ring. Elements of A are called *regular* if they are not zero divisors. Let $R(A)$ denote the group of regular elements of A under the multiplication operation of A.

We can then define the *total ring of fractions* of A by

$$\mathrm{Frac}(A) = R(A)^{-1}A,$$

the localization of A where we invert all regular elements. If A is an integral domain, then $\mathrm{Frac}(A)$ is the usual field of fractions of A. Because we only invert the regular elements of A, the canonical homorphism $A \to \mathrm{Frac}(A)$ is injective and therefore A can be thought of as a subring of $\mathrm{Frac}(A)$. We can use this total ring of fractions to define meromorphic functions for schemes that are not necessarily integral.

Definition 2.2. Let X be a scheme and $\mathcal{O}_X$ be its structure sheaf of regular functions. We define the sheaf $\mathcal{R}_X$ on X to be the sheaf of groups such that for all open $U \subset X$

$$\mathcal{R}_X(U) = \{a \in \mathcal{O}_X(U) \,|\, a_x \in R(\mathcal{O}_{X,x}) \text{ for all } x \in U\}.$$

Thus $\mathcal{R}_X(U)$ is the multiplicative subgroup of $\mathcal{O}(U)$ that contains the elements of $\mathcal{O}(U)$ whose images in the stalks of U are regular.

Definition 2.3. Let $\mathcal{K}'_X$ be the presheaf on X such that for all open $U \subset X$,

$$\mathcal{K}'_X(U) = \mathcal{R}_X(U)^{-1}\mathcal{O}_X(U),$$

and the restriction maps coincide with the restriction maps of $\mathcal{O}_X$.

Note that because elements of $\mathcal{K}'_X(U)$ are fractions of elements of $\mathcal{O}_X(U)$ and because $\mathcal{O}_X(U)$ is a subring of $\mathcal{K}'_X(U)$, the extension to $\mathcal{K}'_X(U)$ of the restriction maps on $\mathcal{O}_X(U)$ is defined and unique. Therefore there is a unique presheaf $\mathcal{K}'_X$ defined as above.

Definition 2.4. The *sheaf of stalks of meromorphic functions* on X is the sheaf $\mathcal{K}_X$ associated to the presheaf $\mathcal{K}'_X$.

Because the canonical morphism $\mathcal{K}'_X \to \mathcal{K}_X$ induces isomorphisms on the stalks, the elements of $\mathcal{K}_X(U)$ are locally fractions of elements of $\mathcal{O}_X(U)$. The following examples illustrate that $\mathcal{K}_X$ generalizes the notion of rational functions on integral schemes.

Example 2.1. We say a scheme X is *integral* if it is irreducible as a topological space and if at each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is an integral domain. We say that $x \in X$ *specializes* to $y \in X$ if $y \in \overline{\{x\}}$. A point $\xi \in X$ is called *generic* if $x \neq \xi$ implies that x does not specialize to ξ. The irreducible components of X are precisely $\{\overline{\{\xi\}} \mid \xi \text{ is a generic point of } X\}$.

Let X be an integral scheme. Then there is a unique generic point ξ. The *field of rational functions* of X, denoted $K(X)$, is the stalk $\mathcal{O}_{X,\xi}$. Then $\mathcal{K}_X$ is the constant sheaf $K(X)$.

Note that in any open affine subscheme $\mathrm{Spec}(A) \subset X$ containing ξ, the definition of generic point implies that ξ is generic in $\mathrm{Spec}(A)$. Because X is integral, A is an integral domain. Therefore $\xi = (0) \in \mathrm{Spec}(A)$ so $\mathcal{O}_{X,\xi} \cong A_{(0)}$, where $A_{(0)}$ denotes the localization of A at the ideal (0). Thus $K(X) = \mathcal{O}_{X,\xi}$ is the field of fractions of A, and $K(X)$ is a field. This example shows that the sheaf of stalks of meromorphic functions generalizes rational functions on X.

Example 2.2. Let $X = \mathrm{Spec}(A)$ where A is an integral domain. Then the generic point of X is the ideal (0) and $K(X) = \mathrm{Frac}(A)$. Therefore $\mathcal{K}_X$ is the the constant sheaf $\mathrm{Frac}(A)$.

More specifically, let X be an integral affine algebraic variety over field k so $A = k[x_1, \ldots, x_n]/I$ for a prime ideal I. Then we see that the elements of $\mathcal{K}_X(U)$ are fractions of polynomials on the connected components of U. Thus $\mathcal{K}_X$ generalizes the classical notion of rational functions on an affine algebraic variety.

2.3 Cartier Divisors on Schemes

Definition 2.5. Let X be a scheme. Then the *group of Cartier divisors* on X is

$$\mathrm{Div}(X) = H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*),$$

the 0th cohomology group of X with values in $\mathcal{K}_X^*/\mathcal{O}_X^*$. Note that $\mathcal{K}_X^*$ and $\mathcal{O}_X^*$ denote the sheaves of the groups of units of $\mathcal{K}_X$ and $\mathcal{O}_X$, respectively.

Note that as a consequence of the definition of Čech cohomology,

$$H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong \mathcal{K}_X^*/\mathcal{O}_X^*(X),$$

the global sections of $\mathcal{K}_X^*/\mathcal{O}_X^*$.

