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Exploration of Curvature Through Physical Materials

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Abstract

Parametric equations are commonly used to describe surfaces. Looking at parametric equations does not provide tangible information about an object. Thus through the use of physical materials, an understanding of the limitations of the materials allows someone to gain a broader understanding of the surface. A Möbius strip and Figure 8 Klein bottle were created through knitting due to the precision and steady increase in curvature allowed through knitting. A more standard Klein bottle was created through crochet due to the ease in creating quick increases in curvature. Both methods demonstrate the change in curvature for both surfaces where the Möbius strip and Figure 8 Klein bottle have slower changes in curvature, but the classic Klein bottle has a quicker change in curvature.

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Chapter 1

Introduction

Mathematics has been studied countless different ways, ranging from graphical representations to pure numerical representations to physical representations. Using physical materials as a means of explaining difficult mathematical concepts has slowly become a staple in education systems over the past few years. As a result, it is vital for one to understand the best ways to communicate math through this medium. Some examples of physical materials that can be used to explain math include paper, balls, and liquid. I have personally chosen to explore the use of knitted materials to explain the mathematical concept of curvature.

Studies have shown that the use of physical materials in classrooms has increased the interest and response of students. With the increased interest students are more likely to pay attention and retain the mathematical concepts they learned in that class. Being able to present students with tangible examples also solidifies the usefulness of mathematics, especially at a younger age. With a new found appreciation for math, students are more likely to pursue it in higher education. In addition to increasing the number of students who continue to pursue mathematics, using physical mediums has an artistic benefit.

Artists all over the world have utilized mathematical concepts in their art work. A few examples include M.C. Escher, Piet Mondarian, and Leonardo Da Vinci. All of these artists have taken inspiration from different fields of mathematics to create beautiful pieces of art. By combining the literal with the abstract, one can define new meaning between the theoretical world and the physical world. As such, it is pivotal to understanding new levels of mathematics to be able to represent it through multiple mediums.

In this paper, I will explore the different ways to represent curvature through knitted and crocheted materials. Using yarn to knit mathematical objects has provided greater insight into the changes of curvature on a single surface than a parametric equation can. Parametric equations are able to give exact values for each position on a surface, but being able to see a physical representation and understand how to get from point A to point B presents the audience with a fuller understanding of the movement of the surface. I decided to focus on the Möbius band and the Klein bottle. By understanding the limitations of both knitting and crocheting, I have developed a wider understanding of the difference between the Möbius band and the Klein bottle.

Knitting and crochet has specifically been used in classrooms to explain hyperbolic surfaces. Due to the ease of increasing and decreasing stitches, mathematicians can quickly create surfaces that possess both positive and negative curvature.

Figure 1. Example of a hyperbolic surface represented through crochet. Image from *The Institute for Figuring*

The flexibility of yarn is shown in Figure 1 where we see areas of slow changes in curvature right next to areas with large changes in curvature.

While the shape might seem difficult to comprehend, one may trace the direction of curvature onto the surface to help explain the different slices of the surface.

Figure 2. Crocheted hyperbolic surfaces with curvature indicated in yellow yarn. Image from *The Institute for Figuring*

With the addition of yellow yarn in Figure 2, the movement of the surface is much more apparent and the surface becomes much easier to comprehend. Other surfaces that have been recreated through yarn include the torus and lorenz curve. These surfaces have both negative and positive curvature.


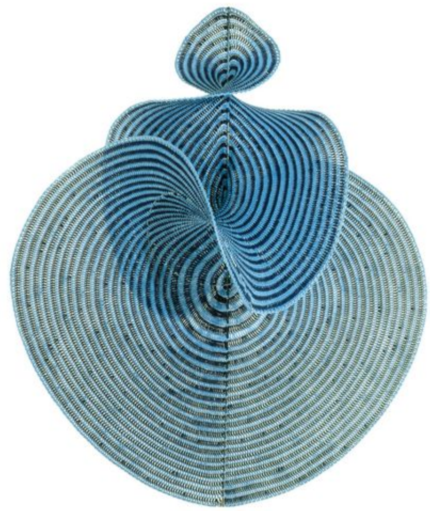

Figure 3. Knitted example of the torus (left) and crocheted example of the lorenz curve (right). Tori image from *S Goldstine*, lorenz curve image from *Eat a Bug Blog*

Figure 3 further demonstrates the versatility of yarn and reiterates flexibility of knitting. The curvature of the torus surface is detailed in both stripes and color blocks where you can see the smooth curve lines changing from positive to negative curvature. The image on the right is of the lorenz curve that has been crocheted where the changes in curvature occur on a much larger scale. This speaks to the slow and precise change in curvature obtained by knitting and a quick and precise change in curvature obtained by crochet.

