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Abstract: A generalized method for solving an undamped second order, linear ordinary differential equation with constant coefficients is presented where the non-homogeneous term of the differential equation is represented by Fourier series and a solution is found through Laplace transforms. This method makes use of a particular partial fraction expansion form for finding the inverse Laplace transform. If a non-homogeneous function meets certain criteria for a Fourier series representation, then this technique can be used as a more automated means to solve the differential equation as transforms for specific functions need not be determined. The combined use of the Fourier series and Laplace transforms also reinforces the understanding of function representation through a Fourier series and its potential limitations, the mechanics of finding the Laplace transform of a differential equation and inverse transforms, the operation of an undamped system, and through programming insight into the practical application of both tools including information on the influence of the number of terms in the series solution.

1 Introduction

Typical presentation of Laplace transforms in differential equation coursework involves the solution of a constant coefficient, ordinary differential equation (ODE) with an impulse or single pulse non-homogeneous term [2, 5]. Fourier series representations of a function are often introduced simply as a means to replicate function values, and are seldom tied directly to the solution of ODEs. Instead, the series are commonly relegated to definitions of the Fourier integral and the Fourier transform[2, 6]. However, for a particular undamped form of the second order, constant coefficient, linear ODE frequently found in differential equation curriculum and in the modeling of physical systems, the concepts of the Laplace transform and the Fourier series may be combined when the non-homogeneous term meets certain Fourier series representation criteria. Exercising this method can provide for a deeper understanding of both topics. Programming of the method enlightens Students to the effects of the number of terms in the series representation, removing the infinite series as abstract concepts. The method can serve as a generalized solution procedure for differential equations and non-homogeneous term functions of a specific form since the Laplace transforms associated with the use of the Fourier series representation are fixed and new transforms and inverse transforms are not required despite different forms of the non-homogeneous terms. The concepts described in this work stemmed from the preparation of the manuscript regarding the special case of partial fraction expansion of Laplace transform resulting from course preparation for the Advanced Mathematics I and II courses at the United States Army Armament Graduate School[4]. In this work, the concept is explained and then applied for a harmonic forcing function, a case of resonance, and then a triangular pulse.

Undamped second order, linear ODEs with constant coefficients in the form provided in (1.1) can be used to describe the operation of a wide variety of systems including a mass and spring, a pendulum, a torsion string, a column of oscillating liquid, an oscillating buoyant object, and current in an electronic circuit without resistance.

$$y'' + \xi^2 y = f(t). \tag{1.1}$$

Undamped systems are often studied to estimate the maximum response to a given input or to illicit information about the natural frequencies of a system. The homogeneous solutions to differential equations of this form will involve a harmonic solution, or solution comprised of cosine and sine functions. The non-homogeneous term in the differential equations of this form, $f(t)$, represents an input into the system. Common inputs may already be harmonic in nature. However, many loading functions are non-harmonic including square, triangular, or other non-sinusoidal waves, pulses, impulses, and even irregular inputs that may vary in magnitude or timing.

Representation of these input non-harmonic, non-homogeneous term functions through a Fourier series when possible can provide significant advantages when solving a differential equation through a Laplace transform based method. Application of standard methods like undetermined coefficients or variation of parameters may become cumbersome for piecewise continuous type functions to which Laplace transforms are commonly applied. Instead of determining the Laplace transform of each unique function each time a new problem is solved, once the input function is written in the Fourier series form, the Laplace transform of the constant, cosine, and sine based functions comprising the Fourier series are well known and the inverse transforms can be found. The Laplace transform of the solution, $y(t)$, takes a consistent form with the most complex terms involve a denominator containing the product of two terms of the form $(s^2 + \alpha^2)\ (s^2 + \beta^2)$.The inverse transforms of terms of this form can be readily found through methods described in [4] as the partial fraction expansion involves the decoupling of terms with s and a constant in the numerator. Although the forcing functions are restricted by requirements for the Fourier series representation and care must be taken to ensure sufficient series terms not only for series representation of the function, but also for the solution of the differential equation, this combined use method can prove a useful tool in solving ODEs of this common form. Further, actual programming of the algorithm provides a means to gain a more acute understanding of the Fourier series and the Laplace transform. After describing the Fourier series, the solutions for one constant, one cosine, and one sine function non-homogeneous term will be addressed, followed by application of the Fourier series to a general non-homogeneous term in the differential equation and the subsequent solution of the differential equation for a general series based non-homogeneous term.

