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An Exposition of Kasteleyn's Solution of the Dimer Model

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1 Introduction

At first glance, one of the more surprising trends in combinatorics is the prevalance of scenarios of the following type: given a collection of regions in $\mathbb{R}^n$, place them in the plane without overlap so that they completely cover a specified region. Determining the existence of and enumerating these covers are two examples of what are broadly referred to as tiling problems.

Despite their surface-value simplicity and purely intrinsic motivation, tiling problems are often very difficult and subtle, and have incredible flexibility to encode information. The enumerations of one-dimensional tilings correspond to linear, constant-coefficient recurrence relations of any order. In higher dimensions, the applications are considerably less elementary, but some have been found and put to good use, for instance, in the enumeration of the so-called "alternating-sign matrices" (see [1] for a thorough treatment).

In general, an arbitrary tiling problem often doesn't have enough symmetry to be mathematically tractable, so we have to restrict the problem considerably. In this paper, we will be concerned only with 2-dimensional *domino* tilings. A domino is a rectangle made by putting two squares of fixed size side-by-side, and we are allowed to use as many of these as we need to cover some fixed region.

One thing that is not particularly obvious about tiling problems in general, and domino tilings in particular, is that they are extremely sensitive to changes in boundary conditions. Consider the Aztec diamond:

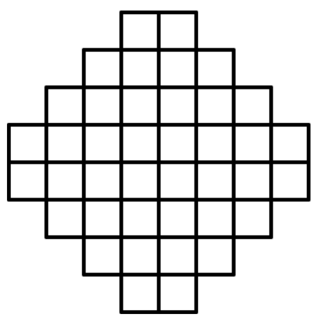

Figure 1: An Aztec diamond of order 4.

It is a commonly-cited fact in the tiling literature that there are $2^{n(n+1)/2}$ domino tilings of this region [3], but if you remove one of the middle rows, there is only one. If you then remove one of the middle columns, there are none at all. Such phenomena are common: Thurston has written an excellent expository paper [8] about a combined algebraic and geometric attack on determining the existence of a domino tiling.

All of this is to say that we are not going to be able to cover even 2-dimensional domino tilings in anywhere near general terms. We will focus on the case that we would think would have the simplest geometry: How many ways can we tile a rectangular board using dominos ? Despite the apparent simplicity of this scenario from a geometric perspective, as an enumeration problem, it is much more difficult than that of the Aztec diamond.

However, as the title suggests, we do not intend to prove anything new in this paper. We don't have to, because Kasteleyn, interested in applications to statistical mechanics (the titular "dimer model") discovered an exact formula in 1961:

Theorem (Kasteleyn [5]). Suppose $\mathcal{T}(m, n)$ is the set of domino tilings of a board of width m and height n. Without loss of generality, let m be even; then the size of $T(m, n)$ is

$$2^{mn/2} \prod_{k=1}^{m/2} \prod_{\ell=1}^{n} \left(\cos^2 \frac{\pi k}{m+1} + \cos^2 \frac{\pi \ell}{n+1} \right)^{1/2}.$$

The natural reaction to this theorem is probably disbelief. Handed a random product formula involving squares of cosines, it is reasonable to doubt that this answer should be an integer. Indeed, in this case, the integrality alone seems to be quite deep. To our knowledge, the second half of the proof we provide here is the simplest known way to show that this product formula always produces integers.

Kasteleyn's original proof is about six pages long, and it is rather terse. Moreover, it assumes a good bit of knowledge of graph theory and tensor products, its motivation is steeped in physical language, and its intuition leans heavily on the solution to the Ising model. In this paper, we trace Kasteleyn's essential argument while simplifying as much as possible and providing introductions to the relevant materials as they arise. In particular, Section 2.3 is adapted from Nicholas Loehr's proof in [6].

We do preserve a notable complexity from the original paper, because it does not significantly increase the complexity of the proof. The main result of the paper is more general than the theorem above, as it is a sort of "generating function" version, where we partition the tilings based on the number of horizontal and vertical dominos.

Theorem (Kasteleyn [5]). Let α be the number of horizontal dominos and $\beta = \frac{1}{2}mn - \alpha$ be the number of vertical dominos used in a domino tiling of a board of width m and height n. Without loss of generality, let m be even; then the weighted sum of the tilings has a product formula:

$$\sum_{\text{tilings}} x^\alpha y^\beta = 2^{mn/2} \prod_{k=1}^{m/2} \prod_{\ell=1}^{n} \left(x^2 \cos^2 \frac{\pi k}{m+1} + y^2 \cos^2 \frac{\pi \ell}{n+1} \right)^{1/2}.$$

Setting $x = y = 1$ removes the weighting and recovers the original theorem, as we would expect.

The high-level overview of the argument is this: we first encode the problem into a matrix in such a way that the total weight of the tilings is counted by a determinant-like object called the Pfaffian. Then we observe that this matrix is similar to a block-diagonal matrix, and its determinant may be calculated explicitly to be the square of the product formula. It is a classical result by Muir [7] that $\det(A) = [\mathrm{Pf}(A)]^2$ for skew-symmetric matrices A of even dimension, and so we recover the theorem.

In practice, this means that the proof splits cleanly into two parts. One is combinatorial, in which we determine that there is a matrix D with the right structure to tell us something about the tilings. The other is linear-algebraic, in which we forget the combinatorial interpretation of D and do anything necessary to find its determinant.

With this in mind, we may as well see the matrix; you may find it useful to know where we are headed as we go through the combinatorial side of the proof. It is most clearly defined as an $n \times n$ matrix of blocks; it has the form

$$D = \begin{bmatrix} xH_m & yV_m & & & & & \\ -yV_m & xH_m & yV_m & & & & \\ & -yV_m & xH_m & yV_m & & & \\ & & -yV_m & xH_m & & & \\ & & & & \ddots & & \\ & & & & & xH_m & yV_m \\ & & & & & -yV_m & xH_m \end{bmatrix},$$

where V_m and H_m are the following $m \times m$ matrices (and recall m is even):

$$H_m = \begin{bmatrix} & 1 & & & & & \\ -1 & & 1 & & & & \\ & -1 & & 1 & & & \\ & & -1 & & & & \\ & & & & \ddots & & \\ & & & & & & 1 \\ & & & & & -1 & \end{bmatrix} \qquad V_m = \begin{bmatrix} -1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}.$$

The reason why H_m and V_m are so named is because they correspond, in some sense, to the horizontal and vertical adjacencies of the squares in the board. We could make this more precise by considering the problem graph-theoretically, but for the sake of this discussion we will not dive into those details. However, we should note that for H_m the signs are a technical restriction, demanded because the Pfaffian is defined only on skew-symmetric matrices. However, the signs on V_m have fairly deep combinatorial significance, which we explore in Section 2.3.