So far we have discussed manipulating curvature through yarn and briefly touched on the different objects that have been made with yarn. In addition to the lorenz curve and torus, there are many other objects that haven't been explored yet. While the M*ö*bius band has been made with yarn, I want to specifically explore the difference between knitting and crocheting a M*ö*bius band. Additionally, since the M*ö*bius band and Klein bottle are so closely related, I explored the similarities between the two with respect to knitting and crochet.

Chapter 2

The History of the Klein Bottle

The idea of the Klein bottle was constructed by Felix Klein, a German mathematician, in 1882. However, to understand how the Klein bottle came into existence, we need to understand what was happening in the field of geometry around that time. Geometry is a branch of mathematics that has been around for centuries and looks at the relationships between objects. There are two main branches of mathematics that look at the relationship between shapes. Geometry and topology both look at the relationship between two shapes to determine how similar, or different, they truly are despite their appearances. Within the branch of geometry, there is both Euclidean and projective geometry. The main difference between these two branches of geometry and topology is the allowed transformations onto an object that still results in the new object and original objects being equivalent.

In Euclidean geometry, the allowed transformations include moving the object around in 3D space and flipping the object horizontally or vertically. Any other transformations (such as stretching or bending) result in a change in shape, in other words the original object and transformed object are no longer *congruent*. In projective geometry, you can be looking at a single object from different angles and they would still be considered equivalent. For instance suppose you are contemplating a pear that's sitting on a table. When you are still deciding to eat it, you are rather far and the pear seems small, but when you give in and decide to eat the pear you pick it up and it's suddenly much larger. In projective geometry, these two images of the pear are equivalent. Note that if we were in Euclidean geometry, the pear would seem to have expanded in both the x and y directions and thus it's image would no longer be considered congruent. In topology, any sort of continuous change inflicted upon an object will result in

the two objects being equivalent. For instance, if you start with a circle and then manipulate it to represent a triangle, it shows that the circle and triangle are topologically the same object. Conversely, a circle and figure 8 are not the same since you cannot manipulate a figure 8 into a circle without breaking the connection in the middle.

The Klein bottle lives in topological space where it can be obtained from a cylinder. First, start with a cylinder where one opening of the cylinder is labeled as A and the other opening is labeled B, both traveling in the same direction. Image stretching the B opening and bending it to cross itself and then align with the A opening so that they are still traveling in the same direction.

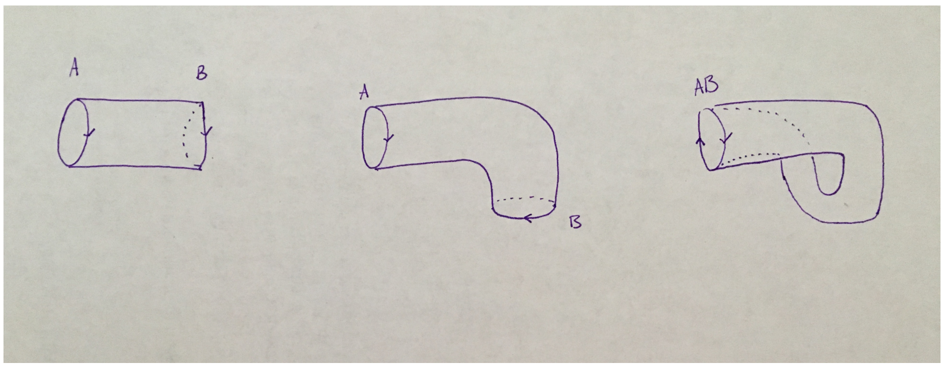

Figure 4. Diagram demonstrating the transformation from a cylinder to a Klein bottle

While the Klein bottle and cylinder are not topologically equivalent, a Klein bottle can still be viewed as an object obtained by the cylinder. Another way to think about the Klein bottle is to imagine two Möbius strips and then attach one edge of a Möbius strip to the same edge on the second strip. Once you do this, you arrive at the Klein bottle.