Definition 2.6. The elements of $D \in \mathrm{Div}(X)$ are called *divisors*. A divisor $D \in \mathrm{Div}(X)$ is called *principal* if it is in the image of the canonical homomorphism

$$H^0(X, \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

A divisor is called *effective* if it is in the image of the canonical homomorphism

$$H^0(X, \mathcal{O}_X \cap \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

Let $D \in \mathrm{Div}(X)$. By the definition of $\mathrm{Div}(X) \cong \mathcal{K}_X^*/\mathcal{O}_X^*(X)$ and the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$, D can be represented by $\{(U_i, f_i)\}_i$, where $\{U_i\}_i$ is an open cover of X, each f_i is a fraction of regular elements of $\mathcal{O}_X(U_i)$ and for every i, j there exists $f \in \mathcal{O}_X(U_i \cap U_j)^*$ such that

$$f_i = f_j f$$

when restricted to $U_i \cap U_j$.

Suppose D_1 and D_2 are represented by $\{(U_i, f_i)\}_i$ and $\{(V_j, g_j)\}_j$, respectively. Then $D_1 = D_2$ if for each i, j there exists $f \in \mathcal{O}_X^*(U_i \cap V_j)$ such that

$$f_i = g_j f$$

when restricted to $U_i \cap V_j$. Thus we see that divisors on X can be thought of locally as fractions of regular functions up to multiplication by regular functions with multiplicative inverses.

A divisor $D \in \mathrm{Div}(X)$ is effective if it can be represented by $\{(U_i, f_i)\}_i$ with each $f_i \in \mathcal{O}_X(U_i)$, and D is principal if it can be represented by $\{(X, f)\}$.

Definition 2.7. Define the group $\mathrm{CaCl}(X)$ by

$$\mathrm{CaCl}(X) = \mathrm{Div}(X)/\sim,$$

where $D_1 \sim D_2$ if $D_1 - D_2$ is principal.

Note that when X is Noetherian and reduced, $\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$, where the Picard group $\mathrm{Pic}(X)$ is the group of invertible $\mathcal{O}_X$-modules under the operation of tensor product.

Suppose X is a Noetherian regular integral scheme. Then X is normal. If $x \in X$ such that $\dim \mathcal{O}_{X,x} = 1$, we say that x has codimension 1. Thus $\mathcal{O}_{X,x}$ is a normal Noetherian local ring of dimension 1 and thus a principal ideal domain. Therefore $\mathcal{O}_{X,x}$ is a local principal ideal domain and thus a discrete valuation ring. Thus $\mathcal{O}_{X,x}$ has an associated normalized valuation that can be used to define the multiplicity of a divisor at a point of codimension 1. These multiplicities capture the intuitive notion of zero or pole order for meromorphic functions. These multiplicities can be used to define a map from $\mathrm{Div}(X)$ to the group of formal sums of codimension 1 points in X. These formal sums of codimension 1 points are called Weil divisors, and when X is a Noetherian regular integral scheme, this map gives an isomorphism between the group of Cartier divisors and Weil divisors.

Therefore when X is a Noetherian regular integral scheme, divisors can be thought of as locally meromorphic functions up to multiplication by units or as formal sums of codimension 1 points. The coefficients in the formal sum correspond to the zero or pole order of the divisor, and thus effective divisors, which are locally regular, correspond to formal sums with nonnegative coefficients. This formal sum interpretation of divisors is analogous to the divisors on finite graphs defined in Baker and Norine (2007) and thus the arithmetical graph divisors discussed in this thesis.

Chapter 3

Arithmetical Graph Laplacian

3.1 Laplacians and Connectedness

To reduce the Frobenius number problem for R to finding $g(M)$ for an arithmetical graph (G, M, R), we can construct an arithmetical graph such that $\Lambda_M = \Lambda_R$. We now discuss some results that will allow us to do this.

We will show results that are analogous to a result for Laplacians of undirected multigraphs. First we will characterize the quadratic form over $\mathbb{Q}$ that corresponds to M.

Proposition 3.1. Let $(G, M, R), R = (r_1, \ldots, r_n)$ satisfy the definition of an arithmetical graph except G is not necessarily connected. Let G have vertices $v_1, \ldots, v_n$ and edge multiset $E(G)$. Then for any $x = (x_1, \ldots, x_n)^\top \in \mathbb{Q}^n$,

$$x^\top M x = \sum_{\{v_i, v_j\} \in E(G)} \left(x_i \sqrt{\frac{r_j}{r_i}} - x_j \sqrt{\frac{r_i}{r_j}} \right)^2 .$$

Note that we account for edge multiplicities in the sum over $E(G)$.

Proof. Let $M = \operatorname{diag}(\delta_1, \ldots, \delta_n) - A$, where A is the adjacency matrix of G. Then because $MR = 0$ for any $i \in \{1, \ldots, n\}$

$$\delta_i = \sum_{\{v_i, v_j\} \in E(G)} \frac{r_j}{r_i}.$$

Thus

$$\begin{aligned}
x^\top M x &= \sum_{i=1}^{n} x_i \left(x_i \delta_i - \sum_{\{v_i,v_j\}\in E(G)} x_j \right) \\
&= \sum_{i=1}^{n} \sum_{\{v_i,v_j\}\in E(G)} x_i \left(x_i \frac{r_j}{r_i} - x_j \right) \\
&= \sum_{\{v_i,v_j\}\in E(G)} x_i \left(x_i \frac{r_j}{r_i} - x_j \right) + x_j \left(x_j \frac{r_i}{r_j} - x_i \right) \\
&= \sum_{\{v_i,v_j\}\in E(G)} \left(x_i \sqrt{\frac{r_j}{r_i}} - x_j \sqrt{\frac{r_i}{r_j}} \right)^2 .
\end{aligned}$$