2 The Combinatorial Half

We lay out a slightly more detailed outline before proceeding. First, Pfaffians are like determinants in that they are a sum over permutations. Therefore, in order to use these objects we must first convert our question about tilings into a question about permutations, which we do in section 2.1. In the next section we explain the Pfaffian in more detail and why it is the obvious choice for counting these tilings, and give a naïve guess at D.

We will observe that the guess at D we made is off by some sign factors, so we will try to alter the signs of D so that all of the terms are counted positively. To do this, we must compute the sign of the permutation associated to each tiling. Section 2.3 culminates with the most sophisticated result of the paper, which shows how to go directly from a tiling to the sign of its permutation, without going through the messy process of determining the cycle notation. Section 2.4 draws together all of the information we have, produces the correct D, and concludes the combinatorial half of the proof.

2.1 Conversion to a Permutation Problem

Although we are counting unlabelled tilings, in the beginning we label everything. Every one of the mn squares in the board gets a label, ordered lexicographically. The rows get numbers, the columns get numbers. Each half of each domino gets a number, in a sensible way. More precisely, we line the dominos up and the d^{th} domino has one end labelled $2d-1$ and the other labelled $2d$. Throughout the document, we refer to these as the odd and even half-dominos.

Note that in making these labels, we are implicitly dividing the set $[mn] = \{1, 2, \ldots, mn\}$ into two distinct partitions: One is the "domino" partition $[mn/2] \times [2]$ and the other is the "position" partition $[n] \times [m]$.

We produce these labels so that we can begin to associate permutations to the tilings. Clearly there are $mn/2$ tiles on the board and mn half-dominos, and so we think of a tiling as a special kind of permutation on $[mn]$. As an arbitrary but useful convention, we will think of our partitions as being from the domino partition of $[mn]$ to the position partition, where where $\sigma(k)$ is the position where the k^{th} half-domino is placed. Therefore, while we will use permutations $\sigma : [mn] \to [mn]$, it is often helpful to pretend that the domain and the codomain are genuinely different sets, according to the relevant partition, explicitly

$$\sigma : [mn/2] \times [2] \to [n] \times [m].$$

2.1.1 Example calculations

To get a better understanding of this definition, it helps to have some examples. Therefore, just to be concrete, we will take $m = 6$ and $n = 3$. The tiling corresponding to the identity

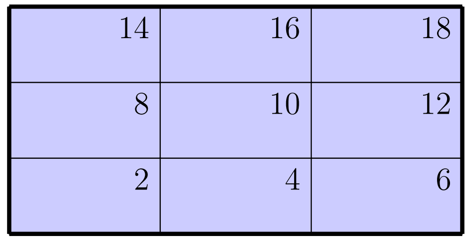

Figure 2: The identity tiling. For clarity, only the even half-dominos are labelled.

permutation is given at the top of the following page and it is as generic as one might expect.

However, while there is certainly only one tiling associated with the identity, you may notice that the identity is not the only permutation that corresponds to this tiling. For instance the permutation (13)(24) swaps the locations of the first two dominos, but preserves the unlabelled structure. Perhaps this is not entirely clear: cycle notation is not the best way to see these permutations. In list form, this permutation is

$$3, 4;\ \ 1, 2;\ \ 5, 6;\ \ 7, 8;\ \ \cdots 17, 18$$

where the semicolons are simply to remind us that the domain of σ, corresponding to the position of the numbers in the list, is split up into dominos. Even simpler, the permutation (56) flips one of the dominos around.

$$1, 2;\ \ 3, 4;\ \ 6, 5;\ \ 7, 8;\ \ \cdots 17, 18.$$

Let us look at a more interesting example.

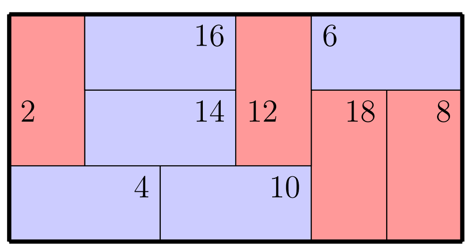

Figure 3: Tiling associated with $\Sigma' = 13, 7;\ 1, 2;\ 18, 17;\ 6, 12;\ 3, 4;\ 16, 10;\ 8, 9;\ 14, 15;\ 5, 11$.

You should take some time to convince yourself that this tiling really is associated with Σ'. However, again we can clearly see that this permutation is not unique. In fact, it is not too hard to determine there are exactly $(\frac{mn}{2})!\, 2^{mn/2}$ permutations corrospending to any tiling.

Why, then, can we not say there are $\dfrac{(mn)!}{(\frac{mn}{2})!\, 2^{mn/2}} = (mn-1)!!$ ways of tiling the board? Unfortunately, it is not the case that every permutation is associated with a tiling. For a simple example, we can take $m = n = 2$, and $\tau = 1, 4;\ 2, 3$. This attempts to send one end of the first domino to the lower-left corner and the other end of the domino to the upper-right corner. But of course a domino cannot do this without breaking.

Since we have no hope of creating a mapping from permutations to associated tilings, we will instead find a canonical representative among the permutations corresponding to each tiling, so that we can create a mapping from tilings to permutations.

2.1.2 Uniquely specifying a permutation

Naïvely, it would be nice if our permutation were monotonic, but of course only the identity is monotonic in the strictest sense. What we can do, however, is have monotonicity in each of the components of our domino partition. More intuitively, we first demand that the larger number of every domino (which is even) goes to the larger position:

$$\sigma(2d-1) < \sigma(2d) \qquad \text{for all } \ 1 \leq d \leq mn/2. \tag{1}$$

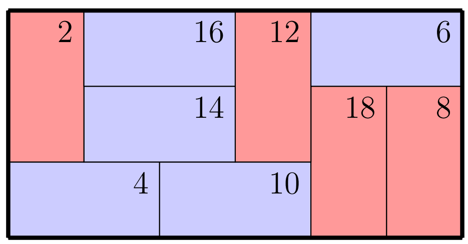

Figure 4: This permutation is a "minimal" change of Σ' to comply with (1).