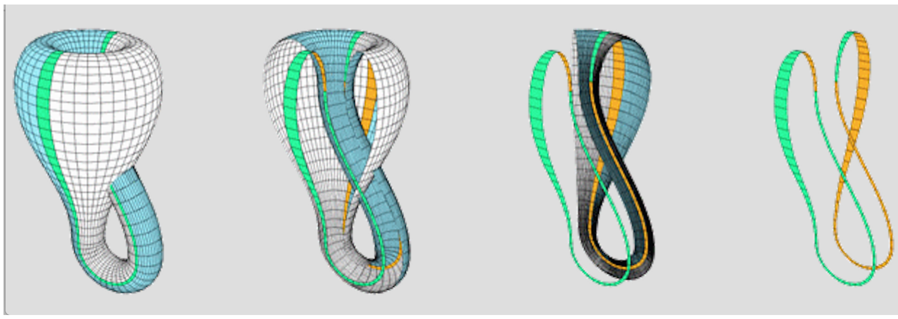

Figure 5. Diagram demonstrating the transformation from two Möbius strips to a single Klein bottle. Image from *Impossible World*

As evident from Figure 5, the Möbius strip plays a rather significant role in understanding how the Klein bottle was developed. The Möbius strip was discovered by a mathematician named August Möbius, who was born in 1790. Möbius was a German mathematician. Most of his work was in astronomy. In fact, Möbius studied astronomy at the University of Göttingen under Carl Gauss. During his time at school, he started to discover a passion in projective geometry and explored the idea of inscribed tetrahedra. Through his research, he discovered that when he formed two inscribed tetrahedra together he ended up with a surface that was one sided. In other words, if you image an ant crawling on the surface of a band, the ant would be able to walk everywhere on the band without having to cross over an edge or a border. To the naked eye, this looks like a ring that has been sliced, twisted and then put back together. While the Möbius band is named after Möbius, it was Johann Benedict Listing, another German mathematician, who got the credit first. At the time Listing was also delving into the study of topological objects and in fact coined the term 'topology' in 1847. Due to his work with topological objects, his fame and mention of the Möbius strip overshadowed Möbius' original findings. It was not until Möbius' memoir was discovered in 1868 that the public learned that Möbius was in fact the first one to uncover the properties of a Möbius strip.

Understanding the different ways to construct a Klein bottle helps understand the different ways one can represent it, or try to knit it. First, let's look at how each point on the Möbius surface and the Klein bottle are defined. Equations as from *MathWorld*, a Wolfram Web Resource.

Below are the parametric equations that describe the Möbius band surface
$x(u, v)$ = (1 + $\frac{v}{2}$ cos($\frac{u}{2}$))cos(u)
$y(u, v)$ = (1 + $\frac{v}{2}$ cos($\frac{u}{2}$))sin(u)
$z(u, v)$ = $\frac{v}{2}$sin($\frac{u}{2}$)
for $0 \leq u < 2\pi$ and -1 $\leq v \leq 1$.

Below are the parametric equations that describe the figure 8 Klein Bottle surface
$x(r, v, \theta)$ = (r + cos($\frac{\theta}{2}$)sin(v)-sin($\frac{\theta}{2}$)sin($2v$))cos(θ)
$y(r, v, \theta)$ = (r + cos($\frac{\theta}{2}$)sin(v)-sin($\frac{\theta}{2}$)sin($2v$))sin(θ) $z(r, v, \theta)$ = sin($\frac{\theta}{2}$)sin(v)+cos($\frac{\theta}{2}$) sin($2v$)
for $0 \leq \theta < 2\pi$, $0 \leq v < 2\pi$, and $r > 2$.