□

We see that when $R = (1, \ldots, 1)$ this result reduces to a well known result for the graph Laplacian:

$$x^\top M x = \sum_{\{v_i,v_j\}\in E(G)} \left(x_i - x_j \right)^2 .$$

We can consider x to be a function on the vertices of G. For the usual Laplacian, we see that as a quadratic form the Laplacian measures how far from constant x is on the connected components of G. In other words, the Laplacian measures how much x differs from a multiple of $(1, \ldots, 1)$ on each component. Proposition 3.1 shows that the Laplacian of an arithmetical graph, noting that we temporarily are allowing the graph to be disconnected, measures how much x differs from a multiple of R on each connected component. We can use this result to prove another one that is analogous to the case of the usual Laplacian.

Proposition 3.2. Let (G, M, R) satisfy the definition of an arithmetical graph except G is not necessarily connected. Then considering M as a linear transformation on $\mathbb{Q}^n$ over $\mathbb{Q}$,

$$\dim \ker M = k,$$

where k is the number of connected components of G.

Proof. Let $R = (r_1, \ldots, r_n)$. For any $x = (x_1, \ldots, x_n)^\top \in \ker M$,

$$0 = x^\top M x = \sum_{\{v_i,v_j\}\in E(G)} \left(x_i \sqrt{\frac{r_j}{r_i}} - x_j \sqrt{\frac{r_i}{r_j}} \right)^2 .$$

Thus for any $\{v_i, v_j\} \in E(G)$

$$x_i = x_j \frac{r_i}{r_j}.$$

In particular for any connected subset of vertices $\{v_{i_1}, \ldots, v_{i_l}\}$, we have $x_{i_1}, \ldots, x_{i_l}$ are determined by only one value. Thus

$$\dim \ker M \leq k.$$

Let $G_1, \ldots, G_k$ be the connected components of G. Then define

$$x_j^{(i)} = \begin{cases} r_j, & v_j \in G_i \\ 0, & v_j \notin G_i \end{cases}.$$

Then for each $i \in \{1, \ldots, k\}$ we have $x^{(i)} = (x_1^{(i)}, \ldots, x_n^{(i)})^\top$ are in $\ker M$ and are linearly independent. Therefore

$$\dim \ker M = k.$$

□

This leads us to a result that helps us construct our desired arithmetical graph.

Proposition 3.3. Let $R = (r_1, \ldots, r_n)$, $\gcd(r_1, \ldots, r_n) = 1$ and $\{b_1, \ldots, b_{n-1}\}$ be a basis for Λ_R. Let M_R' be the matrix with column $i \neq n$ is b_i and column n has entries 0. Let $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $M_R = M_R' P$ is symmetric and has nonpositive nondiagonal entries. Then there exists a unique graph G such that (G, M_R, R) is an arithmetical graph with $\Lambda_{M_R} = \Lambda_R$.

Proof. Because M_R is symmetric and has nonpositive nondiagonal entries, there is a unique graph G and diagonal matrix $\mathrm{diag}(\delta_1, \ldots, \delta_n)$ such that $M_R = \mathrm{diag}(\delta_1, \ldots, \delta_n) - A$, where A is the adjacency matrix of G.

Because $P \in \mathrm{GL}_n(\mathbb{Z})$,

$$\Lambda_{M_R} = \operatorname{im} M_R = \operatorname{im} M_R' = \Lambda_R.$$

Because $\operatorname{im} M_R \subset \Lambda_R$, the columns of M_R are in Λ_R. Then because M_R is symmetric this implies that the rows of M_R are in Λ_R. Thus $M_R R = 0$.

Last we show that G is connected. Suppose that G is not connected. Then by Proposition 3.2 there exist linearly independent $b_n, b_{n+1} \in \mathbb{Q}^n$ such that

$$M_R b_n = M_R b_{n+1} = 0.$$

Note also that because $b_1, \ldots, b_{n-1}$ correspond to the standard basis vectors of $\mathbb{Z}^n \cong \Lambda_R$, they are also linearly independent. We now consider $b_1, \ldots, b_{n+1} \in \mathbb{Q}^n$, $M_R : \mathbb{Q}^n \to \mathbb{Q}^n$. Because any $\mathbb{Q}$ linear combination equal to 0 gives a $\mathbb{Z}$ linear combination equal to 0 by multiplying the coefficients by a common denominator, $\{b_1, \ldots, b_{n-1}\}, \{b_n, b_{n+1}\}$ are both linearly independent sets in $\mathbb{Q}^n$ as a $\mathbb{Q}$ vector space. Also because $b_1, \ldots, b_{n-1} \in \operatorname{im} M_R$ and $b_n, b_{n+1} \in \ker M_R$ where M_R is considered as a $\mathbb{Q}$ vector space linear transformation,

$$n = \dim \mathbb{Q}^n = \dim \operatorname{im} M_R + \dim \ker M_R \geq (n-1) + 2 > n.$$

Thus G is connected. Therefore G is the unique graph such that (G, M_R, R) is an arithmetical graph with $\Lambda_{M_R} = \Lambda_R$. $\square$

Therefore for any $R = (r_1, \ldots, r_n)$ with $\gcd(r_1, \ldots, r_n) = 1$, the problem of computing the Frobenius number can be reduced to finding a basis $\{b_1, \ldots, b_{n-1}\}$ of Λ_R, $P \in \mathrm{GL}_n(\mathbb{Z})$ that gives an M_R as above, and computing $g(M_R)$.