Secondly, we demand that the dominos themselves (not the half-dominos) are monotonically increasing in some sense. We choose this to mean increasing in the smaller (odd) half, for reasons that will become clear later:

$$\sigma(1) < \sigma(3) < \sigma(5) < \cdots < \sigma(mn-1). \tag{2}$$

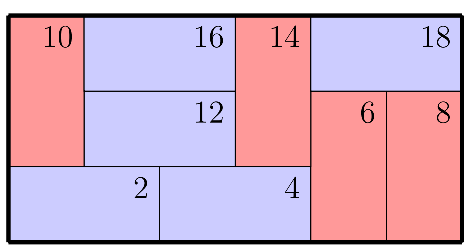

Figure 5: This permutation $\Sigma = 1,2;\ 3,4;\ 5,11;\ 6,12;\ 7,13;\ 8,9;\ 10,16;\ 14,15;\ 17,18$ corresponds to the same tiling as Σ' but also complies with the monotonicity conditions (1) and (2).

It is worth taking the time to convince yourself that for any (unlabelled) tiling, only one of the permutations corresponding to it also satisfies both (1) and (2). It is also worth observing that these constraints alone are not sufficient to guarantee a permutation corresponds to a tiling. Indeed, the example $\tau = 1, 4;\ 2, 3$ from before satisfies $1 < 4$ and $2 < 3$ together with $1 < 2$, and yet there is no tiling associated with τ. We will discuss the constraints which are both necessary and sufficient later in section 2.3, when the distinction is more urgent.

2.2 The Pfaffian

One might wonder why these particular constraints were chosen over any other selection mechanism of the relevant permutations. The reason is that we wanted the permutations we chose to be compatible with the definition of the Pfaffian.

We give a formal definition of the Pfaffian before discussing it at some length.

> **Definition.** Suppose $A = [A(i, j)]$ is a skew-symmetric $2L \times 2L$ matrix. Then the **Pfaffian** of A is $\mathrm{Pf}(A) = \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{d=1}^{L} A\big(\sigma(2d-1), \sigma(2d)\big)$, where the sum is over all permutations of $\{1, 2, \ldots, 2L\}$ satisfying (1) and (2).

The name was coined by Cayley [2] in honor of Johann Friedrich Pfaff, the doctoral advisor of Möbius and Gauss [4]. Importantly, the Pfaffian is a number we assign to a matrix in much the same way as a determinant is. It captures information about all of the entries in A in some holistic way and tries to retain as much information as it can while only being a single number.

Let us peer a little deeper into the definition. This has the same general feel as the determinant, in that it is a signed sum over permutations, where the summands are products of matrix elements. However, beyond the familiar form, the two concepts are quite different. Most evidently, the sum does not include all permutations: there is a definition which does use all permutations, but the skew-symmetry consolidates $N!\,2^N$ of the summands together and ultimately produces the same result.

More subtly, the elements of A are chosen with quite a different procedure than they are in the determinant formula. Instead of taking the value of the domain and codomain into account, we instead look at the two ends of the domino, and we look at the row corresponding to the location of the odd half, and the column corresponding to the location of the even half. (Of course, there is nothing magic about this order; we could just as easily use the other one and take the negative at the end; the important thing is, as usual, consistency.)

2.2.1 Calculations for small matrices

To demystify this number a bit more, it helps to look at some actual matrices. Let us consider the simplest case: $L = 1$. Then a generic skew-symmetric 2×2 matrix is

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}.$$

Since $L = 1$, the product term simply becomes $A(\sigma(1), \sigma(2))$. Moreover, any σ satisfying (1) and (2) has $\sigma(1) = 1$, and so the only permutation of $\{1, 2\}$ in the sum is the identity. Hence,

$$\operatorname{Pf}\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a_{12} = a.$$

For a more substantial example, let $L = 2$. A generic skew-symmetric 4×4 matrix looks like

$$\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}.$$

We know all the relevant σ will fix 1; we can enumerate them by looking at the image of 2. If $\sigma(2) = 2$ then $2 < \sigma(3) < \sigma(4)$, and so σ is the identity. If $\sigma(2) = 4$, then $1 < \sigma(3) < \sigma(4) < 4$ and so $\sigma = (243)$. Finally, if $\sigma(2) = 3$, then $1 < \sigma(3) < \sigma(4)$ and so $\sigma = (23)$.

Therefore, the product terms of the Pfaffian are

$$\mathrm{id} \mapsto af \qquad (243) \mapsto cd \qquad (23) \mapsto be.$$

Including the sign terms and summing over σ gives

$$\operatorname{Pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + cd.$$

Reading into this formula more than might be wise, we could notice that

$$\operatorname{Pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = a \cdot \operatorname{Pf}\begin{bmatrix} 0 & f \\ -f & 0 \end{bmatrix} - b \cdot \operatorname{Pf}\begin{bmatrix} 0 & e \\ -e & 0 \end{bmatrix} + c \cdot \operatorname{Pf}\begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix},$$

where the sub-Pfaffians are formed by removing both rows and columns in which the factor appears; for instance b appears in the first and third row and column, and removing both of these leaves the submatrix $\begin{bmatrix} 0 & e \\ -e & 0 \end{bmatrix}$. Although we might think this is a curiosity of small L, it turns out that this observation holds in general. So one has a "Laplace expansion" method for computing the Pfaffian; it is really quite determinant-like in nature in terms of actual calculation. With this intuition in mind, we return to our counting problem.

2.2.2 A near miss

Recalling that we are trying to associate to each $m \times n$ board a matrix D of size mn, we can interpret each matrix element as a pair of board positions. When σ corresponds to a tiling, the matrix elements under consideration correspond to dominos. However, we know from the remarks at the end of the previous section that not every permutation allows us to consider the pair $(\sigma(2d-1), \sigma(2d))$ to be the ends of a domino.