Below are the parametric equations that describe the classic Klein Bottle surface
$x(u, v)$ = -$\frac{2}{15}$cos(u)(3cos(v)-30sin(u) + 90cos^4(u)sin(u)-60cos^6(u)sin(u)+5cos(u) cos(v)sin(u))
$y(u, v)$ = -$\frac{1}{15}$sin(u)(3cos(v)-3cos^2cos(v)-48cos^4(u)cos(v)+48cos^6(u)cos(v)-60sin (u)+5cos(u)cos(v)sin(u)-5cos^3cos(v)sin(u)-80cos^5(u)cos(v)sin(u)+80cos^7(u)cos (v)sin(u))
$z(u, v)$ = $\frac{2}{15}$(3+5cos(u)sin(u))sin(v))
for $0 \leq u < \pi$, and $0 \leq v < 2\pi$

Chapter 3

The Basics of Yarn Work

Many mathematicians express curvature either through equations or through 3D modeling; however, there is another way to visualize it. Physical materials, such as yarn, can be used to help people visualize and interact with different types of curvature. In fact, hyperbolic knitting is used in many classrooms to teach students about positive and negative curvature. Just as there are with any medium, knitting and crocheting have their limitations. In order to understand how to properly represent surfaces using yarn, I have tested the limitations of both knitting and crocheting.

Knitting is the craft of interlocking loops of yarn with two separate needles, while crochet is the craft of interlocking yarn with a single hooked needle. As a result, the different style of interlocked patterns obtainable by each method is different. While one can knit a sweater or crochet the same sweater, the interlocking loops for each is different. One way to quantify the difference in interlocking loops is by using a stitch *gauge*. The gauge indicates the number of stitches and number of rows one can make per inch given a specific type of yarn, needle, or hook used. A pattern knitted will have a larger gauge than a pattern crocheted given the same yarn and size needle/hook used. As such, there is more precision in a knit pattern than in a crochet pattern; however, it is easier to increase size and width with crochet.

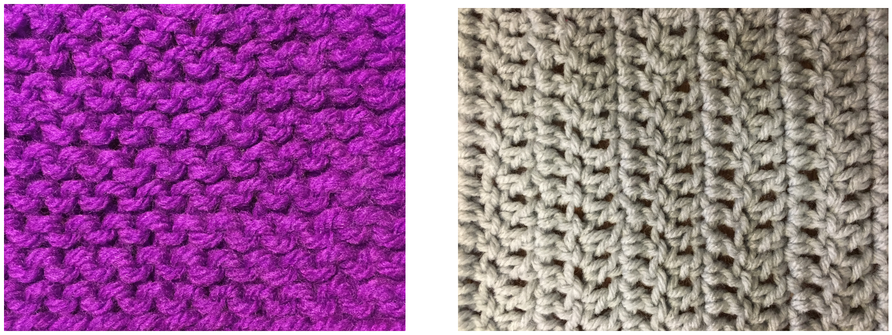

Figure 6. (A) shows gauge pattern for knitted square, (B) shows gauge pattern for crocheted square.

As indicated in Figure 6, crochet patterns have a much larger gauge for the same yarn used and needle size than that of a knitted pattern. Another way to think about this is that the area of a single standard stitch in knitting has less area than a single standard stitch in crochet. Thus, in order to increase the rate of curvature quickly, crochet is the more appropriate method to use since each additional stitch increases the total area of your surface more than an additional single standard knit stitch would. Conversely, If you want to slowly increase the rate of curvature and create smoother curves, knitting is the method to use. But before we can fully understand the difference between rate of curvature for both knitting and crocheting, we need to understand what curvature is and how it is expressed.

As mentioned earlier, one way to think about curvature is how much a surface or line deviates from the *flat* representation. With that thought, there are two types of properties: extrinsic and intrinsic. Extrinsic properties are those that depend on the coordinate space in which the surface, or curve, is embedded in. Embedding is the representation of a geometric object. Intrinsic properties are those that can be measured within the surface without reference to the space surrounding it. Distance and speed are examples of intrinsic properties. As outlined earlier, topology is the study of geometric properties and their relationship with a surface's surroundings, and thus we will be referring to extrinsic curvature properties from here on out.