2 Fourier series based forcing function

For differential equations of the form of (1.1) a general non-homogeneous term, $f(t)$, may be applied, where in this equation each term has been divided by the coefficient of the original y'' coefficient. This non-homogeneous function can be expressed as a Fourier Series under certain restrictions. The Fourier series representation of a function, f(t), with period $2L$ is given by the standard expression set in (2.1b) and (2.1c) [6, 3].

$$y'' + \xi^2 y = f(t) \text{ with,} \tag{2.1a}$$

$$f(t) = a_o + \sum_{n=1}^{\infty} \left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right); \tag{2.1b}$$

$$\text{where } a_o = \frac{1}{2L}\int_{-L}^{L} f(t)dt; \quad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\omega_n t)\,dt; \tag{2.1c}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\omega_n t)\,dt; \;\; \omega_n = \frac{n\pi}{L}.$$

Examining these functions, an even function will not involve any sine terms, for the b coefficients will be zero, and an odd function will not involve any sine terms as the a coefficients will be zero. The triangular pulse input function described later in this work, is an example of an even function where no sine terms will be present in the series representation. With the general expression for the coefficients, this reduction in terms will be calculated naturally during the evaluation of the coefficient integration. Fourier series can be used to represent many functions under the restriction of the Dirichlet conditions. The Dirichlet conditions for a function $f(t)$[7] are:

1. $f(t)$ must have a finite number of minima and maxima.
2. $f(t)$ must have a finite number of discontinuities and no infinite discontinuities.

It can be shown, that if the limit in the expression in (2.2) exists, then function $f(t)$ must also satisfy the Dirichlet conditions.

$$\lim_{B\to\infty} \left| \int_0^B f(t)dt \right|. \tag{2.2}$$

If the Dirichlet conditions are satisfied [8, 7], then,

$$\int_{-L}^{L} f(t)cos(\omega t)dt,$$

$$\int_{-L}^{L} f(t)sin(\omega t)dt,$$

each approach zero as ω approaches infinity, indicating convergence of the series. By the Weierstrass test, the Fourier series can be shown to be uniformly convergent. Uniform convergence means that the integration of the function can be represented by the term by term integration of the series [6]. The differentiation of the function can be represented by the term by term differentiation of the series provided the second derivative of the function is defined, continuous or has a right hand and left hand limit over the domain of interest[3]. The Fourier series representation involves an infinite sum. However, in practice an infinite sum cannot be achieved. An estimate on the square of the error with a finite number of terms, N_s, in a Fourier series representation can be made using Parseval's identity[6],

$$E_s = \int_{-L}^{L} f(t)^2 dt - L \sum_{n=1}^{N_s} \left[2a_o^2 + (a_n^2 + b_n^2) \right]. \tag{2.3}$$

Potential Fourier series-based function sets include discontinuous and/or repeated or irregular pulse functions as well as continuous functions. For conditions where the Fourier series representation of a function is defined, a differential equation in the form of (1.1) can be solved upon determining the Fourier series representation the function. The homogeneous solution and then the non-homogeneous solution associated with each of the non-homogeneous terms in the series will be described next along with a discussion on estimating the error in the solution using a finite number of terms in the series.

3 Second order linear, ordinary differential equation with constant coefficients

A second order, linear, ODE with constant coefficients is given in (1.1) with $y(t)$ subject to the initial conditions $y(0) = \Phi$ and $y'(0) = \Psi$. The Laplace transform of this differential equation can be taken and used to find the solution. However, with the goal of application to the Fourier series representation of the non-homogeneous term, the description focuses on a specific non-homogeneous term form that includes one term for each of the functions found in the general Fourier series expansion. The solutions associated with the cosine and sine terms can then be summed for the solution with the Fourier series representation as explained in further detail later in this work.

3.1 Harmonic input function for constant coefficient ODE with Laplace transform and a special case of partial fraction expansion

Consider the constant coefficient second order ODE of the form given in (3.1) with the initial conditions of $y(0) = \Phi$ and $y'(0) = \Psi$ and where the forcing functions involve two harmonic functions of frequency, ω, along with a constant a_o,

$$y'' + \xi^2 y = a_o + a\cos(\omega t) + b\sin(\omega t). \tag{3.1}$$

The homogeneous form of the differential equation will be addressed first.Taking the Laplace transform of the differential equation yields:

$$s^2Y(s) - sy(0) - y'(0) + \xi^2 Y(s) = \frac{a_o}{s} + \frac{as}{s^2+\omega^2} + \frac{b\omega}{s^2+\omega^2}.$$

Substituting the initial conditions gives:

$$(s^2+\xi^2)Y(s) = \Phi s + \Psi + \frac{a_o}{s} + \frac{as}{s^2+\omega^2} + \frac{b\omega}{s^2+\omega^2}.$$

