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Using a Sand Tank Groundwater Model to Investigate a Groundwater Flow Model

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Keywords: Aquifer, hydraulic head, Groundwater Flow Model, sand tank, separation of variables, Fourier's Law, IVBVP, no-flow boundary conditions, mixed boundary conditions
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Abstract: A Sand Tank Groundwater Model is a tabletop physical model constructed of plexiglass and filled with sand that is typically used to illustrate how groundwater water flows through an aquifer, how water wells work, and the effects of contaminants introduced into an aquifer. Mathematically groundwater flow through an aquifer can be modeled with the heat equation. We will show how a Sand Tank Groundwater Model can be used to simulate groundwater flow through an aquifer with a no-flow boundary condition.

1 Introduction

One of the first partial differential equations that is encountered in an introductory course on boundary value problems is the one-dimensional heat equation. Typically, the one-dimensional heat equation is derived and introduced as a model for heat flow through a rod or bar made up of a homogeneous heat-conducting material with insulated lateral surface. Following section 2.1 of [16], consider a rod such as the one depicted in Figure 1.

![Figure 1: Rod of heat-conducting material with insulated lateral surface.](http://www.codee.org/)
Assume that the rod has uniform cross-section (perpendicular to the $x$–axis and that the temperature is the same at any point of this cross-section. With this assumption, we can consider the temperature $u$ in the rod to be a function of position along the $x$–axis and time $t$, i.e. $u = u(x, t)$. The diagram in Figure 1 shows heat flowing from right to left through the rod (we are assuming that temperature $T_1$ is greater than temperature $T_0$), along with a small representative cross-sectional slice of width $\Delta x$ and cross sectional area $A$.

There are four possible contributions of heat to this cross-sectional slice:

1. Heat flow into the right face, $-q(x + \Delta x, t)A$;
2. Heat flow out of the left face, $-q(x, t)A$;
3. Heat generation within the slice\(^1\), $g(x, t)A\Delta x$; and
4. Heat storage within the slice, $\rho c A \Delta x \frac{\partial u}{\partial t}$.

Here, $q(x, t)$, which is taken to be positive when heat flows to the right, is the heat flux or amount of heat per unit time per unit area flowing through the rod’s cross-section at position $x$ at time $t$, with units $[q] = \text{heat/(time \times area)}$. The quantity $g(x, t)$ describes the rate at which heat is generated within the slice, with units $[g] = \text{heat/(time \times volume)}$, $\rho$ is the density, with units $[\rho] = \text{mass/volume}$, and $c$ is the heat capacity per unit mass, with units $[c] = \text{heat/(mass \times temperature)}$. Both density $\rho$ and heat capacity $c$ can depend on $x$ and $t$, but since we are considering a rod made up of homogeneous material, we can assume that these parameters are constant, \([16]\).

Using the Law of Conservation of Energy, which applied to the cross-sectional slice indicates that the heat flow into the right face plus the heat generation within the slice is equal to the heat flow out of the left face plus the heat storage in the slice, \([16]\), we see that for this scenario,

$$-q(x + \Delta x, t)A + g(x, t)A\Delta x = -q(x, t)A + \rho c A \Delta x \frac{\partial u}{\partial t},$$

or, rearranging terms in (1.1) and dividing by $A\Delta x$,

$$\frac{-q(x + \Delta x, t) + q(x, t)}{\Delta x} = \rho c \frac{\partial u}{\partial t} - g(x, t).$$

Letting $\Delta x \to 0$ in (1.2), we have

$$-\frac{\partial q}{\partial x} = \rho c \frac{\partial u}{\partial t} - g(x, t).$$

Equation (1.3) provides a relationship between the heat flux $q$ and temperature $u$. In order to get an equation that involves only the temperature $u$, we can use Fourier’s Law,

$$q = -\kappa \frac{\partial u}{\partial x},$$

\(^1\)”Heat generation” accounts for heat entering or leaving the slice by means such as convection, radiation, resistance to electrical current, chemical reaction, or nuclear reaction, \([16]\).
which relates the gradient of the temperature to the heat flux. The proportionality quantity $\kappa$, known as the thermal conductivity, may depend on $x$ and $t$, but since our rod is homogeneous, $\kappa$ can be assumed to be constant, [16]. The minus sign in (1.4) indicates that heat flows from higher temperatures to lower temperatures, [16]. Substituting $-\kappa \frac{\partial u}{\partial x}$ for $q(x, t)$ in (1.3) yields

$$\frac{\partial}{\partial x} \left( \kappa \frac{\partial u}{\partial x} \right) = \rho c \frac{\partial u}{\partial t} + g(x, t).$$  