The previous chapter briefly spoke about the two representations of the Klein bottle. The figure 8 model, which can be thought of as two Mӧbius bands connected to one another. The more traditional representation of the Klein bottle can be thought of as a cylinder stretched and reattached

to itself. Both the Mӧbius strip and the Klein bottle are non-orientable, which means if you start walking in place on the surface you can eventually end up back at your original spot, but reflected. This means, that any definition given to a single point on the surface does not hold for every point on that surface and so neither the Mӧbius strip nor the Klein bottle have a traditional front and back. Alternatively, the curvature of a Mӧbius strip can be represented by a parameterization based on the topological features of the Mӧbius strip, as demonstrated in an earlier chapter. The Klein bottle is also non-orientable, and thus does not have a single definition of the surface, but rather different ones depending on location on the surface. There are two representations of the Klein bottle: the figure 8 and the more common representation. The parameterization of the figure 8 model very closely resembles that of the Mӧbius strip where the only difference is that the Mӧbius is rotated about a line and the Klein figure 8 is rotated about a circle. The parameterization of the common representation of the Klein bottle is different, but does not give any more insight into the surface than the topological reasoning. With a foundation in yarn work and a general idea of how to control area, one can start exploring the ways to represent curvature with yarn.

Chapter 4

Curvature Through Physical Materials

Now that we have discussed curvature in general and the specific curvature of both the Möbius strip and Klein bottle, we can start exploring how to create these surfaces using yarn. To start, I looked at the basic ways to increase the rate of stitches, or curvature, through both crochet and knitting. The most basic method for knitting, or crocheting, a surface is to create a base chain. This base chain consists of loops. To then continue to the second row, you simply make a single new knot into each loop on your chain row. To continue to the third row, you continue to add a single new knot into each loop on the second row.

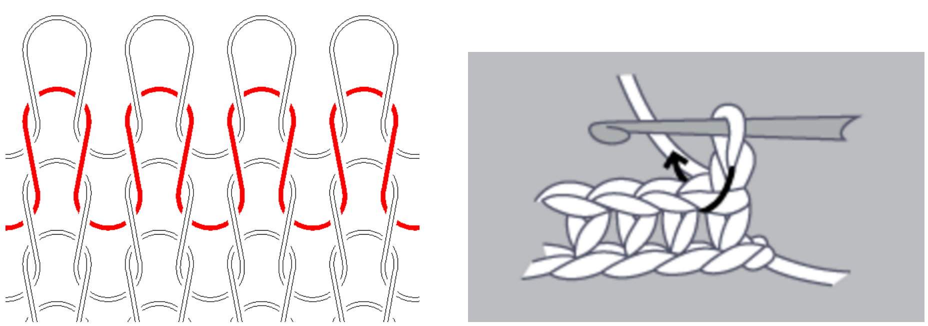

Figure 7. Stitch diagrams demonstrating how to increase each previous loop with a new knot. Left shows a knit pattern and the right shows a crochet pattern. Knit pattern from *Compile Yarn*, crochet pattern from *Yarn Standards*

Since it is easier to increase curvature through crochet, I continued exploring methods of changing, and controlling curvature with crochet samples only. Figure 7 refers to a standard increase, where both the knit and crochet pattern will result in a rectangular surface. In order to create surfaces with varying curvature (hyperbolic surfaces) more easily, I decided to create surfaces in rounds. The idea of advancing to each new row, now called rounds, is the same as the one described earlier. However, now our starting chain row will form a circle rather than a line, as shown in Figure 8.

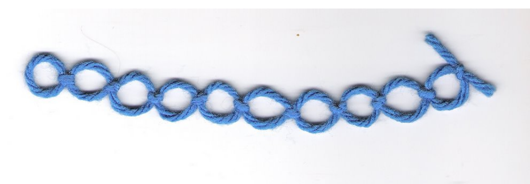


Figure 8. Starting chain loops, both for crochet, where the left will produce a rectangular surface and the right will produce a circular surface. Rectangular chain loop image from *B.Hooked Crochet*, and circular chain loop from *Martian Mischief*

Once you have a base, you can increase or decrease stitches as you like to create a variety of different shapes. The three basic shapes I will discuss will be a circle with zero curvature, one with positive curvature, and one with negative and positive curvature. In order to create a circle with zero curvature, you crochet a new knot into each of the loops on the chain base then you will stitch two new knots into each loop on the second row. You need to increase the second row to two stitches per loop in order to maintain the area needed to be filled, of sorts. An easier way to think about this is that the circumference of the second row is twice that of the first row, and thus to keep the surface flat, one will need to create twice as many new knots.