Fortunately, the matrix D is yet to be defined; although the Pfaffian will consider all permutations of the desired form, the permutations will not contribute to the sum if any of the matrix elements in the product $\prod_d D(\sigma(2d-1), \sigma(2d))$ are zero. Therefore, we can manipulate matrix entries to our advantage, so that only the pairs corresponding to tilings actually contribute to the sum.

The plan is to make almost all of the matrix entries zero, except those pairs of board positions which could hold a domino. We call these **domino pairs**. We can now make precise the idea we described in the paragraph before: if for some σ we find that $(\sigma(2d-1), \sigma(2d))$ is not a domino pair, this matrix entry should be zero; this will cause any product which contains it to be zero and hence the σ term does not contribute to the sum. Conversely, if $(\sigma(2d-1), \sigma(2d))$ is a domino pair, then this matrix entry should not be zero, and so any permutation consisting exclusively of domino pairs will contribute to the sum.

Since we weight horizontal and vertical dominos differently, we need some way to distinguish between these two. This is straightforward and it leads to the characterization of domino pairs in general. If a domino is horizontal, then the even half is directly to the right of the odd half, i.e. $\sigma(2d) = \sigma(2d-1)+1$ whenever the odd half is not an endpoint: $\sigma(2d-1) \neq rm$ (for any $1 \leq r \leq n$). Similarly, if a domino is vertical, then the even half is directly above the odd half. In the ordering of board positions, this is m spaces away, so $\sigma(2d) = \sigma(2d-1)+m$ whenever the odd half is not in the top row: $\sigma(2d-1) \leq nm - m$.

This leads us to our first guess at D. Label all of the matrix entries at horizontal domino pairs with an x and all the matrix entries at vertical domino pairs with a y, adjust signs to make it antisymmetric, and force all other entries to be zero. The product term then works out to be

$$\prod_{d=1}^{mn/2} D\big(\sigma(2d-1),\sigma(2d)\big) = \begin{cases} x^\alpha y^\beta & \sigma \text{ is a tiling} \\ 0 & \sigma \text{ is not a tiling} \end{cases}.$$

Here α and β are, as in the theorem statement, the number of horizonal and vertical dominos used in σ, respectively. With this as our D', we remove the noncontributing terms from the Pfaffian calculation to arrive at a very near miss:

$$\mathrm{Pf}(D') = \sum_{\text{tilings}} \mathrm{sgn}(\sigma) x^\alpha y^\beta.$$

Unfortunately, it is not true that all domino tilings have positive sign. Therefore, in order to correct D' to the proper matrix D, we must find a way to express $\mathrm{sgn}(\sigma)$ in a more palatable form.

2.3 Sign of a Tiling

One way to calculate the sign of a permutation $\sigma : [mn] \to [mn]$ is by considering its inversion number $\mathcal{I}_\sigma$, which is defined as $\mathcal{I}_\sigma = \sum_{k=1}^{mn} \mathcal{I}_\sigma(k)$, where $\mathcal{I}_\sigma(k)$ are the partial inversion numbers $\#\{p > k : \sigma(p) < \sigma(k)\}$. Intuitively, the inversion number shows how much a permutation differs from the identity. Rigorously, the sign of a permutation σ is $(-1)^{\mathcal{I}_\sigma}$, and so the parity of the inversion number determines the sign.

Again, an example may be helpful. We use the permutation Σ associated with the tiling we considered in Section 2.1:

$$\Sigma = 1,2;\ 3,4;\ 5,11;\ 6,12;\ 7,13;\ 8,9;\ 10,16;\ 14,15;\ 17,18.$$

$$\begin{aligned} \mathcal{I}_\Sigma &= (0+0)+(0+0)+(0+5)+(0+4)+(0+3)+(0+0) \\ &\quad +(0+2)+(0+0)+(0+0) \\ &= 14 \end{aligned}$$

Therefore, $\mathrm{sgn}(\Sigma) = (-1)^{\mathcal{I}_\Sigma} = 1$. We make some observations and may suspect that they generalize. Every $\mathcal{I}_\Sigma(k)$, where k corresponds to the odd half of any domino, or the even half of a horizonal domino, contributed 0 to the total inversion number. Therefore, it appears that we only need to consider those k which are associated with the even part of a vertical domino.

2.3.1 Combinatorial interpretation of the sign

The remainder of this section is devoted to a proof of the following lemma:

Lemma. The inversion number of σ has the same parity as the number of dominos in odd-numbered columns.

We begin by formalizing the observations made in the calculation of $\mathcal{I}_\Sigma$ above. We need to determine, for any given domino d, how many half-dominos p from dominos later than d are placed in smaller board positions than the half-dominos of d. In other words, we must compare $\sigma(p)$ with $\sigma(2d)$ and $\sigma(2d-1)$ for those $p > 2d$ and $p > 2d-1$.

First, we calculate the partial inversion numbers $\mathcal{I}_\sigma(2d-1)$ for the odd half-dominos. If p is odd, then by (2) we have that $\sigma(p) > \sigma(2d-1)$. If p is even, we apply (1) before (2) and find that $\sigma(p) > \sigma(p-1) \geq \sigma(2d-1)$. Hence, we have that $\{p > 2d-1 : \sigma(p) < \sigma(2d-1)\} = \varnothing$.

Next, suppose that the d^{th} domino is horizontal. Then $\sigma(2d) = \sigma(2d-1)+1$ and so if $p > 2d$ then $\sigma(p)+1 > \sigma(2d-1)+1 = \sigma(2d)$. Hence $\sigma(p) \geq \sigma(2d)$ but since $p \neq 2d$ the inequality becomes strict. Therefore, again, we have that $\{p > 2d : \sigma(p) < \sigma(2d)\} = \varnothing$.