(1.5)

Since thermal conductivity $\kappa$ is constant, we can divide (1.5) by $\kappa$ to get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho c}{\kappa} \frac{\partial u}{\partial t} - \frac{g(x, t)}{\kappa}.$$  

Finally, with thermal diffusivity $k$ defined by

$$k = \frac{\kappa}{\rho c},$$

we arrive at the one-dimensional heat equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t} - \frac{g(x, t)}{k}.$$  

(1.6)

A common application of the heat equation is to find the temperature $u(x, t)$ at any point $x$, at any time $t$, along the rod with a known initial temperature and specified boundary conditions, such as a fixed temperature at each end. In some texts, such as [16], section 2.5, one may also find applications that involve a fixed temperature at one end and the other end insulated (a no-flow condition).

One application of the heat equation that we have not found in most boundary value problem texts is groundwater flow through an aquifer, which is a water bearing porous medium such as sand, through which water flows easily, [5, 7]. Instead of temperature, we are interested in hydraulic head $h(x, t)$ (which can be thought of as the height of the water level, relative to a reference point, such as sea level), at any point $x$, at any time $t$, along the aquifer. Just as with the heat conducting rod, for applications, we need to specify appropriate boundary conditions and initial condition. Instead of fixed temperatures (or insulated boundary) at each end and a known initial temperature distribution, there would be fixed head levels (or a no-flow boundary) and known head levels at each point of the aquifer at a given time.

Table 1 illustrates that each aspect of heat flow in a rod has an analog in the groundwater flow through an aquifer setting. As was done for the heat-conducting rod, we will assume that the aquifer is homogeneous, so that the parameters are constants that depend on the specific aquifer material.
Heat Flow in Rod

<table>
<thead>
<tr>
<th>Heat flux, (q(x, t))</th>
<th>Volumetric flux, (q(x, t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature, (u(x, t))</td>
<td>Head level, (h(x, t))</td>
</tr>
<tr>
<td>Fourier’s Law, (q = -\kappa \frac{\partial u}{\partial x})</td>
<td>Darcy’s Law, (q = -K \frac{\partial h}{\partial x})</td>
</tr>
<tr>
<td>Thermal conductivity, (\kappa)</td>
<td>Hydraulic conductivity, (K)</td>
</tr>
<tr>
<td>Heat generation, (g(x, t))</td>
<td>Recharge, (R(x, t))</td>
</tr>
<tr>
<td>Heat storage, (pcA\Delta x \frac{\partial u}{\partial t})</td>
<td>Groundwater storage, (S_s A\Delta x \frac{\Delta h}{\Delta t})</td>
</tr>
<tr>
<td>Density ((\rho)), Heat capacity ((c))</td>
<td>Specific storage ((S_s))</td>
</tr>
<tr>
<td>Thermal diffusivity, (k = \frac{\kappa}{\rho c})</td>
<td>Hydraulic diffusivity, (k = \frac{K}{S_s})</td>
</tr>
</tbody>
</table>

Table 1: Aspects of heat flow in a rod vs. groundwater flow in an aquifer, \([3, 8, 16, 21]\).

For groundwater flow, recharge \(R(x, t)\) describes the volume of water added per unit time per unit volume of the aquifer, with \([R] = \text{volume}/(\text{time} \times \text{volume})\) and specific storage \(S_s\) is the volume of water added to, or released from, storage per unit volume of aquifer per unit change in hydraulic head, with \([S_s] = \text{volume}/(\text{volume} \times \text{length})\). Darcy’s Law, the analog of Fourier’s Law, \((1.4)\), in the groundwater flow setting,

\[
q = -K \frac{\partial h}{\partial x}
\]  

\((1.7)\)

shows that the volumetric flux or volumetric flow rate per unit cross-sectional area of the aquifer, \(q\), is proportional to the gradient of the hydraulic head, \([3, 8]\), with proportionality quantity, \(K\), the hydraulic conductivity. Since the units for volumetric flux \(q\) are \([q] = \text{volume}/(\text{time} \times \text{area})\), it follows that the units for the hydraulic conductivity \(K\) in \((1.7)\) are \([K] = \text{length}/\text{time}\). Typical values for hydraulic conductivity \(K\) in sand range from 0.0000009 m/sec to 0.0002 m/sec, \([3, 6]\).