When $R = (r_1, r_2)$, the arithmetical graph with two vertices and $r_1 r_2$ edges has the property that $\Lambda_M = \Lambda_R$. Thus by the discussion in Section 3.2, such a P always exists when $n = 2$. It should be noted that when $n = 2$, there is a known closed form solution to the Frobenius number problem.

We know of no algorithm for finding such a P in general. Using the Smith normal form decomposition of M'_R, a $P \in \mathrm{GL}_n(\mathbb{Z})$ can be founds such that $M'_R P$ is symmetric, but we are not guaranteed that the signs of the entries will be as desired to construct an arithmetical graph.

3.2 Matrix Images and the Action of $\mathrm{GL}_n(\mathbb{Z})$

Suppose we are given $R = (r_1, \ldots, r_n)$ with $\gcd(r_1, \ldots, r_n) = 1$ and we want to calculate the Frobenius number by finding an arithmetical graph (G, M_R, R) such that $\Lambda_M = \Lambda_R$. We have shown above that if we have a matrix M'_R whose columns span Λ_R, we can find M_R by finding a certain $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $M_R = M'_R P$ satisfies the properties described above. We will show in this section that if there exists an arithmetical graph (G, M, R) such that $\Lambda_M = \Lambda_R$, there must exist a $P \in \mathrm{GL}_n(\mathbb{Z})$ such that $M = M'_R P$. We will show this by showing that the orbits of the action of $\mathrm{GL}_n(\mathbb{Z})$ by right multiplication on $\mathbb{Z}^n$ are the equivalence classes of matrices that have the same image. In fact this is a well known result even if we replace $\mathbb{Z}$ with any euclidean domain. We include a proof for completeness.

Let R be a Euclidean domain with norm $f : R \setminus \{0\} \to \mathbb{N}$. Let $A = (a_1, \ldots, a_n) \in R^n$ be a multiset. Then for any $q \in R$, we will use $i \mapsto i - qj$ to denote the operation on A that replaces a_i with $a_i - qa_j$. We will use $i \leftrightarrow j$ for the operation on A that swaps a_i and a_j. For any unit $u \in R$ we will use $i \mapsto ui$ to denote the operation on A that replaces a_i with ua_i. For reasons that will be made clear in this section, we will call operations on A of these types *column operations*.

Proposition 3.4. Let $(A) \subset R$ be the ideal generated by the entries of A. If $1 \in (A)$, there is a finite number of column operations turning A into $(1, 0, \ldots, 0)$.

Proof. We use the standard proof to show that Euclidean domains are principle ideal domains. The key idea is that when using the Euclidean algorithm, each step can be considered a column operation. We describe this algorithm.

We will assume that there are at least two nonzero entries of A. If there are not, we can skip to the part of the algorithm discussed in the next paragraph. Let a_i, a_j with $f(a_i) \geq f(a_j)$ be the entries of A with greatest norms. Then there exists $q, r \in R$ such that

$$a_i = qa_j + r$$

and $r = 0$ or $f(r) < f(a_j) \leq f(a_i)$. We apply $i \mapsto i - qj$ to A to get A_1. If A_1 has at least two nonzero entries, we repeat this step. Note that because $a_i = (a_i - qa_j) + qa_j$, we have $(A_1) = (A)$. Also

$$f(a_i - qa_j) = f(r) < f(a_1), \qquad \text{or} \qquad a_i - qa_j = r = 0,$$

so the sum of the norms of the nonzero entries of A_1 is strictly less than the sum of the norms of the nonzero entries of A. Thus this process must terminate in a finite number of steps. Therefore after a finite number of column operations, we have A_2 such that all but one entry is zero.

If the ith entry of A_2 is the nonzero entry, we apply $1 \leftrightarrow i$ to A_2 to get A_3. Then because $(A_3) = (A) = (1)$, the the first entry of A_3 must be a unit u. Therefore we can apply $1 \mapsto u^{-1}1$ to A_3 to get $(1, 0, \ldots, 0)$. ☐

Let $C = (c_1, \ldots, c_n)$ with each $c_i \in R^k$, where we are considering R^k as a module over R. We will use the same notation as above to denote the same column operations on C. Note that we still have addition, subtraction, and scalar multiplication in R^k, so this notion of column operation is defined for lists of elements of R^k. Also each column operation on C does not change the span.

Proposition 3.5. Let $\text{span}(C)$ be the submodule of R^k generated by the entries of C. If $\text{span}(C) = R^k$, there is a finite number of column operations turning C into a list containing precisely the standard basis vectors of R^k and 0 vectors.