As we suspected from the calculation above, we have reduced the problem to the case where the d^{th} domino is vertical. We begin by observing that $\mathcal{I}_\sigma(2d) \leq m-1$. This is because if $p > 2d$ then $p > 2d-1$, and since p only contributes to $\mathcal{I}_\sigma(2d)$ when $\sigma(p) < \sigma(2d)$, it follows that

$$\sigma(2d-1) < \sigma(p) < \sigma(2d) = \sigma(2d-1) + m.$$

However, not all of the $m-1$ half-dominos will contribute to $\mathcal{I}_\sigma(2d)$; symbolically, the extras are the p such that $\sigma(2d-1) < \sigma(p) < \sigma(2d)$, but for which $p < 2d$. More intuitively, these are the half-dominos which appear between the two halves of the d^{th} domino, but which occur on earlier dominos. By the definition of the ordering, these can only be even halves of vertical dominos: since dominos are indexed by their odd halves, an odd half between the two halves of d necessarily occurs on a later domino. The diagram below elucidates this idea a bit more:

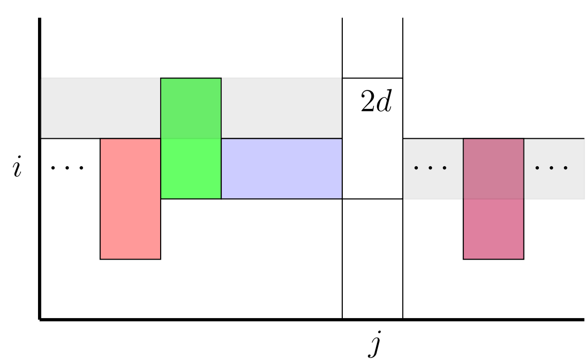

Figure 6: The half-dominos that do not contribute to $\mathcal{I}_\sigma(2d)$ are the even halves of all dominos whose odd halves occur before the odd half of domino d; we color these green and dark pink.

We will define each of the colors rigorously. Suppose $\sigma(2d) = \sigma(2d-1) + m = (i+1)m + j$:

- There are p_D dark-pink vertical dominos $c < d$ with even halves on row i such that $\sigma(2c-1) < \sigma(2d-1) < \sigma(2c) < \sigma(2d)$.
- Similarly there are g green dominos. These satisfy the same inequalities but their even halves are on row $i+1$.
- The other vertical dominos with even halves on row i are light-pink, there are p_L such dominos $c < d$ with $\sigma(2c) < \sigma(2d-1)$.
- Finally, there are b blue horizontal dominos $c < d$ on row i. (The horizontal dominos $c > d$ on row i we leave uncolored)

Therefore, as we argued above, $\mathcal{I}_\sigma(2d) = (m-1) - p_D - g$. Since we are only concerned with the parity, we get the following equivalences mod 2:

$$\mathcal{I}_\sigma(2d) \quad \equiv \quad (m+1) + p_D + g \quad \equiv \quad 1 + p_D + g \quad \equiv \quad 1 + p_D + g + 2b.$$

To complete the proof, we need a basic fact: the number of vertical dominos with odd halves on any row r is even. This is easy to prove by induction: because rows have even length, it must clearly be true for the first row. If it is true for row i then an even number of locations in row $i+1$ are taken up by the even halves of the dominos with odd halves on row i, and therefore the statement is true for row $i+1$ as well.

Since m is even, it follows that the number of vertical dominos with even halves on row r is also even, which for us means that $p_D + p_L \equiv 0$, or in other words $\mathcal{I}_\sigma(2d) \equiv 1 + p_L + g + 2b$.

This is a very significant change, because any half-dominos on row i to the left of $2d-1$ are either the even half of a vertical domino, or the odd half of a vertical domino, or some half of a horizontal domino. The number of such halves correspond directly to the terms in the above equivalence. Therefore:

$$j \;=\; p_L + g + 2b + 1 \quad \equiv \quad \mathcal{I}_\sigma(2d).$$

Since j is the number of the column in which the domino is located, we have

$$\mathcal{I}_\sigma = \sum_{\text{v. dom.}} \mathcal{I}_\sigma(2d) \;\equiv\; \sum_{\text{v. dom.}} j \;\equiv\; \#\{\text{vertical dominos in odd columns}\}$$

which is precisely the desired result. □

2.4 Finishing Up

We have now shown a direct way to go from a tiling to the sign of its permutation without any tedious calculation. With this knowledge, we will update our guess for D to produce the correct matrix. Remember that our first guess was to set all of the entries corresponding to horizontal domino pairs to x and the entries corresponding to vertical domino pairs to y; this was not quite right because the $\operatorname{sgn}(\sigma)$ term remained.

However, we now know that if $\beta = \beta_o + \beta_e$, where β_o is the number of vertical dominos in odd rows and β_e is the number of vertical dominos in even rows, then our old guess can be written as

$$\operatorname{Pf}(D') = \sum_{\text{tilings}} (-1)^{\beta_o} x^\alpha y^\beta = \sum_{\text{tilings}} (-y)^{\beta_o} x^\alpha y^{\beta_e}$$

Therefore, we can guarantee a positive sign if we replace all of the y in odd-numbered columns with $-y$. Hence, we update our guess by defining all entries of D to be zero, except the entries corresponding to horizontal domino pairs:

$$D(a, a+1) \quad = \quad x \qquad\qquad a+1 \neq km \text{ for any } k,$$

and to even-column vertical domino pairs:

$$D(2b, 2b+m) \quad = \quad y \qquad\qquad 2b+m \leq nm,$$

and to odd-column vertical domino pairs:

$$D(2b-1, 2b-1+m) \quad = \quad -y \qquad\qquad 2b+m \leq nm.$$

Finally, by the antisymmetry conditions, we are also forced to have

$$\begin{aligned}
D(a+1, a) &\quad = \quad -x &\qquad& a+1 \neq km \text{ for any } k,\\
D(2b+m, 2b) &\quad = \quad -y &\qquad& 2b+m \leq nm,\\
D(2b-1+m, 2b-1) &\quad = \quad y &\qquad& 2b+m \leq nm.
\end{aligned}$$

First, this really is an update to D' in that it precisely assigns x to each horizontal domino pair, and $\pm y$ to each vertical domino pair depending on parity. Second, this D is precisely the one described in the introduction.

Moreover, by using these signs, we have proven the combinatorial half of Kasteleyn's formula:

Theorem. Suppose that D is as defined above. Then

$$\operatorname{Pf}(D) = \sum_{\text{tilings}} x^\alpha y^\beta,$$

where the sum is over all tilings of an $n \times m$ board (m even) with dominos, where α is the number of horizontal dominos and β is the number of vertical dominos.