Using Darcy’s Law, \((1.7)\), and the idea of continuity (conservation of mass), \([21]\), one can use essentially the same argument as above for deriving the one-dimensional heat equation\(^2\) to obtain a partial differential equation that describes hydraulic head level \(h\) at any point in an aquifer, at any time, known as the one-dimensional groundwater flow equation,

\[
\frac{\partial^2 h}{\partial x^2} = \frac{1}{K} \frac{\partial h}{\partial t} - \frac{R(x, t)}{K},
\]  

\((1.8)\)

where hydraulic diffusivity \(k\) depends on the material in the aquifer through which the groundwater is flowing. From Table 1, hydraulic diffusivity \(k\) is defined by

\[
k = \frac{K}{S_s},
\]

so it follows that the units for hydraulic diffusivity are \([k] = \text{length}^2/\text{time}\). One can expect that for sand, hydraulic diffusivity \(k\) ranges from approximately 0.000884903 m\(^2\)/sec to 1.563076923 m\(^2\)/sec, \([3]\). As we see, the one-dimensional groundwater flow equation, \((1.8)\), is the “same” as the one-dimensional heat equation, \((1.6)\).

\(^2\)For a historical discussion of Darcy’s Law and a detailed derivation of equation \((1.8)\), see reference \([3]\).
One way to illustrate groundwater flow is via a physical Sand Tank Groundwater Model. A Sand Tank Groundwater Model or sand tank, such as the one pictured in Figure 2, from Ball State University’s Department of Geological Sciences, “is an educational device constructed ... of sturdy layered sand lenses to represent a sliced section of earth. Through the use of water tinted with food coloring or grape Kool-Aid, it is possible to observe a wide range of groundwater movements, [11].” The sand tank is designed so that water can flow between narrow rectangular recharge columns on either end into or out of the aquifer portion in the middle via small holes at the base of each column. Figure 3 provides an illustration of the components in the sand tank seen in Figure 2.

![Figure 2: Sand Tank Groundwater Model.](image)

![Figure 3: Sand Tank Groundwater Model Components: Underground Tank (UT), Leaky Lagoon (LL), Stream (ST), Bedrock Aquifer, Sand Lenses (SL1 - SL5), Shallow Wells (S1 - S4), Artesian Well (AW), Pumping Wells (P1 - P2), Deep Wells (W1 - W3), Drain Outlets (D1 - D2), Recharge Columns (R1 - R2), Access Holes into Aquifer (H1 - H2).](image)

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3In geology, a lens is a body of ore or rock that is thick in the middle and thin at the edges, like a doubly convex lens, [1].
Typically, sand tanks are used for outreach or educational purposes to "demonstrate a variety of groundwater flow principles, including water table levels, how groundwater supplies are replenished, how groundwater flows through different geologic materials, and how contaminants from a variety of sources can leak into groundwater, [22]. As an example, for the sand tank in Figure 3 one adds clean "groundwater" to the aquifer via the holes at the base of the recharge columns until the water level appears in the "leaky lagoon" and "stream". Then "contaminants" (food coloring) can be introduced into the aquifer via the "underground storage tank", "bedrock aquifer", or the opening at the top of one of the various wells. The contaminants will move with the groundwater through the aquifer as actions are taken, such as pumping a well via a vacuum pump or letting water "flow" out of the stream by opening up the stream’s outlet (outlet D2 in Figure 3).

For additional details on the set up, operation, and description of components in typical sand tanks, see these guides [13, 15] and these videos [12, 20].

These physical models can also be used in more advanced classroom settings for "introducing flow nets and quantifying groundwater flow, for groundwater modeling and calibration via a spreadsheet, and for dye-transport modeling. Students learn from the sand tank through interaction and visualization, and by collecting and interpreting data. This integrates the theories from lectures into the sand-tank laboratories, enhancing students’ skills and understanding", [14]. The types of courses where these concepts from hydrology are introduced are geologic engineering, geology, environmental science, mining engineering, and chemical engineering, [14, 17]. Groundwater flow can be modeled using basic numerical methods techniques via standard software such as Visual Modflow or spreadsheets, [9, 14], that require a minimal amount of differential equations.