Solving for Y(s):

$$Y(s) = \frac{\Phi s}{s^2+\xi^2} + \frac{\Psi}{s^2+\xi^2} + \frac{a_o}{s(s^2+\xi^2)} + \frac{as}{(s^2+\xi^2)(s^2+\omega^2)} + \frac{b\omega}{(s^2+\xi^2)(s^2+\omega^2)}. \tag{3.2}$$

Now, the inverse transform of the first two terms can be found from standard textbook Laplace transform tables. The inverse of the third term can be found from a simple partial fraction expansion.

$$\frac{a_0}{s(s^2+\xi^2)} = \frac{a_0}{\xi^2}\frac{1}{s} - \frac{a_0}{\xi^2}\frac{s}{(s^2+\xi^2)},$$

yielding an inverse transform of,

$$a_o\frac{1}{\xi^2}\left(1-\cos(\xi t)\right).$$

The inverse transform of the fourth and fifth terms, often not in standard textbook Laplace transform tables, can be found through a special case of partial fraction expansion [4]. Let $P = a$ and $Q = b\omega$. Then,

$$\frac{Ps+Q}{(s^2+\xi^2)(s^2+\omega^2)} = \frac{As+B}{s^2+\xi^2} + \frac{Cs+D}{s^2+\omega^2}.$$

To complete the inverse transform, two conditions must be considered. The first is when the forcing function oscillation or applied frequency, ω, differs from the natural frequency of the system, ξ, or $\xi \neq \omega$. The second is when the forcing function oscillation or applied frequency is the same as the natural frequency of the system or $\xi = \omega$.

3.1.1 Case where natural frequency and applied frequency differ

Assuming $\xi \neq \omega$, or the case where the applied forcing function frequency differs from the natural frequency of the system, utilizing partial fraction expansion, the A and C coefficients of s are decoupled from the B and D coefficients [4]. For an expression of this form, the expansion coefficients are:

$$A = \frac{P}{\omega^2-\xi^2}; \quad B = \frac{Q}{\omega^2-\xi^2}; \quad C = -\frac{P}{\omega^2-\xi^2}; \quad D = -\frac{Q}{\omega^2-\xi^2}.$$

The resulting partial fraction expansion is simply:

$$\frac{Ps+Q}{(s^2+\xi^2)(s^2+\omega^2)} = \frac{1}{(\omega^2-\xi^2)}\frac{Ps+Q}{(s^2+\xi^2)} - \frac{1}{(\omega^2-\xi^2)}\frac{Ps+Q}{(s^2+\omega^2)}. \tag{3.3}$$

Taking the inverse transform of (3.2), with (3.3) substituted into (3.2), the solution for $y(t)$ is:

$$\begin{aligned} y(t) =&\Phi \cos(\xi t) + \frac{1}{\xi}\Psi \sin(\xi t) + a_o \frac{1}{\xi^2}(1-\cos(\xi t)) \\ &+ \frac{1}{(\omega^2-\xi^2)}\left[P(\cos(\xi t)-\cos(\omega t)) + Q\left(\frac{1}{\xi}\sin(\xi t) - \frac{1}{\omega}\sin(\omega t)\right)\right]. \end{aligned} \tag{3.4}$$

With any sine and cosine forcing function with a frequency differing from the natural frequency of the system, the solution for $y(t)$ can be found using (3.4).

3.1.2 Case where natural frequency and applied frequency are the same

Now, consider the case where the natural frequency of the system and the applied frequency related to the forcing function are equal, $\xi = \omega$. The governing equation is:

$$y'' + \xi^2 y = a_o + a\cos(\xi t) + b\sin(\xi t).$$

Taking the Laplace transform, and solving for $Y(s)$ yields:

$$Y(s) = \frac{\Phi s}{s^2+\xi^2} + \frac{\Psi}{s^2+\xi^2} + \frac{a_o}{s(s^2+\xi^2)} + \frac{as}{(s^2+\xi^2)^2} + \frac{b\omega}{(s^2+\xi^2)^2}. \tag{3.5}$$

The special partial fraction expansion with the decoupled pairs of coefficients just described also comes about for the fourth and fifth terms in (3.5). Letting $P = a$ and $Q = b\omega$, the partial fraction expansion of the last two terms in (3.5) can be written as [4]:

$$\frac{Ps+Q}{(s^2+\xi^2)^2} = \frac{Ps}{(s^2+\xi^2)^2} + \frac{Q}{(s^2+\xi^2)^2}. \tag{3.6}$$