3 The Algebraic Half

As before, we will make note of the structure of this half before proceeding. Unfortunately, the narrative for this section is not as clean; this is true in our exposition as well as in both Kasteleyn's and Loehr's proofs. However, we have some reason to believe that this directionlessness is not inherent in the argument, and we provide some hopeful remarks in Section 4.

The tractability of the algebraic half hinges on the fact that D can be expressed as a certain sum of tensor products, where the same matrix occurs in opposite factors in each term. Although the resulting tensor product of eigenvectors does not quite diagonalize the matrix, it does ultimately make it block-diagonal with a block size of 2×2. We introduce tensor products in Section 3.1 along with the expression for D, and in section 3.2 we produce the diagonalization. Finally, in section 3.3 we evaluate the determinant and join the two halves together to complete the proof.

3.1 A Happy Accident

The tensor product is foundational to the work we will do throughout this section, so we will take a momentary respite from the main work of this proof to give the unfamiliar reader a crash course.

Definition. The **tensor product** (or Kronecker product) $A \otimes B$ of two matrices $A = [a_{ij}]_{i,j=0}^{n}$ and B is given by the following block form:

$$\begin{bmatrix} a_{11}B & a_{12}B & & a_{1n}B \\ a_{21}B & a_{22}B & & a_{2n}B \\ & & \ddots & \\ a_{n1}B & a_{n2}B & & a_{nn}B \end{bmatrix}.$$

A remark about terminology: a *tensor product* is properly a very technical construct in multilinear algebra, which we are not going to get remotely close to. Therefore, when realized as a matrix, some prefer to call this the Kronecker product. However, the essence of the general tensor product is simply to make this notion coordinate-free, so we will use the generic term.

The tensor product allows us to more compactly describe a broad, useful class of block matrix constructions. Moreover, because it is simply a matrix constructed in blocks, it is essentially the nicest operator that we can imagine on the set of matrices. It acts very well with all of the other matrix operations, and the relevant results are taken from [6], without proof, on the next page.

Proposition. For $n \times n$ matrices A and B, $m \times m$ matrices C and D, invertible matrices M and N, and scalar α, the tensor product has the following properties:

- $(A+B) \otimes C = A \otimes C + B \otimes C$.
- $A \otimes (C+D) = A \otimes C + A \otimes D$.
- $\alpha A \otimes C = \alpha(A \otimes C) = A \otimes \alpha C$.
- $(A \otimes C)(B \otimes D) = AB \otimes CD$.
- $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$.

The reason why the notion of the tensor product helps us out considerably is because if we recall the horizontal and vertical matrices that make up D from the introduction:

$$H_k = \begin{bmatrix} & 1 & & & & & \\ -1 & & 1 & & & & \\ & -1 & & 1 & & & \\ & & -1 & & & & \\ & & & & \ddots & & \\ & & & & & & 1 \\ & & & & & -1 & \end{bmatrix} \qquad V_m = \begin{bmatrix} -1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix},$$

then we can write D in a very simple form using the tensor product:

$$D = \begin{bmatrix} xH_m & yV_m & & & & \\ -yV_m & xH_m & yV_m & & & \\ & -yV_m & xH_m & & & \\ & & & \ddots & & \\ & & & & xH_m & yV_m \\ & & & & -yV_m & xH_m \end{bmatrix} = I_n \otimes xH_m + H_n \otimes yV_m.$$

What is particularly interesting about this construction is that H_k shows up as a factor of both summands. Because of the tensor product's tendancy to play nice with matrix operations, it also works very well with diagonalization.

Suppose that we can diagonalize H_k to the diagonal matrix $\Lambda_k = U_k^{-1} H_k U_k$. Then if we diagonalize in both factors, $\tilde{\Lambda} = (U_n \otimes U_m)^{-1} D (U_n \otimes U_m)$, we can imagine that the multiple occurences of H_k may work to our advantage. In particular, $\tilde{\Lambda}$ ends up being block-diagonal, which we can prove by a straightforward application of the tensor properties.

$$
\begin{aligned}
\tilde{\Lambda} &= (U_n \otimes U_m)^{-1}(I_n \otimes xH_m + H_n \otimes yV_m)(U_n \otimes U_m) \\
&= (U_n^{-1} \otimes U_m^{-1})(I_n \otimes xH_m)(U_n \otimes U_m) + (U_n^{-1} \otimes U_m^{-1})(H_n \otimes yV_m)(U_n \otimes U_m) \\
&= (U_n^{-1} I_n U_n \otimes xU_m^{-1} H_m U_m) + (U_n^{-1} H_n U_n \otimes yU_m^{-1} V_m U_m) \\
&= x(I_n \otimes \Lambda_m) + y(\Lambda_n \otimes U_m^{-1} V_m U_m)
\end{aligned}
$$

Because both I_n and Λ_n are diagonal matrices, it follows that $\tilde{\Lambda}$ is also block-diagonal. (If the remaining factor $U_m^{-1} V_m U_m$ was also diagonal, we would have a full diagonalization, but as of yet we cannot say anything this strong.) This "diagonalization" of D is of particular importance for us because of the following classical theorem:

Theorem (Muir [7]). If A is a $2L \times 2L$ skew-symmetric matrix, $\det(D) = [\mathrm{Pf}(D)]^2$.

A full diagonalization is as good as computing the determinant, and so the block-diagonalization is a good first step. In the next section, we will display an explicit form for $\tilde{\Lambda}$ by finding Λ_k and U_k, as well as calculate the remaining factor $U_m^{-1} V_m U_m$. We will find that we have not fully diagonalized the matrix, but we have gotten close enough to compute the determinant.

3.2 Diagonalization

We begin by noting that H_k is skew-symmetric, and therefore we can apply the Spectral Theorem over $\mathbb{C}$: it is diagonalizable into k mutually orthogonal eigenspaces, which must necessarily be one-dimensional. Therefore, it is reasonable to seek a diagonalization of H_k.