Sand tanks can also be used in an “unconventional manner” to tie together groundwater flow modeling and more advanced differential equations concepts and solution techniques. In [3], by replacing a sand tank’s drain plug (outlet drain D1 at the base of the left recharge column in Figure 3) with a drain tube, emptying the tank of as much water as possible, and pouring green food colored water into the right recharge column at a constant rate, the modified sand tank was used to physically simulate one-dimensional groundwater flow through an aquifer, starting with an initial head level close to the bottom of the aquifer. Since the observed well head levels in deep wells W1 – W3 didn’t reach the height of the sand lenses, they were able to model an aquifer comprised of essentially homogeneous material with fixed head levels at each boundary. Using a cell phone and Tracker software, [2], well head levels were collected from the three deep wells W1 – W3 over time, for approximately seven seconds. The data was then used to investigate mathematical models for one-dimensional groundwater flow in this case. For a detailed discussion of the experimental process, set-up, and models considered, see [3, 4].

By leaving a sand tank in unmodified state (i.e. leaving in the drain plug D1 in Figure 3) we were able to physically simulate one-dimensional groundwater flow through an aquifer with a fixed head level at one boundary (right) and no-flow at the other boundary (left). Again, since the majority of the observed well head levels in deep wells W1 – W3 didn’t reach the height of the sand lenses, we considered the aquifer to be comprised of essentially homogeneous material. Instead of pouring colored water into the right recharge column at a constant rate, we added a drop of green food coloring to each well and used an upside down bottle with stopper and tube to establish a fixed head level and
introduce clean water at a fixed rate into the aquifer via the access hole at the base of the right recharge column. Using a video camera instead of a cell phone, we were able to film the sand tank and use Tracker to collect approximately 70 seconds worth of well head data from the resulting video recording.

Figure 4 shows a screenshot from Tracker, with the three deep wells W1 (left), W2 (middle), and W3 (right), indicated by respective colors red, green, and pink), located at $x_L = 3.25$ in, $x_M = 12$ in, and $x_R = 18.5$ in, respectively. The horizontal length of the aquifer is $a = 23.75$ in, with fixed head level $H_1 = 9.6875$ inches at the right boundary, $x = a$. Well head data collected at each of these wells over time (406 measurements for each well, taken approximately once every $5/30$ second) as well as additional sand tank measurements are provided in the accompanying Excel file (WellHeadLevels2024.xlsx) and PDF file (SandTankDimensions.pdf). Copies of the MP4 video file (Trial 2a - Trim.mp4) and Tracker TRK file (Trial 2a - Trim.trk) used to collect this data are also provided. In the Tracker file, which requires the MP4 video file, the water level starts to rise at 3.503 seconds, slowing down as time increases. We collected data from 3.503 seconds until 71.071 seconds, corresponding to frames 64 to 2194, until we were unable to discern changes in head level at the right well, due to the colored water in the well being indistinguishable from the color of aquifer material surrounding the well.5

Figure 4: Collecting well head data via Tracker.

This paper is organized as follows: in Section 2, we investigate several variations of a model based on the one-dimensional groundwater flow equation using the data collected from our sand tank; in Section 3, we test our model with a “real-world” application involving the time it takes for contaminants to reach wells drilled for drinking water in the sand tank; in Section 4, we discuss further questions that have arisen and implications for public policy. We also provide, in Appendix 1: Solving the IVBVP and Appendix 2: Revised IVBVP, detailed solutions of the models presented in Section 1.  

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4Technically, for our video camera, the frame rate was 29.97 frames per second.

5This data may also be collected as a separate project via the provided MP4 video file or a sand tank and video camera (if available), along with Tracker.
2 A Model for our Sand Tank Aquifer

Figure 5 provides an illustration of our sand tank aquifer. A mathematical model for one-dimensional groundwater flow through this aquifer consists of the groundwater flow equation (1.8) along with appropriate boundary conditions and an initial condition. To simplify our model, we will assume that there is no recharge term in equation (1.8).

\[ \frac{\partial^2 h}{\partial x^2} = \frac{1}{k} \frac{\partial h}{\partial t}, \quad \text{for } 0 < x < a, t > 0, \quad (2.1) \]

\[ h(a, t) = H_1, \quad \text{for } t > 0. \quad (2.2) \]

\[ \frac{\partial}{\partial x} h(0, t) = 0, \quad \text{for } t > 0, \quad (2.3) \]

\[ h(x, 0) = f(x) \quad \text{for } 0 < x < a. \quad (2.4) \]

Here, boundary condition (2.2) indicates a fixed head level of $H_1$ at the right end of the aquifer and using Darcy’s Law (1.7), we see that boundary condition (2.3) indicates that there is no water flowing through the left end of the aquifer. The initial head level at time $t = 0$ is given by equation (2.4). To guarantee a solution to the IVBVP (2.1)–(2.4), we assume that $f(x)$ is \textit{sectionally smooth} on $[0, a]$, i.e. $f$ has at most a finite number of removable jumps, discontinuities, and corners, with the function and its derivatives being continuous between such points, [16].