Proof. Let $A = (a_1, \dots, a_n) \in R^n$ where a_i is the first entry of c_i. The first standard basis vector is in the span of C, so $1 \in (A)$. Thus by Proposition 3.4 there is finite number of column operations that turns A into $(1, 0, \dots, 0)$. We apply these operations to C to get a C_1 in which the first entry of the first vector is 1 and the first entry of the remaining vectors in 0. Because column operations do not change the span,

$$\text{span}(C_1) = \text{span}(C) = R^k.$$

Thus in particular, the second standard basis vector is in the span of C_1. Because any linear combination of vectors in C_1 that includes the first vector of C_1 has a nonzero first entry, the second standard basis vector is in the span of the 2nd through nth vectors of C_1. We can thus use a finite number of column operations that do not use the first vector of C_1 to get a C_2 such that the second entry of the second vector is 1 and the second entry of the 3rd through nth vectors is 0. Because these operations did not use the first vector of C_1, the first entries of the 2nd through nth vectors of C_2 are also 0. By the same argument, we can repeat this process until we get $C_k = (c'_1, \dots, c'_n)$ such that c'_i has jth entry 0 for $j < i$ and 1 for $j = 1$. Thus we can use column operations with c'_k to make the kth entries of all vectors 0. We repeat this with what c'_i has become with i ranging from $k-1$ to 1. Thus after a finite number of column operations, we have a list containing precisely the standard basis vectors of R^k and 0 vectors $\square$

We see that this process used in the above proof can be considered a version of row reduction that works for finitely generated free R^k modules. We now use this to characterize square matrices with entries in R that have the same image.

Proposition 3.6. Let M, N be n by n matrices with entries in R and consider them as module homomorphisms $R^k \to R^k$. Then $\text{im}\, M = \text{im}\, N$ if and only if there exists $P \in \text{GL}_n(R)$ such that

$$N = MP.$$

Proof. One direction is clear because right multiplication by an element of $\mathrm{GL}_n(R)$ does not change the image.

Suppose that $\operatorname{im} M = \operatorname{im} N$. Let $b_1, \ldots, b_k \in R^n$ be a basis for $\operatorname{im} M = \operatorname{im} N$. Let B be the n my n matrix with first k columns $b_1, \ldots, b_k$ and remaining columns 0.

We will show that there exists $P_1 \in \mathrm{GL}_n(R)$ such that $B = MP_1$. Let $\phi : \operatorname{im} M \to R^k$ be the module isomorphism sending b_i to the ith standard basis vector of R^k. Let c_i be the image under ϕ of the ith column of M. Let $C = (c_1, \ldots, c_n) \in R^k$. Because $\operatorname{span}(c_1, \ldots, c_n) = R^k$, by Proposition 3.5 there exists a finite number of column operations on C that give the list C' with the standard basis vectors of R^k. Note that we can guarantee that the standard basis vectors are ordered in ascending order in the beginning of the list and are followed by the 0 vectors. To see this, either see the proof of Proposition 3.5 or swap columns.

We can apply the column operations that turn C into C' to the columns of M. Because of the definitions of the operations and the fact that ϕ is an isomorphism, applying these operations to M is equivalent to taking the image of the columns under ϕ, applying the operations to C and then using the inverse of ϕ on the result. Thus these column operations turn M into B. Because these column operations can each be performed by right multiplication by an element of $\mathrm{GL}_n(R)$, there exists $P_1 \in \mathrm{GL}_n(R)$ and similarly $P_2 \in \mathrm{GL}_n(R)$ such that

$$B = MP_1, \qquad B = NP_2.$$

Therefore

$$N = MP_1P_2^{-1}.$$

$\square$

Proposition 3.7. Let M, N be n by n matrices with entries in $\mathbb{Z}$. Then as integer lattices $\operatorname{im} M = \operatorname{im} N$ if and only if there exists $P \in \mathrm{GL}_n(\mathbb{Z})$ such that

$$N = MP.$$

Proof. $\mathbb{Z}$ is a Euclidean domain, so this follows from Proposition 3.6. $\square$

Chapter 4

Chip-Firing on Arithmetical Graphs

4.1 Chip-Firing on an Arithmetical Graph

In this section we characterize arithmetical graphs (G, M, R) with the property that $\Lambda_M = \Lambda_R$. To describe the lattice Λ_M, it is helpful to consider a chip-firing game played on the vertices of the graph G. We describe this chip-firing game, which when $R = (1, \ldots, 1)$ has the same legal moves as the chip-firing game described in Baker and Norine (2007).

Let (G, M, R) be an arithmetical graph, $R = (r_1, \ldots, r_n)$ and $v_1, \ldots, v_n$ be the vertices of G that correspond to the multiplicities $r_1, \ldots, r_n$, respectively, and let a_{ij} be the number of edges between v_i and v_j. Also let A be the adjacency matrix of G and

$$M = \mathrm{diag}(\delta_1, \ldots, \delta_n) - A.$$

The starting state of the chip-firing game is a divisor $D = (c_1, \ldots, c_n) \in \mathbb{Z}^n$. In the context of chip-firing, we consider the state D to be the state where each vertex v_i has $c_i r_i$ chips. Note that we allow vertices to have a negative number of chips. For each v_i there are two corresponding legal moves:

- For each v_j adjacent to v_i, move $a_{ij} r_j$ chips from v_i to v_j.
- For each v_j adjacent to v_i, move $a_{ij} r_j$ chips from v_j to v_i. In other words, reverse the legal move described above.

After a legal move, the game is in state $D' = (c'_1, \ldots, c'_n)$, where each v_i has $c'_i r_i$ chips. Note that because (G, M, R) is an arithmetical graph, a legal move corresponding to v_i changes the number of chips at v_i by

$$\sum_{v_j \text{ adjacent to } v_i} a_{ij} r_j = \sum_{j \neq i} a_{ij} r_j = \delta_i r_i.$$

Therefore $D' \in \mathbb{Z}^n$ and is another divisor.