We begin with a theoretical computation of the eigenvalues and eigenvectors. Since we have an explicit form for H_k, we can write down the system of equations determined by considering each coordinate of the equation $H_k x = \lambda x$:

$$
\begin{aligned}
x_2 &= \lambda x_1 \\
-x_1 + x_3 &= \lambda x_2 \\
-x_2 + x_4 &= \lambda x_3 \\
&\vdots \\
-x_{k-2} + x_k &= \lambda x_{k-1} \\
-x_{k-1} &= \lambda x_k
\end{aligned}
$$

We can attempt to solve this system naïvely, and the result is quite pretty: the last equation gives that $x_k = -\frac{1}{\lambda} x_{k-1}$ and so the second-to-last equation gives that $-x_{k-2} = (\lambda + \frac{1}{\lambda}) x_{k-1}$

and so $x_{k-1} = -1/(\lambda + 1/\lambda)x_{k-2}$. It is not difficult to see by induction that working all the way back to the second equation gives

$$x_2 = \frac{-x_1}{\lambda + \frac{1}{\lambda + \frac{1}{\ddots \lambda}}},$$

where there are $k-1$ fraction bars. But from the first equation we have that $x_2 = \lambda x_1$, and so by dividing through by x_1 we have

$$0 = \lambda + \frac{1}{\lambda + \frac{1}{\lambda + \frac{1}{\ddots \lambda}}},$$

where there are k instances of λ. Note it is safe to divide through by x_1 because if $x_1 = 0$ then every entry of the vector is zero, and 0 is not an eigenvector.

Unfortunately, producing the rational function given by this continued fraction would take us too far afield, and even once we have it, there is no guarantee that we will be able to solve it efficiently. Therefore, we will do as Kasteleyn did and simply state the answer.

3.2.1 Eigenvalues and eigenvectors

From the theoretical calculation above, we know that if we have a number λ and a sequence of numbers (x_a) for $0 \leq a \leq k+1$, these will represent an eigenvalue and an eigenvector of H_k if they satisfy the proper algebraic relations. Specifically, these are that $x_0 = x_{k+1} = 0$ and all other x_a satisfy the recurrence $-x_{a-1} + x_{a+1} = \lambda x_a$.

Define a matrix U_k and a list of complex numbers as follows:

$$U_k(a,b) = i^a \sin\left(\frac{\pi ab}{k+1}\right) \qquad \lambda_k(b) = 2i\cos\left(\frac{\pi b}{k+1}\right).$$

If we think of each column of the matrix U_k, that is, $U_k(a,b)$ for any fixed b, as being a vector, then the point of the following lemma is to show that it satisfies the algebraic conditions needed to be an eigenvector with eigenvalue $\lambda_k(b)$:

Lemma. For $1 \leq a, b \leq k$ we have

$$-U_k(a-1,b) + U_k(a+1,b) = \lambda_k(b)U_k(a,b).$$

The proof, unfortunately, is relatively unenlightening. Beginning on the left-hand side and expanding the definition of the $U_k(a,b)$ and $\lambda_k(b)$, we factor out imaginary terms:

$$-i^{a-1}\sin\left(\frac{\pi ab - \pi b}{k+1}\right) + i^{a+1}\sin\left(\frac{\pi ab + \pi b}{k+1}\right) = i^{a+1}\left[\sin\left(\frac{\pi ab - \pi b}{k+1}\right) + \sin\left(\frac{\pi ab + \pi b}{k+1}\right)\right].$$

The proof completes by applying the identity, $\sin(x+y)+\sin(x-y) = 2\sin(x)\cos(y)$ for $x = \pi ab/(k+1)$ and $y = \pi b/(k+1)$:

$$i^{a+1}\left[\sin\left(\frac{\pi ab - \pi b}{k+1}\right) + \sin\left(\frac{\pi ab + \pi b}{k+1}\right)\right] = 2i^{a+1}\sin\left(\frac{\pi ab}{k+1}\right)\cos\left(\frac{\pi b}{k+1}\right).$$

□

We already saw how this lemma is actually a statement about eigenvectors, but we can go even further and bundle this information together into a single matrix equation. Doing this, we find that $H_k U_k = U_k \Lambda_k$, where Λ_k is the diagonal matrix with (b, b) entry equal to $\lambda_k(b)$. Therefore, $U_k^{-1} H_k U_k = \Lambda_k$.

To complete our goal of determining $\tilde{\Lambda}$, we need to evaluate $U_m^{-1} V_m U_m$. It is not clear that this should be anything particularly nice, but it ends up about as clean as one could hope:

Lemma. For $1 \leq a, b \leq k$ we have

$$(-1)^a U_m(a, b) = U_m(a, m+1-b).$$

This time we start on the right hand side and use the sum-of-angles identity:

$$i^a \sin\left(\frac{\pi a(m+1) - \pi ab}{m+1}\right) = i^a\left[\sin(\pi a)\cos\left(\frac{\pi ab}{m+1}\right) - \cos(\pi a)\sin\left(\frac{\pi ab}{m+1}\right)\right]$$

The first term vanishes, and $\cos(\pi a) = (-1)^a$, so the result follows. □

Again, by putting this into a matrix equation, the left side is simply $V_m U_m$. The right side requires a bit more thought, but we eventually see that it is simply the (a, b) entry of U_m right-multiplied by the permutation matrix that sends b to $m+1-b$. As a matrix, this looks like the identity but with the ones going along the *anti-diagonal*, so we call it J_m.

We have thus shown that $V_m U_m = U_m J_m$, or in other words $U_m^{-1} V_m U_m = J_m$.

Finally, we round out the block-diagonalization by writing an explicit form for the blocks of $\tilde{\Lambda} = x(I_n \otimes \Lambda_m) + y(\Lambda_n \otimes J_m)$. The block whose bottom-right entry is (bm, bm) is given by:

$$\begin{bmatrix} x\lambda_m(1) & & & & & & & y\lambda_n(b) \\ & x\lambda_m(2) & & & & & y\lambda_n(b) & \\ & & \ddots & & & \iddots & & \\ & & & x\lambda_m(\frac{m}{2}) & y\lambda_n(b) & & & \\ & & & y\lambda_n(b) & x\lambda_m(\frac{m}{2}+1) & & & \\ & & \iddots & & & \ddots & & \\ & y\lambda_n(b) & & & & & x\lambda_m(m-1) & \\ y\lambda_n(b) & & & & & & & x\lambda_m(m) \end{bmatrix}.$$

Remember that m is even, which means this form is always meaningful: the two diagonals are guaranteed to "miss" in the middle.