To solve (2.1)–(2.4), one can use the standard technique of separation of variables. In [4], this technique is used to solve an IVBVP similar to (2.1)–(2.4), with no-flow boundary condition (2.3) replaced with a constant head level boundary condition, namely $h(0, t) =$ constant, for $t > 0$. We found that the solutions to both IVBVP are nearly the same, with the differences being the steady-state found for each, the appearance of sines or cosines in the solution, and the type of Fourier coefficients for each solution. (For completeness,
in Appendix 1: Solving the IVBVP, we provide the solution that appears in [4], slightly modified as needed for our situation). One finds that

\[ h(x, t) = H_1 + \sum_{n=1}^{\infty} b_n \cos \left( \frac{(2n - 1)\pi}{2a} x \right) e^{-\left(\frac{(2n-1)\pi}{2a}\right)^2kt}, \]  

(2.5)

where

\[ b_n = \frac{2}{a} \int_0^a (f(x) - H_1) \cos \left( \frac{(2n - 1)\pi}{2a} x \right) dx. \]  

(2.6)

We note that a problem similar to (2.1)–(2.4) is presented as an example and solved in [16]. With boundary conditions (2.2) and (2.3) swapped, i.e. fixed head level at the left end \((x = 0)\) and no-flow at the right end,

\[ h(0, t) = H_1, \quad \text{for} \; t > 0. \]  

(2.7)

\[ \frac{\partial}{\partial x} h(a, t) = 0, \quad \text{for} \; t > 0, \]  

(2.8)

one finds that the solution is

\[ h_1(x, t) = H_1 + \sum_{n=1}^{\infty} b_n \sin \left( \frac{(2n - 1)\pi}{2a} x \right) e^{-\left(\frac{(2n-1)\pi}{2a}\right)^2kt}, \]  

(2.9)

where

\[ b_n = \frac{2}{a} \int_0^a (f(x) - H_1) \sin \left( \frac{(2n - 1)\pi}{2a} x \right) dx. \]  

(2.10)

By swapping \(x = 0\) and \(x = a\) in our sand tank model, (2.9), (2.10) can be used as the solution to this revised physical set-up.

Another solution alternative for (2.1)–(2.4) is to make the change of variable \(h_2(x, t) = h_1(a - x, t)\). Then

\[ h_2(x, t) = H_1 + \sum_{n=1}^{\infty} b_n \sin \left( \frac{(2n - 1)\pi}{2a} (a - x) \right) e^{-\left(\frac{(2n-1)\pi}{2a}\right)^2kt}, \]  

(2.11)

with \(b_n\) given by equation (2.10). Applying the sine sum identity to the terms in the sum in equation (2.11) and making the substitution \(y = a - x\) in (2.10), one finds that

\[ h_2(x, t) = H_1 + \sum_{n=1}^{\infty} b_n \left[ (-1)^{n+1} \cos \left( \frac{(2n - 1)\pi}{2a} x \right) \right] e^{-\left(\frac{(2n-1)\pi}{2a}\right)^2kt}, \]  

(2.12)

and

\[ b_n = \frac{2}{a} \int_0^a (f(a - y) - H_1)(-1)^{n+1} \cos \left( \frac{(2n - 1)\pi}{2a} y \right) dy. \]  

(2.13)

Combining equations (2.12) and (2.13), it follows that \(h_2(x, t)\) solves (2.1)–(2.3), with initial condition (2.4) replaced by

\[ h(x, 0) = f(a - x) \quad \text{for} \; 0 < x < a. \]  

(2.14)

If the initial condition (2.4) is a constant head level at each point, i.e. \(f(x) \equiv H_0\) for all \(0 < x < a\), then \(h(x, t) = h_2(x, t)\) for all \(0 < x < a\) and \(t > 0\).
3 Testing Our Model

To check our model, we use Mathematica\(^6\) to compare head level data for each well (left, middle, and right) collected from the sand tank to our model (2.5), (2.6), after specifying model parameters and initial data. The length of the aquifer is \(a = 23.75\) in, head level at the right boundary when \(x = a\) is measured to be

\[
H_1 = 9.6875\text{ in,} \tag{3.1}
\]

and since initial head levels at each well are 1.02919 in, 1.16587 in, and 1.39256 in, at the left, middle, and right wells, respectively, we choose the initial head level to be

\[
f (x) \equiv H_0 = \frac{1.02919 + 1.16587 + 1.39256}{3} = 1.19587\text{ in,} \tag{3.2}
\]

for \(0 < x < a\). With this choice of initial condition (3.2), the \(b_n\) coefficients in (2.5) are found with (2.6) to be

\[
b_n = \frac{2}{a} \int_0^a (H_0 - H_1) \cos \left( \frac{(2n - 1) \pi}{2a} x \right) \, dx = \frac{4(-1)^n(H_1 - H_0)}{\pi(2n - 1)} \tag{3.3}
\]