The goal of this chip-firing game is to to take the chip state on G from D to $0 \in \mathbb{Z}^n$ by a finite sequence of legal moves. Note that in Baker and Norine (2007), the goal of the chip-firing game is to take D to a state $E \in \mathbb{Z}^n$ that is an effective divisor, a goal that is closely related to Riemann-Roch theorem for finite graphs. We have made the goal of our chip-firing game to reach state 0 because this goal is specifically related to when $\Lambda_M = \Lambda_R$. Note that with either goal, the key idea of this chip-firing game is to illustrate which divisors are equivalent in $\text{Pic}(\Lambda_M)$.

Proposition 4.1. $\Lambda_M = \Lambda_R$ if and only if for every starting state D with $\deg_R(D) = 0$, the chip-firing game can be won. In fact two divisors $D, D' \in \mathbb{Z}^n$ are equivalent in $\text{Pic}(\Lambda_M)$ if and only if there is a finite sequence of chip-firing moves that takes D to D'.

Proof. We will prove the second statement by showing that $D \in \Lambda_M$ if and only if there is a finite sequence of chip-firing moves taking $0 \in \mathbb{Z}^n$ to D. Let $e_i \in \mathbb{Z}^n$ be the column vector with ith entry 1 and remaining entries 0. By the definition of the legal chip-firing moves, the first move corresponding to v_i is equivalent to adding $M(-e_i)$ to the state and the other move is equivalent to adding Me_i to the state. Therefore $D \in \Lambda_M$ is equivalent to the existence of a finite sequence of chip-firing moves taking 0 to D. This proves the second statement of the proposition.

To show that the first statement of the proposition is true, we note that the total number of chips on the graph G in state $D = (c_1, \ldots, c_n)$ is

$$\sum_{i=1}^{n} c_i r_i = D \cdot R = \deg_R(D).$$

Therefore the game can be won for every D with $\deg_R(D) = 0$ if and only if every D with $\deg_R(D) = 0$ is equivalent to 0 in $\text{Pic}(\Lambda_R)$, which is true if and only if $\Lambda_M = \Lambda_R$. □

To illustrate the use of thinking of arithmetical graphs in the context of chip-firing, we give some examples to show that this perspective is helpful in determining when an arithmetical graph (G, M, R) has the property $\Lambda_M = \Lambda_R$.

Example 4.1. Let $R = (1, 2, 1)$ and G be the graph on the vertices v_1, v_2, v_3 defined by the figure below.

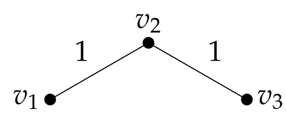

Figure 4.1 G labeled with edge multiplicities

Then for arithmetical graph (G, M, R), $\Lambda_M = \Lambda_R$.

Proof. Suppose $D = (c_1, c_2, c_3) \in \mathbb{Z}^3$ with $\deg_R(D) = 0$. This corresponds to a state with c_1 chips on v_1, $2c_2$ chips on v_2, and c_3 chips on v_3. The move associated with v_1 can be used c_1 times, noting that we use the correct move according to the sign of c_1, and the move associated with v_3 can be used c_3 times. This brings us to a state in which v_1 and v_3 have 0 chips. Because chip-firing moves preserve the number of chips, this state also has 0 chips at v_2. Therefore a sequence of moves brings D to $0 \in \mathbb{Z}^3$, so $\Lambda_M = \Lambda_R$. $\square$

Example 4.2. Let $R = (1, 3, 3, 1)$ and G be the graph on the vertices v_1, v_2, v_3, v_4 defined by the figure below.

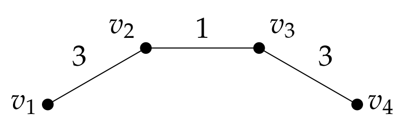

Figure 4.2 G labeled with edge multiplicities

Then for the arithmetical graph (G, M, R), $\Lambda_M \neq \Lambda_R$.

Proof. Let $D = (1, 0, 0, -1)$. Then $\deg_R(D) = 0$. The only moves that move chips from v_1 are the moves associated with v_1 and v_2, and both of these change the number of chips at v_1 by a multiple of 3. Therefore there is no sequence of moves that takes D to $0 \in \mathbb{Z}^4$, so $\Lambda_M \neq \Lambda_R$. $\square$

4.2 Lattices Induced by Simple Graphs

In this section we consider when $R = (1, \ldots, 1)$. In this setting, we can completely characterize which graphs G give an arithmetical graph (G, M, R) with $\Lambda_M = \Lambda_R$.

Proposition 4.2. If $R = (1, \ldots, 1)$ and (G, M, R) is an arithmetical graph, then $\Lambda_M = \Lambda_R$ if and only if G is a tree.

Proof. Suppose $\Lambda_M = \Lambda_R$, and let m be the number of edges and n be the number of vertices in G. Then

$$g(\Lambda_M) = g(1, \ldots, 1) = 0.$$

Then by Example 1.4

$$m - n + 1 = g(\Lambda_M) = 0,$$

so G is a connected graph with $n - 1$ edges. Therefore G is a tree.

For the other direction, suppose G is a tree. Let v be a vertex in G and $D \in \mathbb{Z}^n$ with $\deg_R(D) = 0$. We will show that there is a finite sequence of chip-firing moves that takes D to $0 \in \mathbb{Z}^n$ that does not use the moves associated with v. We will show this by induction on the number n of vertices of G. When $n = 1$, the only $D \in \mathbb{Z}^n$ with $\deg_R(D) = 0$ is 0, so the base case holds.