Now, the inverse transform of the first term in (3.6) can be found by applying the derivative property that $\mathscr{L}(tf(t)) = -F'(s)$. Letting $g(t) = \frac{P}{\xi}\sin(\xi t)$, the transform becomes $G(s) = \frac{P}{(s^2+\xi^2)}$. Taking the derivative with respect to s, $G'(s) = \frac{-2sP}{(s^2+\xi^2)^2}$, yielding just an extra 2 and negative sign from the original term, so that $\mathscr{L}(\frac{P}{\xi}t\sin(\xi\, t)) = \frac{2sP}{(s^2+\xi^2)^2}$. Therefore:

$$\mathscr{L}^{-1}\left(\frac{Ps}{(s^2+\xi^2)^2}\right) = \frac{P}{2\xi}t\sin(\xi t). \tag{3.7}$$

The inverse transform of the second term in (3.6) can be found by applying the integral property : $\mathscr{L}\left(\int_0^t f(t)dt\right) = \frac{1}{s}\mathscr{L}(f(t))$. Setting $\gamma(t) = \frac{Q}{2\xi}t\sin(\xi t)$, then $\mathscr{L}(\gamma(t)) = \frac{2sQ}{(s^2+\xi^2)^2}$. The integral of $\gamma(t)$ can be found : $\int_0^t \gamma(t)\,dt = \frac{Q}{2\xi}\left[\frac{1}{\xi^2}\sin(\xi t) - \frac{t}{\xi}\cos(\xi t)\right]$. Only an extra 2 appears from the Laplace transform needed. Applying the integral property,

$$\mathscr{L}^{-1}\left(\frac{Q}{(s^2+\xi^2)^2}\right) = \frac{Q}{2\xi}\left[\frac{1}{\xi^2}\sin(\xi t) - \frac{t}{\xi}\cos(\xi t)\right].$$

The solution for the case where the non-homogeneous forcing function term frequency matches the natural frequency of the system is:

$$y(t) = \Phi\cos(\xi t) + \frac{1}{\xi}\Psi\sin(\xi t) + a_o\frac{1}{\xi^2}(1-\cos(\xi t)) + \frac{P}{2\xi}t\sin(\xi t) + \frac{Q}{2\xi}\left[\frac{1}{\xi^2}\sin(\xi t) - \frac{t}{\xi}\cos(\xi t)\right]. \tag{3.8}$$

Hence, the solution to the differential equation for the conditions when the forcing frequency and the natural frequency are equal is known through (3.8).

3.1.3 Application to Fourier series based forcing function

Now that the solutions to the homogeneous form of (1.1) and a non-homogeneous function with a single constant, sine, and cosine function have been determined for both the cases of non-natural frequency and natural frequency non-homogeneous function frequencies, the solutions can be applied to a Fourier series representation of the forcing function or non-homogeneous term. Once the non-homogeneous function has been expressed in Fourier series form, the use of the combination of the Fourier series representation of the forcing function with a Laplace transform differential equation solution method provides a regimented, repeatable solution process for any appropriate non-homogeneous function.

Upon calculation of the coefficients of the Fourier series representation as per (2.1b), a solution can be determined for each of the cosine and sine terms in the Fourier series summation using the results in (3.4) and (3.8). Because the differential equation is linear, the the summation of the term-wise solutions associated with each Fourier frequency is also a solution to the non-homogeneous form of the differential equation. Hence, the solution to (2.1a) can be expressed by (3.9):

$$y(t) = \Phi\cos(\xi t) + \frac{1}{\xi}\Psi\sin(\xi t) + a_o\frac{1}{\xi^2}(1-\cos(\xi t)) + \sum_{n=1}^{\infty} Y_n. \tag{3.9}$$

where the Y_n are the solutions for terms related Fourier series cosine and sine forcing functions. Hence, for each term in the summation, a check needs to be performed whether the frequency in the Fourier series matches the natural frequency of the system, yielding the two cases in (3.10),

$$\xi \neq \omega_n; \quad Y_n = \frac{1}{(\omega_n^2 - \xi^2)}\left[P(\cos(\xi t) - \cos(\omega_n t)) + Q\left(\frac{1}{\xi}\sin(\xi t) - \frac{1}{\omega}\sin(\omega t)\right)\right], \tag{3.10a}$$

$$\xi = \omega_n; \quad Y_n = \frac{P}{2\xi}t\sin(\xi t) + \frac{Q}{2\xi}\left[\frac{1}{\xi^2}\sin(\xi t) - \frac{t}{\xi}\cos(\xi t)\right]. \tag{3.10b}$$

3.1.4 Algorithm and series solution

As described, the error in the series representation of a function by a finite number of terms can be estimated using Parseval's Identity in (2.3) which gives the square of the error. However, the number of terms in the series that yield a given error for the Fourier series representation will not yield the same error level in terms of the solution to the differential equation. Some explanation of the algorithm for the implementation of the method is therefore required.