Computing coefficients \(b_n\) via (3.3), with (3.1) and (3.2) and setting \(t = 0\) in (2.5), we can determine an appropriate number of terms in the sum in equation (2.5) for our model. Using graphical, (root mean square error) RMSE, and square error comparisons, we find that a partial sum with 50 terms in our model should be sufficient. Figure 6 compares our model at time \(t = 0\) sec, \(h(x, 0)\), to the initial head level \(f(x)\) on the interval \(0 < x < a\).

For the hydraulic diffusivity \(k\), since we don’t know specifically what type of sand is in the aquifer, we choose a value for \(k\) to get a good graphical match between model and data, followed by an application of Mathematica’s FindMinimum command to minimize RMSE.\(^7\) Figure 7 shows that with \(k = 3.01356\) in\(^2\)/sec (= 0.00194429934 m\(^2\)/sec) and an RMSE of 0.451039 in, while \(k\) falls within the expected range given above, our model overestimates head level at the right well and underestimates the head level at the left well.

One way to take into consideration the discrepancy between the model and measured data at the right well is to treat fixed head level \(H_1\) at \(x = a\) as an unknown parameter to be determined. This approach was used in [4], as the same discrepancy was observed when comparing data collected from a sand tank to a model for groundwater flow through

\(^6\)A Mathematica notebook (GroundwaterFlowModels2024.nb) as well as PDF version (GroundwaterFlowModels2024.pdf) are included as supplements to this paper. All of the numerical calculations and graphs for this paper are included in these files. For those that do not have access to Mathematica, a supplemental Excel file (GroundwaterFlowModels2024.xlsx) that provides numerical calculations and graphs that are similar to the Mathematica results is also included.

\(^7\)At this point, we also compute RMSE between model and measured data for \(m = 1\) to 100 terms in the model to see how different numbers of terms impact the choice of \(k\) and resulting error. What we find is that the resulting RMSE and choice of \(k\) are essentially the same for most choices of \(m\), with the smallest error occurring with \(m = 11\). For this reason, we choose \(m = 11\) terms for our model.
Figure 6: Initial head level.

Figure 7: Comparison of model to measured data.
an aquifer with fixed head levels at each boundary. Possible reasons for this phenomenon that take into account sand tank construction, Torricelli’s Law, [23], and Darcy’s Law are carefully discussed in [4]. Again, starting with choices of $k$ and $H_1$ to get a good graphical fit, followed by an application of Mathematica’s `FindMinimum` command, we obtain an RMSE of 0.297491 in with $k = 3.81486$ in$^2$/sec and $H_1 = 8.53077$ in. It is clear from Figure 8 that the resulting model provides a much better fit for the right well and about the same results for the middle well, but still does not match as well at the left well.

![Figure 8: First revision: variable head level at right boundary.](image)

In [4], to address the discrepancy between measured head level and model at the left well, possibly due to flow rate through the open drain hole at the base of the left end of the aquifer, the fixed head level at $x = 0$ was also treated as a parameter. To see if we can get a better fit at the left well, we make another model revision by also treating the initial head level as another unknown parameter $H_0$ to be determined along with parameters $k$ and $H_1$ to minimize RMSE. Figure 9 shows that with $k = 3.13721$ in$^2$/sec, $H_1 = 8.69686$ in, $H_0 = 1.81506$ in, we get much better match at all three wells. For this revision, RMSE is reduced to 0.196507 in.
Finally, to see if we can reduce the RMSE, we revise our model a third time to include a recharge term. As discussed in Appendix 2: Revised IVBVP, this amounts to incorporating a new steady-state and revised coefficients in our solution, with (2.5) and (2.6) revised as follows

\[
h(x, t) = \frac{a^2 R + 2H_1 K - Rx^2}{2K} + \sum_{n=1}^{\infty} b_n \cos \left( \frac{(2n - 1)\pi}{2a} x \right) e^{-\left(\frac{(2n - 1)\pi}{2a}\right)^2 k t}, \tag{3.4}
\]

with

\[
b_n = \frac{2}{a} \int_{0}^{a} \left( H_0 - \frac{a^2 R + 2H_1 K - Rx^2}{2K} \right) \cos \left( \frac{(2n - 1)\pi}{2a} x \right) \, dx
\]

\[
= \frac{2(-1)^n \left( 8a^2 R + 2K(\pi - 2\pi n)^2(H_1 - H_0) \right)}{\pi^3 K(2n - 1)^3} \tag{3.5}
\]