Now suppose G has n vertices. Let $G_1, \ldots, G_k$ be the connected components of $G - v$. Because each G_i is connected in $G - v$ and because there are no cycles in G, there is exactly one vertex in each G_i that is adjacent to v in G. Let $v_1, \ldots, v_k$ be the vertices of $G_1, \ldots, G_k$, respectively, that are adjacent to v in G.

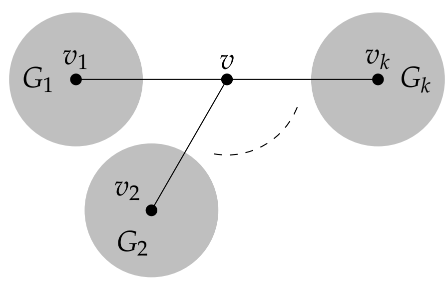

Figure 4.3 G with the connected components of $G - v$.

Let $i \in \{1, \ldots, k\}$ and l be the number of vertices in G_i. Because there is one edge between v_i and v and $R = (1, \ldots, 1)$, we can use the moves associated with v_i to take D to a divisor that has 0 total chips at the vertices of G_i. Because the vertices adjacent to v_i in $G - v$ are in G_i, these moves do not change the number of chips at the vertices in the other connected components of $G - v$. Let $D_i \in \mathbb{Z}^l$ be the resulting divisor restricted to G_i.

Because D_i has 0 total chips in G_i and G_i has $l < n$ vertices, induction gives that we can take D_i to $0 \in \mathbb{Z}^l$ without using v_i. Because all vertices in G_i other than v_i are only adjacent to vertices in G_i, we can use this sequence of moves in G without changing the number of chips at any vertices outside of G_i.

We can then repeat this process for each G_i to get a divisor that has 0 chips at all vertices of each G_i. Because the total number of chips, which equals the degree of D, does not change with chip-firing moves, this process brings the number of chips at v to $\deg_R(D) = 0$. Thus in a finite sequence of chip-firing moves not including the moves associated with v, we can take D to $0 \in \mathbb{Z}^n$. Therefore $\Lambda_M = \Lambda_R$. □

4.3 Lattices Induced by Arithmetical Graphs

In this section, we will use chip-firing to prove a result that gives restrictions on any graph G that gives an arithmetical graph (G, M, R) with $\Lambda_M = \Lambda_R$ for a fixed R. We will then discuss some of the implications this result gives for the structure for such a graph G.

Definition 4.1. Let G be an undirected multigraph on vertices $v_1, \ldots, v_n$. Then the *skeleton* of G, denoted $\mathrm{sk}(G)$ is the simple graph that has vertices $v_1, \ldots, v_n$ and an edge between v_i and v_j if and only if there is at least one edge between v_i and v_j in G.

In the context of chip-firing, $\mathrm{sk}(G)$ determines the directions chips can move, and the multiplicities of the edges in G determine how many chips are moved.

Proposition 4.3. Let (G, M, R), $R = (r_1, \ldots, r_m, s_1, \ldots, s_n)$ be an arithmetical graph with vertices $v_1, \ldots, v_m, u_1, \ldots, u_n$.

If $\Lambda_M = \Lambda_R$ and $\{v_i, u_j\}$ is a bridge in $\mathrm{sk}(G)$ crossing the cut $V = \{v_1, \ldots, v_m\}, U = \{u_1, \ldots, u_n\}$, then

$$\gcd(r_i, s_j) \leq \operatorname{lcm}(\gcd(r_1, \ldots, r_n), \gcd(s_1, \ldots, s_n)).$$

Proof. Consider the possible values of $D = (c_1, \ldots, c_m, b_1, \ldots, b_n) \in \mathbb{Z}^n$ that give a chip-firing state on G with $\deg_R(D) = 0$. Let

$$M = \sum_{k=1}^{m} c_k r_k \qquad \text{and} \qquad N = \sum_{k=1}^{n} b_k s_k$$

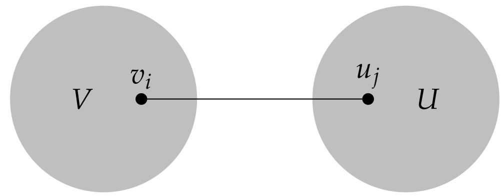

Figure 4.4 $\mathrm{sk}(G)$ and the bridge crossing partition V, U

be the total number of chips on vertices in V and U, respectively. We will prove this result by considering the possible values of M.

First because M is an integer combination of the r_k, M is a multiple of $\gcd(r_1, \ldots, r_m)$. Similarly N is a multiple of $\gcd(s_1, \ldots, s_n)$. Because $\deg_R(D) = 0$, $N = -M$. Thus M is also a multiple of $\gcd(s_1, \ldots, r_m)$. In fact M can be any integer that is both a multiple of $\gcd(r_1, \ldots, r_m)$ and $\gcd(s_1, \ldots, s_n)$. Therefore the possible values of M are the multiples of $\mathrm{lcm}(\gcd(r_1, \ldots, r_n), \gcd(s_1, \ldots, s_n))$ and in particular we can fix $D \in \mathbb{Z}^n$ with $\deg_R(D) = 0$ with

$$M = \mathrm{lcm}(\gcd(r_1, \ldots, r_m), \gcd(s_1, \ldots, s_n)).$$

Because $\deg_R(D) = 0$ and $\Lambda_M = \Lambda_R$, there is a finite sequence of chip-firing moves taking D to $0 \in \mathbb{Z}^n$. Because $\{v_i, u_j\}$ is a bridge in $\mathrm{sk}(G)$, the only moves that transfer chips between V and U are the ones associated with v_i and u_j. The moves associated with v_i transfer a multiple of s_j between V and U and the moves associated with u_j transfer a multiple of r_i between V and U. Therefore because there is a sequence of moves that transfers M chips from V to U, M is a multiple of $\gcd(r_i, s_j)$. Therefore

$$\gcd(r_i, s_j) \leq M = \mathrm{lcm}(\gcd(r_1, \ldots, r_m), \gcd(s_1, \ldots, s_n)).$$

□

In fact even if multiple edges cross a cut of G, this argument can be used to show a weaker result in which $\gcd(r_i, s_j)$ is replaced with the greatest common divisor of the multiplicities of all vertices incident to the edges that cross the cut.