Figure 9. Illustration showing the increase in circumference from the first round to the second round is two fold. Image is adapted from *Moogly-blog*

Looking at Figure 9, we see that the purple band is the radius of the first round while the orange band is the radius of the second round. Here, the orange band is essentially twice the length of the purple band. Now suppose the purple band has length r, thus the circumference of the first round is $2\pi r$. Since the orange band is twice the length of the purple one, we know the length is $2r$, which correlates with a circumference of $4\pi r$, which agrees with the previous method of increasing the new knot count to two per stitch on the first round. With this in mind, to create a surface with positive curvature, you would simply continue to crochet a single new knot into each loop for both rounds. Since the second row will not have enough knots to cover the area to lie flat, the shape will be forced to curve upward at the edges. In order to create a surface with negative curvature, you create two new knots for each loop in both the first and the second

round. Due to the increase in area, the surface is no longer able to lay flat, thus forcing the yarn to naturally curve upward and downward.

Figure 10. (A) crochet swatches displaying zero (top left), positive (top right), and negative curvature (bottom).

With a solid foundation in how to represent curvature using yarn, I attempted to create a Möbius strip and both representations of the Klein bottle. There are two ways to approach creating the Möbius strip. One way is by thinking of the strip as a band with a single incision, twisted, and then reattached to itself. This approach was simple to create, and since I didn't need to increase the number of stitches, but rather just twist the model, I decided to move forward with knitting. This way, one can clearly see the slow change in curvature of a Möbius strip.

Chapter 5

Möbius strip and Klein Bottle in Yarn

I started a Möbius strip by knitting a rectangular swatch, twisting it midway and then reconnecting it with itself. This is the simplest way to create a Möbius strip. Another way to is to use circular needles. Circular needles allow for one to knit in a circular fashion. This is different from the circular crochet since in crochet you work from the center of the circle outwards. With circular needles you can create cylinders that work outwards in.

Although there are no photos provided, I was able to knit a Möbius band using circular needles in such a way that required no stitching two edges together. This allowed for a seamless creation of the Möbius band that did not require stitching two edges together. As such, it is much more representative of an actual Möbius band, shown in Figure 12.

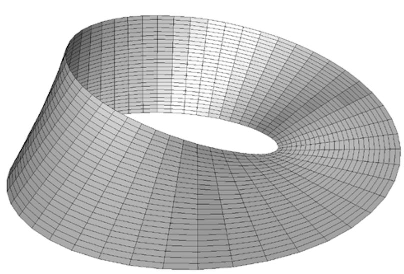

Figure 12. Mathematical model of Möbius strip. Image from *Paul Bourke*

Now let's explore the different ways to create the Klein bottle. As mentioned before there are two forms of the Klein bottle: the classic and the figure 8. Since the curvature in the Klein bottle changes pretty rapidly and

in order to create the lack of an inside, I wanted to start with the little spout and increase my stitches as necessary. The challenge here was understanding when and by how much to increase the stitches especially since you don't want to increase too quickly and create a ruffle. As such, when I first started with the cylinder I created a circular base and stitched one new knot into each loop of the previous to force the yarn to curve upward and form a hollow tube. After this I increased the number of stitches into the loops; however, unlike earlier, I alternated the number of stitches I increased by. I alternated between two new stitches and one new stitch in order to maintain the increase in surface but to also control it enough so that the yarn won't start to flutter.

Figure 13. Multiple angles of crocheted Klein bottle

Figure 13 shows the crocheted Klein bottle I created from different angles. The far left image shows the side view of the Klein bottle showing where the cylinder crosses itself to reconnect with the other opening. The middle

image is of the bottom of the Klein bottle showing it is indeed hollow and does not just close. The far right photo is from the top and offers a second view of where the Klein bottle intersects itself.

The other form of the Klein bottle was the figure 8 model. When thinking about this one, I thought of the cross section.