In addition to specifying the constant coefficients of the homogeneous form of the differential equation and the initial conditions, the starting and ending times of the analysis and the timestep size or total number of timesteps or divisions must be specified, N_{step}. In this work a 50 divisions of the time period of interest were utilized. An acceptable error for the Fourier series representation of the function alone, ϵ_{smax}, must be set as well.In this work, the ϵ_{smax}, was set to 1e-05. An acceptable error level in the solution of the differential equation must also be defined, ϵ_{ymax}. This error level is set to 1e-05.

Initially, N_o=20 terms were used in the series with the coefficients calculated and Parseval's Identity applied to estimate the error. The number of terms is then doubled until the desired level of ϵ_{smax} is reached. The final number of terms in the summation is given by N_P.

Next, the solution of the differential equation is carried out using N_P terms, following (3.9) and (3.10). Once the solution is determined over the time period of interest, the number of terms is doubled. The two solutions at each of the timesteps, 1 through N_{step}, calculated are stored and then compared. Let i be the counter for the timesteps, j be the counter for the solution with the modified summation terms, N_j be the number of terms in the summation for solution j, the solution at a given time i for the two summation cases are $Y_{j-1,i}$ and $Y_{j,i}$ respectively,

$$Y_{j-1,i} = \sum_{n=1}^{N_{j-1}} \left(a_{n,j-1} \cos\left(\omega_{n,j-1} t_i\right) + b_{n,j-1} \sin\left(\omega_{n,j-1} t_i\right)\right),$$

$$Y_{j,i} = \sum_{n=1}^{N_j} \left(a_{n,j} \cos\left(\omega_{n,j-1} t_i\right) + b_{n,j} \sin\left(\omega_{n,j} t_i\right)\right).$$

An estimate of the error in the solution is given by,

$$\epsilon_y = \frac{1}{N_{step}} \sum_{i=1}^{N_{step}} \left(\left| Y_{j,i} - Y_{j,i-1} \right| \right). \qquad (3.11)$$

The process proceeds, storing and comparing the last two solutions until the error is within the desired limit, ϵ_{ymax}. Such a practice is a standard method to ensure solution independence of the number of terms in the series. [1] The basic algorithm is depicted in Figure 1.

For many functions without jumps or that vary more gradually with time, 20 terms are typically sufficient for the error levels specified. As an example, the Fourier series representation and solution with 2, 4, and 10 terms in the series for a triangular pulse in Section 4.3 is provided in Figure 2. The quick convergence of this solution is seen with the scale altered to show any noticeable difference.

Once the programming of the general solution form has been established, the method can be applied regardless of the input forcing function as long as the function meets

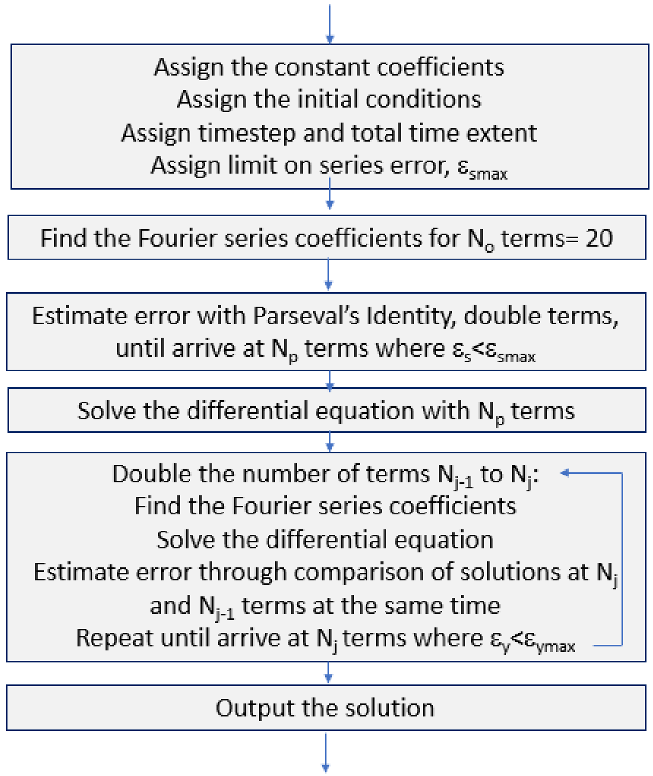


Figure 1: Method algorithm

the Dirichlet conditions. The general applicability of this method allows for a nearly ready-made means of determining the solution to differential equations of the required form using the combination of Laplace transforms and the Fourier Series.