Intuitively, this result shows that arithmetical graphs with $\Lambda_M = \Lambda_R$ cannot have bridges between vertices whose multiplicities share many factors. Because every edge in a tree is a bridge, this result can eliminate many possible graphs G with $\mathrm{sk}(G)$ a tree from having the property $\Lambda_M = \Lambda_R$.

For example, we see that if $R = (1,3,3,1)$ and $\mathrm{sk}(G)$ is the skeleton of the graph in Example 4.2, $\Lambda_M \neq \Lambda_R$.

This is a feature that only occurs for arithmetical graphs with nontrivial vertex multiplicities, as when $R = (1,\ldots,1)$ the result reduces to a trivial one. Therefore we see that although when $R = (1,\ldots,1)$, $\Lambda_M = \Lambda_R$ if and only if G is a tree, there are many values of R such that restrictions are placed on graphs that look like trees. In fact, unlike when $R = (1,\ldots,1)$, there are examples of arithmetical graphs (G, M, R) where G is a tree and $\Lambda_M \neq \Lambda_R$, and there are also examples of arithmetical graphs (G, M, R) where $\mathrm{sk}(G)$ is not a tree and $\Lambda_M = \Lambda_R$.

Example 4.3. Let $R = (1,1,2,2,1,1)$ and G be the graph on vertices v_1, v_2, v_3, v_4 and edge multiplicities 1 defined by the figure below. Then for the arithmetical graph (G, M, R), $\Lambda_M \neq \Lambda_R$.

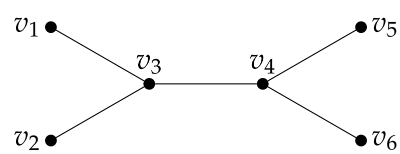

Figure 4.5 Graph G

Proof. Because $\{v_3, v_4\}$ is a bridge in $\mathrm{sk}(G)$ and

$$\gcd(r_3, r_4) = 2 > 1 = \mathrm{lcm}(\gcd(r_1, r_2, r_3), \gcd(r_4, r_5, r_6)),$$

by Proposition 4.3, $\Lambda_M \neq \Lambda_R$. $\square$

Example 4.4. Let $R = (2,3,5)$ and G be the graph on vertices v_1, v_2, v_3 defined by the figure below. Then for the arithmetical graph (G, M, R), $\Lambda_M = \Lambda_R$.

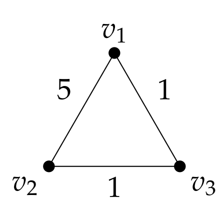

Figure 4.6 Graph G labeled with edge multiplicities

Proof. Define matrices M' and P as

$$M' = \begin{pmatrix} 3 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \text{and} \qquad P = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Note that $M = M'P$. We first show that the columns of M' span Λ_R. Call the first two columns D_1 and D_2, respectively. The columns are contained in Λ_R and the first two are not linearly dependent, so as $\mathbb{Q}$ vector spaces, the span of the columns of M' equals $\Lambda_R \otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore for any $D \in \Lambda_R$, there exists $a, b \in \mathbb{Q}$ such that

$$D = aD_1 + bD_2.$$

Because the last entry of D_1 is 0 and the last entry of D_2 is -1, b is the opposite of the last entry of D. Therefore $b \in \mathbb{Z}$ and $bD_2 \in \mathbb{Z}^3$. Because $D \in \mathbb{Z}^3$, this implies that $aD_1 \in \mathbb{Z}^3$. Then because $\gcd(3, -2) = 1$, this implies that $a \in \mathbb{Z}$. Therefore D is in the span of D_1 and D_2 as an integer lattice, so the columns of M' span Λ_R.

Then because $\det(P) = 1$, $P \in \mathrm{GL}_3(\mathbb{Z})$. Thus

$$\Lambda_M = \operatorname{im} M = \operatorname{im} M'P = \operatorname{im} M' = \Lambda_R.$$

$\square$

Chapter 5

Conclusion and Future Work

We have discussed results that can be used to find arithmetical graphs whose g-number is the Frobenius number of R and how this connection could be used to characterize when g-number and linear rank coincide. To reduce the Frobenius number problem to finding the g-number of arithmetical graphs, it would be of interest to further characterize arithmetical graphs (G, M, R) with $\Lambda_M = \Lambda_R$.

The techniques used in this thesis to characterize these arithmetical graphs have been purely combinatorial. As arithmetical graphs arise from regular models of curves, it is possible that the geometry of the regular models could be leveraged to further characterize when $\Lambda_M = \Lambda_R$. Also as the linear rank of (G, M, R) is always an upper bound for the Frobenius number of R, another possible research direction would be to characterize which arithmetical graphs (G, M, R) minimize linear rank for a fixed R.

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