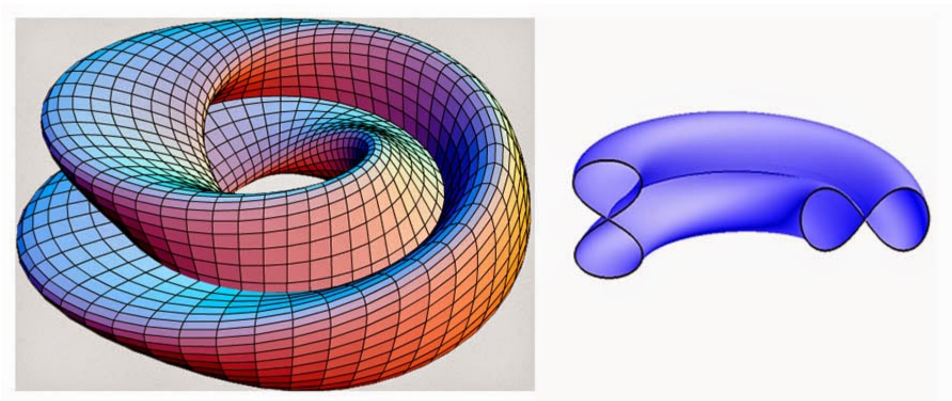

Figure 14. Figure 8 variation of the Klein bottle on the left and the cross section of this variation on right. Figure 8 image from *MathWorld*, cross section from *WikiMedia*

As shown in Figure 14, the cross section of a figure 8 Klein bottle is simply a figure 8, hence the figure 8 Klein bottle variation. As such, I figured the best way to create this version would be knit a long figure 8, twist, and reconnect it with itself. This is very similar to how I made the M*ö*bius strip except now I am working with a figure 8 rather than a rectangle. In order to create the figure 8 shape, I needed to understand how to intersect an object with itself. This is a little different from when I crocheted the Klein bottle since I just created a hole for the cylinder to go through but with this model I will need to physical intersect the object with itself.

The method of intersecting that I arrived at involved kniting a rectangular swatch and then finding the line where I wanted to intersect and pulled each loop one by one through the surface. This was very tedious, but I found it was the most seamless intersecting method.

Figure 15. Diagram showing intersecting rectangular swatch. Left diagram shows how the rectangular swatch intersected itself multiple times and the right diagram shows a close up of how seamless this intersection seems

Now with a method on how to create a figure 8, I started to play around with the best ways to create the figure 8 Klein bottle. My first attempt involved creating a base of 80 chain loops. This figure turned out to be too large and the object looked more like a Möbius strip than the figure 8 Klein bottle. As a result, I decided to reduce the base chain loop count down to 24 loops. This results in a tighter object that more closely resembles the target object.

Figure 16. Left photo shows a top view of the first attempt of the figure 8 Klein bottle and the right photo shows a close of the intersecting section

As shown in Figure 16, the surface more closely resembles a Möbius

strip where there is no clear distinct intersecting line.


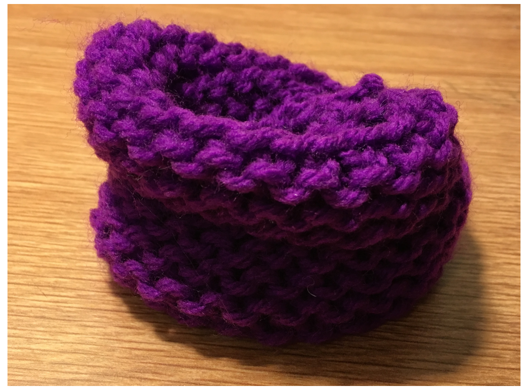

Figure 17. Left photo shows a top view of the second attempt of the figure 8 Klein bottle and the right photo shows a close of the intersecting section

Clear differences between Figure 16 and Figure 17 can be seen, where in Figure 17 the resulting object is much tighter and it is clear where the object intersects itself to create that figure 8.

As we have seen, crocheting the classic Klein bottle allows for the right amount of control and variability in curvature while knitting the figure 8 Klein bottle allows for the smooth change in curvature and precise intersecting.

Chapter 6

Conclusions

Just like everything in life, there are limitations. I explored the limitations of knitted and crocheted materials in trying to figure the best way to create the Möbius strip and the Klein bottle. Through my exploration, I found that knitting increases the area by a small fraction and so there is smoother changes in curvature and thus can create more precise objects. However, the benefits of crocheting is that you can create quick increases in curvature very easily and this allows for one to create an object that has a lot of varying curvature, for example the class Klein bottle. The Möbius strip on the other hand has very similar change in curvature around the whole surface and so this is why knitting is best for this object.

In addition to controlling the change in curvature of an object, knitting allows for more precise detailing and so intersecting itself is more possible with knitting.

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