3.3 Concluding the Proof

In fact, we can diagonalize $\tilde{\Lambda}$ a little further. The blocks of the matrix appear to be made of of X-shapes, or alternatively, a bunch of concentric squares. If we permute the rows and columns of the matrix such that we let the corners of each square form a 2×2 block, then we have determinants which we definitely know how to calculate. We should be mildly concerned that we have a sign error, but this turns out not to be an issue. Every 2×2 block we form requires two transpositions, one row and one column, and so the sign changes cancel out.

We are now on the home stretch. Therefore the determinant of $\tilde{\Lambda}$ is just the product of the determinants of the blocks:

$$\det(\tilde{\Lambda}) = \prod_{m \times m \text{ blocks}} \prod_{2 \times 2 \text{ blocks}} \det \begin{bmatrix} x\lambda_m(a) & y\lambda_n(b) \\ y\lambda_n(b) & x\lambda_m(m+1-a) \end{bmatrix}$$

$$= \prod_{b=1}^{n} \prod_{a=1}^{m/2} \det \begin{bmatrix} x\lambda_m(a) & y\lambda_n(b) \\ y\lambda_n(b) & x\lambda_m(m+1-a) \end{bmatrix}$$

$$= \prod_{b=1}^{n} \prod_{a=1}^{m/2} \Big(x^2\lambda_m(a)\lambda_m(m+1-a) - y^2\lambda_n(b)^2 \Big).$$

We substitute the explicit forms of the eigenvalues and use the identity $\cos(\pi - \theta) = -\cos\theta$ to find that

$$\det(\tilde{\Lambda}) = \prod_{b=1}^{n} \prod_{a=1}^{m/2} \left(-4x^2 \cos \frac{\pi a}{m+1} \cos \frac{\pi(m+1-a)}{m+1} + 4y^2 \cos^2 \frac{\pi b}{n+1} \right)$$

$$= 4^{mn/2} \prod_{b=1}^{n} \prod_{a=1}^{m/2} \left(x^2 \cos^2 \frac{\pi a}{m+1} + y^2 \cos^2 \frac{\pi b}{n+1} \right)$$

Finally, take the square root, cite the Muir result, and apply The Combinatorial Half to conclude our proof:

Theorem. If D is defined as in section 2.3 and $\tilde{\Lambda}$ as in 3.1, then

$$\sum_{\text{tilings}} x^\alpha y^\beta = \mathrm{Pf}(D) = \sqrt{\det(\tilde{\Lambda})} = 2^{mn/2} \prod_{k=1}^{m/2} \prod_{\ell=1}^{n} \left(x^2 \cos^2 \frac{\pi k}{m+1} + y^2 \cos^2 \frac{\pi \ell}{n+1} \right)^{1/2}.$$

where the sum is over all tilings of an $m \times n$ board (m even) with dominos, with α the number of horizontal dominos and $\beta = \frac{1}{2}mn - \alpha$ the number of vertical dominos.

4 Possible Future Work

We have presented here a detailed exposition of Kasteleyn's solution to the dimer model, presented as a counting problem of dominos on a rectangular board. Ultimately, we believe it is possible to prove this result by approaching it from a rather different direction. While we of course cannot give any details yet, we would like to give a high-level summary of the proposed proof.

When we defined D, we essentially agreed that we were counting domino tilings at weight $x^\alpha y^\beta$ and other ways of pairing up board positions at weight 0. We might imagine doing something different by counting domino tilings at weight $x^\alpha y^\beta$ but counting other pairings at *average* weight 0. Ideally, we would be able to construct weights on the tiles that extend to weights of matchings in such a way that the pairings partition into naturally-defined orbits whose total weight is zero, so that

$$\sum_{\text{pairings}} w(P) = \sum_{\text{tilings}} x^\alpha y^\beta + \sum_{\text{orbits}} 0,$$

and therefore the weighted sum of all pairings add up to the desired sum. This idea is not new; in fact, one can interpret the principle of inclusion-exclusion as a counting problem of this sort. Since we essentially built the machinery to talk about pairings in this exposition, it is easy to refer to expanding our "universe" in that direction. In practice, however, we expect to expand in other directions, perhaps to include domino *coverings*, which may overlap or spill across one edge of the board, or even partial coverings which do not cover the entire board.

We have gone over Kasteleyn's proof (and Loehr's simplification) in some detail in hopes that it would provide some insight, and indeed we are optimistic that this will be possible. Our hope essentially comes from the opaqueness of the linear-algebraic half of the proof, which seems like it should have a more useful interpretation from other perspectives.

One of the strongest ideas we can easily observe is that H_k is actually a matrix of some independent significance: it is the adjacency matrix of a directed path graph. Similarly, D is also an adjacency matrix, whose significance is easier to see through the lens of our naïve first guess D'. This corresponds to a directed checkerboard graph, i.e. the direct product of a path graph of length m and a path graph of length n.

The effect of changing signs to D is that the orientation of every other column path becomes flipped. We also note that since m is even there are an equal number of up-oriented columns as down-oriented columns, and there is a strong temptation for us to think of this as some sort of *semi-direct graph product*. This is not a concept explored in the literature, and we hope to search for some low-hanging fruit in this direction, as it is easy to imagine that it may be of independent interest.

Once we are thinking in graph-theoretic terms, a couple other directions open as well. First of all, any linear algebraic calculations in this context could be interpreted through the lens of spectral graph theory. It would be beneficial to see if there are any obvious ways to use this interpretation to make the argument a bit more intuitive.

Secondly, it suggests some topological alterations to the problem which might be theoretically simpler. If the ordinary checkerboard graph gives the theory of tiling a rectangle, then it is easy to join up both ends to produce the (essentially identical) theory of tiling a torus. Kastelyen devotes a section to this variant in his original paper.

However, given that the columns' orientations are alternating, an intriguing second option emerges: perhaps the most natural setting for domino tilings is actually a *Klein bottle*. Using the fundemental polygon, it is simple to see that the orientations caused by this identification would then be consistent in both directions. Since Kasteleyn was interested in physical models, it is not surprising that he did not consider this variant, but it seems plausible that this has some similar and simpler theory.

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