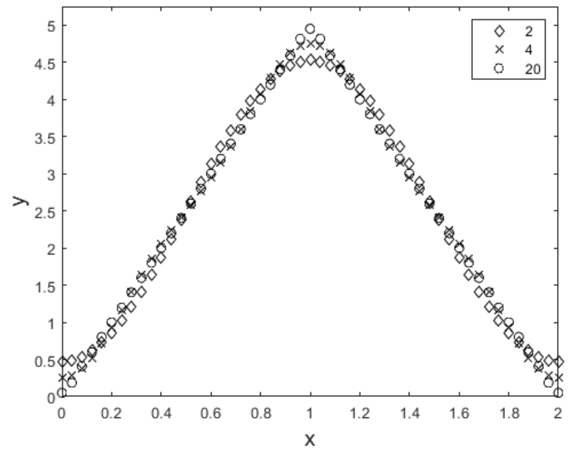
4 Application of the method

To demonstrate the method, the solution of a cosine forcing function at a different frequency than the natural frequency and then at a frequency equal to the natural frequency are presented. Finally, the method is applied to a system with a triangular pulse forcing function showing the utility of the method for more general forcing functions, including functions with discontinuities. As mentioned, with the triangular pulse, or other even functions, the sine Fourier series solution coefficients will be zero and for odd functions, the cosine series solution coefficients will be zero.

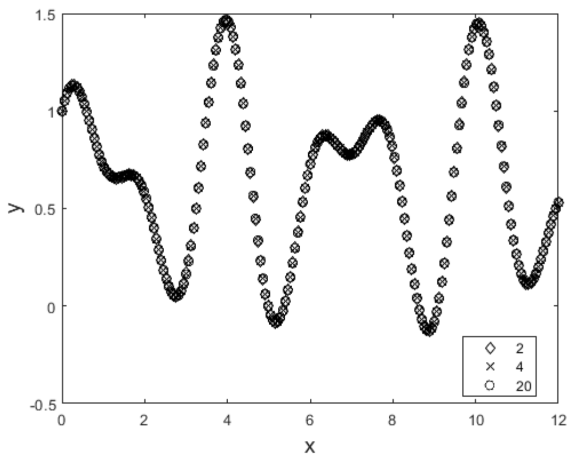
4.1 Validation of the method with independent cosine forcing function

In this first case, consider the IVP

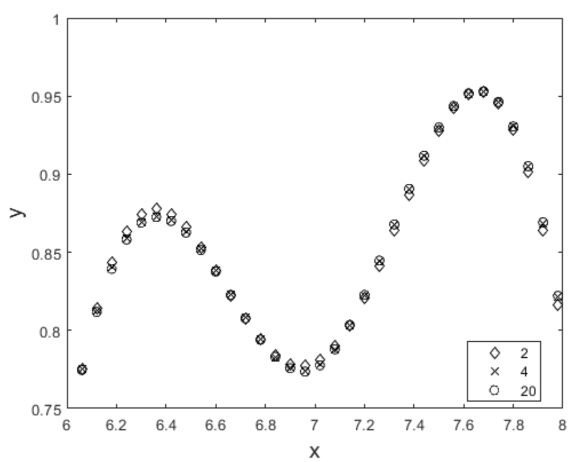
$$\begin{cases} y'' + 3^2 y = 5\cos(2t) \\ y(0) = y'(0) = 1. \end{cases} \tag{4.1}$$



(a) Fourier series representation of triangular pulse



(b) Solution for triangular pulse case with indicated series terms



(c) Solution with refined scale

Figure 2: Solution for triangular pulse case

In the Fourier series expansion of the non-homogeneous cosine term on the right side of the equation with a period of π, so $L = \pi/2$, the frequency of this cosine term, 2, matches with the frequency of one of the terms in the summation of the Fourier series expansion, ω_n, when $n = 1$ as the Fourier series frequencies are given in (4.2).

$$\omega_n = \frac{n\pi}{L}; \;\; L = \frac{\pi}{2}; \;\; \omega_1 = 2. \tag{4.2}$$

By orthogonality, only one term in the Fourier series is non-zero, the cosine term when $n = 1$ and $\omega = 2$. Therefore, comparing with the parameters in (3.9) and (4.1), $\xi = 3,\ y(0) = \Phi = 1; y'(0) = \Psi = 1; a_1 = 5, a_n = 0, n \neq 1, b_n = 0, P_n = a_n$ and $Q_n = b_n\, \omega_n = 0$. From (3.9) and (3.10), the solution to the differential equation is then provided in (4.3):

$$y(t) = \cos(3t) + \frac{1}{3}\sin(3t) + Y_1, \tag{4.3a}$$

$$\text{where} : (\xi = 3) \neq (\omega_1 = 2); \;\; Y_1 = \frac{5}{(2^2 - 3^2)}\left[\cos(3t) - \cos(2t)\right]. \tag{4.3b}$$

The reduction in the solution to a one term Fourier Series expansion will automatically be handled by the calculations of the Fourier series expansion coefficients and so special cases need not be considered. By inspection, the initial conditions are met with this solution. This solution also matches with the solution obtained using the method of undetermined coefficients, the standard method for solving such a differential equation. The solution obtained from code implementing the general technique described without any simplification for a single term is plotted in Figure 3.