With this model revision, we get essentially the same results as our second revision, with \( k = 2.80571 \text{ in}^2/\text{sec} \), \( H_1 = 8.68895 \text{ in} \), \( H_0 = 1.84571 \text{ in} \), \( R = 0.000740298 \text{ sec}^{-1} \), \( K = 0.300134 \text{ in/sec} \) and an RMSE of 0.192617 in. Figure 10 reinforces this, as the graphs are nearly indistinguishable from those in Figure 9.\(^8\)

\(^8\)Comparing the models for the second revision (equations (2.5), (2.6), (3.2), and (3.3)) and the third revision (equations (3.4) and (3.5)), it turns out that since \( \frac{R}{K} = 0.00246656 \text{ in}^{-1} \) is close to zero, the leading terms (steady-states) and Fourier coefficients for both models yield essentially the same values for head levels at each well. The supplementary Excel file, GroundwaterFlowModels2024.xlsx, provides tables and graphs that can be used to compare each model to measured well data at each well. It is clear from these tables and graphs that the choice of fixed head level \( H_1 \) and initial head level \( H_0 \) also have an impact on each model.
4 A “Real-World” Application

From our model comparison to measured data results, it is clear that the heat equation can be used to model groundwater flow through an aquifer. As a final test of our model (we will use the second revision), consider the following “real-world” situation.

Suppose the groundwater in our sand tank aquifer is contaminated at time $t = 0$ sec and we have a concern that it may impact three wells drilled for drinking water. The wells are located at positions in the aquifer corresponding to approximately $x = 4.1$ in, $x = 11.2$ in, and $x = 19.1$ in. The wells draw groundwater from an approximate head level of 4.4 in. Using our model, we can answer the following questions:

1. Does the groundwater reach any of these wells?
2. If so, estimate the time at which the groundwater reaches these wells.
3. How do these model time estimates, if any, compare to the actual time needed for the contaminated groundwater to reach the drinking wells?

Plotting our model head levels at each drinking well location, we see from Figure 11 that the model predicts that the contaminated groundwater will reach the left, middle, and right drinking wells at approximately $t = 48$ sec, $t = 30$ sec, and $t = 4$ sec, respectively.

Using Mathematica’s FindRoot command with our model, we can get more accurate numerical estimates for these times, namely $t = 49.1323$ sec, $t = 30.4799$ sec, and $t = 4.39004$ sec, at the left, middle, and right drinking water wells, respectively. We can then check with Tracker to see when the the groundwater actually reaches a head level of 4.4 in for each drinking water well. What we find is that the actual times are approximately $t = 63.514$ sec, $t = 33.984$ sec, and $t = 7.291$ sec, respectively. We discuss a possible reason for this that leads to a question for future investigation in the next section.
5 Conclusion, Further Questions, and Implications for Public Policy

Using a Sand Tank Groundwater Model for an aquifer and collecting head level data via a video camera and Tracker, we have been able to show that the one-dimensional groundwater flow equation can be used to model the head levels in the aquifer for the case when there is a fixed head level at one boundary and a no-flow condition at the other boundary. The results obtained are surprisingly good (to us), especially considering the fact that the aquifer we worked with does not have uniform material throughout.\footnote{Sand Tank Groundwater Models, which are constructed to simulate various types of material in an aquifer are very expensive (the ones we have worked with are $850 or more) and cannot be easily reconfigured, \cite{18}.}

Regarding the “real-world” question, we did notice that at each drinking water well, the actual times at which the well head level of 4.4 inches is reached in the aquifer are consistently greater than the times predicted by the model. Looking at the Sand Tank Groundwater Model (Figures 2 and 4), the material at the bottom of the aquifer is different than the material at the level where the groundwater enters the drinking water wells. Our model assumes that the aquifer material is homogenous throughout, so a natural question that could be considered in future investigations is can we modify our model to take into consideration different materials in the aquifer at different levels? Another question to consider is, perhaps our model needs to be modified to consider groundwater flow in more than one direction, perhaps there is both horizontal and vertical movement of the groundwater through the aquifer. This would lead to a much more complicated model that is beyond the scope of this investigation.