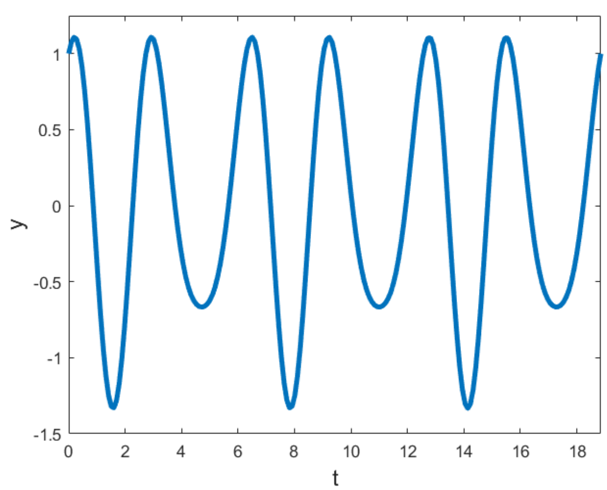


Figure 3: Solution for case with independent forcing function frequency

4.2 Validation of the method with cosine forcing function at the natural frequency

Next, the natural frequency of the system is modified to be the same as the frequency of the applied forcing function so that the second type of condition and solution can be tested. Consider the IVP

$$\begin{cases} y'' + 2^2 y = 5\cos(2t) \\ y(0) = y'(0) = 1. \end{cases} \tag{4.4}$$

The Fourier series expansion of the right side of (4.4) is the same as that in the previous section 4.1 with only the one term in the series expansion for $n = 1$. The applied frequency though is now equal to the system natural frequency. For this problem, the parameters are: $\xi = 2,\ y(0) = \Phi = 1; y'(0) = \Psi = 1; a_1 = 1, a_n = 0, n \neq 1, b_n = 0, P_n = a_n,$ and $Q_n = b_n\omega_n = 0$. From (3.9) and (3.10), for the second case where $\xi = \omega_1$, the solution to the differential equation of this form is then given in (4.5):

$$y(t) = \cos(2t) + \frac{1}{2}\sin(2t) + Y_1, \tag{4.5a}$$

$$\text{where} : \xi = \omega_1 = 2; \qquad Y_1 = \frac{t}{2 \cdot 2}\sin(2t). \tag{4.5b}$$

Again, the solution reached satisfies the initial condition and differential equation and matches the solution found through the method of undetermined coefficients. Once the general solution form is programmed, the algorithm can automatically select the appropriate solution form for the given term in the Fourier series expansion, whether the natural frequency and forcing function frequency differ or whether they are equal. The solution showing the expected resonance features produced by the general algorithm is plotted in Figure 4.

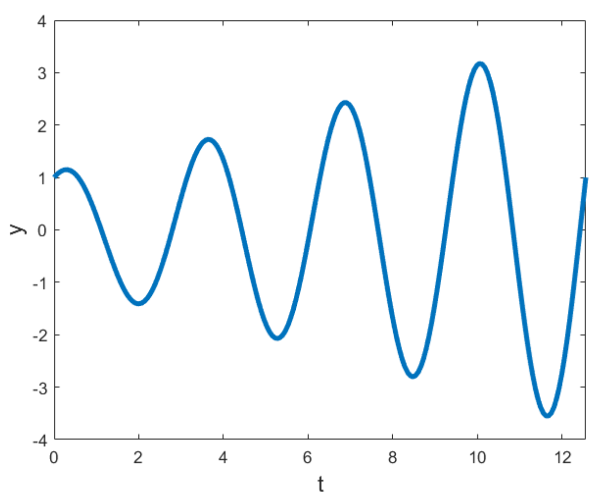


Figure 4: Solution for case with equal natural and forcing function frequency

4.3 Application of the method to a triangular pulse

Finally, in order to demonstrate the capabilities of the concept described, a forcing function that cannot be readily considered using the standard ordinary differential equation techniques is applied: a repeated triangular pulse. Consider the IVP

$$\begin{cases} y'' + 2^2 y = f(t) \\ y(0) = y'(0) = 1 \end{cases} \qquad \text{where } f(t) = \begin{cases} 5t, & t \in [0,1] \\ 5(2-t), & t \in (1,2]. \end{cases}$$

A graph of $f(t)$ is shown in Figure 5. The combined Fourier series/Laplace transform

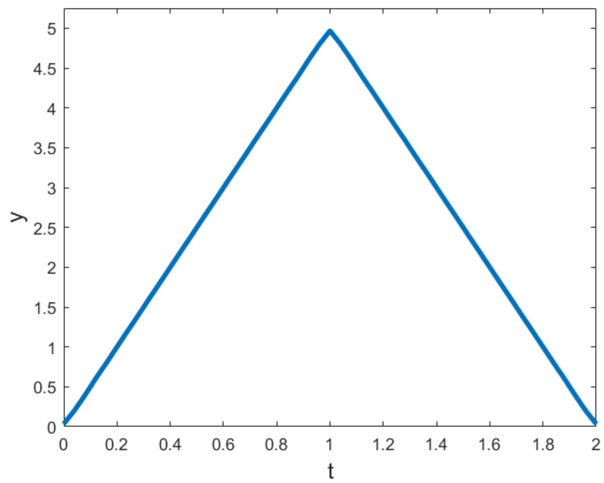


Figure 5: One period of the applied, repeated pulse, f(t)

method described is now implemented in its general form. With the natural frequency of $\xi = 2$, a period of 2, and so an L of 1, together with the information about the discontinuous and repeated forcing function segments and ranges, the solution can be found. The Fourier series coefficients can be calculated from the known information. Once these coefficients have been determined, then the coefficients in the series portion of the solution stemming from the necessary inverse Laplace transforms can be found, making allowances for whether the Fourier series term frequency matches the natural frequency of the harmonic system or whether the frequencies are independent. Hence, upon inserting the information about the forcing function and natural frequency, the solution to the differential equation can be found with relative ease. The solution for the problem presented is shown in Figure 6. Therefore, with the repeated triangular pulse representing any non-standard forcing function, the advantages of the combined method are evident. Also, by programming the basic algorithm, in a non-standard setting a clearer understanding of the Fourier series representation of functions and the Laplace transform techniques can be achieved.

5 Conclusions

In this work, a combined Fourier series-Laplace transform technique is described as a means for solving the ODEs for a harmonic, undamped system where a general forcing function can be represented by a Fourier series. The method allows for a generalized solution technique for any harmonic system where the forcing function may be written in

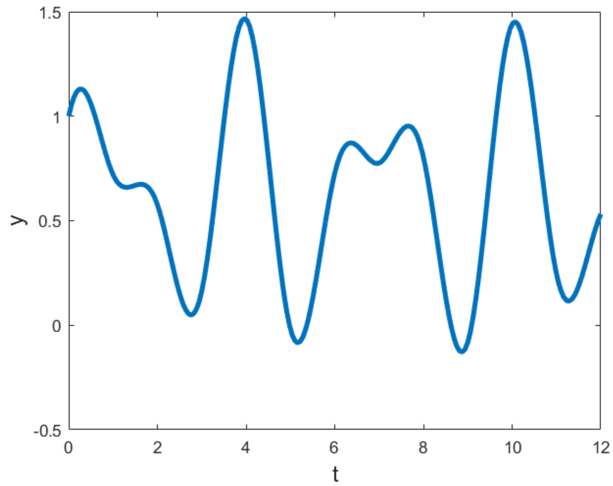


Figure 6: Solution with the given repeated pulse

terms of a Fourier series. This concept will be introduced as supplementary information after the Fourier Series topic is introduced to exhibit the interconnections of the topics discussed in the Advanced Mathematics classes. Because of the wide array of particularly mechanical and electrical systems that can be modeled through this form of an ODE, the technique can also be applied as a general analysis tool for the appropriate forcing functions.

References

[1] F.S. Acton. *Real Computing Made Real.* Princeton University, 1st edition, 1996.

[2] W.E. Boyce and R.C. DiPrima. *Elementary Differential Equations and Boundary Value Problems.* Wiley, 6th edition, 2005.

[3] R. Churchill. *Fourier Series and Boundary Value Problems.* McGraw-Hill, 2nd edition, 1963.

[4] L.A. Florio and R.D. Hanc. Special case of partial fraction expansion with laplace transform application. *CODEE Journal, Vol. 16, Article 2*, 2023.

[5] W. Greenberg. *Advanced Engineering Mathematics.* Prentice Hall, 3rd edition, 1998.

[6] E. Kreyszig. *Advanced Engineering Mathematics.* Wiley, 10th edition, 2010.

[7] I. Sneddon. *Fourier Transforms.* McGraw-Hill, 1st edition, 1951.

[8] E.C. Titchmarsh. *Fourier Integrals.* Oxford, 1st edition, 1948.