As pointed out in the Environmental Protection Agency’s \textit{Handbook of Groundwater Protection and Cleanup Policies for RCRA Corrective Action}, \cite{19}, which “is designed to help ... a regulator, member of the regulated community, or member of the public find and understand EPA policies on protecting and cleaning up groundwater at Resource Conservation and Recovery Act (RCRA) corrective action facilities”, such as Indiana’s

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Comparing_Model_to_Drinking_Well_Head_Level.png}
\caption{Comparing Model to Drinking Well Head Level.}
\end{figure}
Department of Environmental Management, [10], “[g]roundwater supplies drinking water to half of the nation and virtually all people living in rural areas [and] ... supports many billions of dollars worth of food production and industrial activity.” For this reason, it is crucial that policy makers are convinced that the mathematical models used to help investigate and conduct cleanup of sites with contaminated groundwater are valid. Projects such as the one outlined in this paper can be used for this purpose.

Appendix 1: Solving the IVBVP

In this section, we provide a solution of IVBVP

\[
\frac{\partial^2 h}{\partial x^2} = \frac{1}{k} \frac{\partial h}{\partial t}, \quad \text{for } 0 < x < a, t > 0, \tag{5.1}
\]

\[h(a, t) = H_1, \quad \text{for } t > 0. \tag{5.2}\]

\[
\frac{\partial}{\partial x} h(0, t) = 0, \quad \text{for } t > 0, \tag{5.3}
\]

\[h(x, 0) = f(x) \quad \text{for } 0 < x < a. \tag{5.4}\]

that uses the standard separation of variables technique.

Remark: The solution that follows is taken from [4], slightly modified as needed, for our specific IVBVP (5.1)–(5.4). We are including this solution for completeness.

The Steady-State Problem: In order to solve (5.1)–(5.4), we first look at a simpler related problem – the steady-state problem. After a long time, under the same conditions, one would expect the variation in head level in the aquifer to die away. Mathematically, we would have

\[
\lim_{{t \to 0}} h(x, t) = H(x). \tag{5.5}
\]

The function \(H(x)\), called the steady-state head distribution, must satisfy the groundwater flow equation (5.1) and boundary conditions (5.2), (5.3), as these hold for all \(t > 0\). Therefore \(H(x)\) should solve the problem:

\[
\frac{d^2 H}{dx^2} = 0, \quad 0 < x < a, \tag{5.6}
\]

\[H(a) = H_1, \tag{5.7}\]

\[H'(0) = 0. \tag{5.8}\]

Integrating (5.6) twice and imposing conditions (5.7) and (5.8), one finds that the steady-state problem (5.6)–(5.8) has solution

\[H(x) \equiv H_1. \tag{5.9}\]
The Transient Problem: Now that we have found the “long time behavior” of \( h(x, t) \), we need to look for the “rest” of the unknown function \( h(x, t) \). We do so by defining the transient head distribution:

\[
w(x, t) := h(x, t) - H(x).
\]  

(5.10)

Using (5.9) and (5.10), we find the following relations hold:

\[
\begin{align*}
    h(x, t) &= w(x, t) + H(x), \\
    \frac{\partial^2 h}{\partial x^2} &= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 H}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}, \\
    \frac{\partial h}{\partial t} &= \frac{\partial w}{\partial t}, \\
    h(a, t) &= w(a, t) + H(a) = w(a, t) + H_1, \\
    \frac{\partial h}{\partial x}(0, t) &= \frac{\partial w}{\partial x}(0, t) + H'(0) = \frac{\partial w}{\partial x}(0, t).
\end{align*}
\]

Substituting these five relations into (5.1)–(5.4) yields an IVBVP for the transient solution, with homogeneous boundary data and homogeneous PDE:

\[
\begin{align*}
    \frac{\partial^2 w}{\partial x^2} &= \frac{1}{\kappa} \frac{\partial w}{\partial t}, \quad 0 < x < a, t > 0, \\
    w(a, t) &= 0, \quad t > 0, \\
    \frac{\partial w}{\partial x}(0, t) &= 0, \quad t > 0, \\
    w(x, 0) &= f(x) - H_1 = g(x), \quad 0 < x < a, t > 0.
\end{align*}
\]

(5.11)–(5.14)

Note that equation (5.9) is used to get equation (5.14).

Separation of Variables: The Method of Separation of Variables can now be used to solve (5.11)–(5.14), as we have a homogeneous PDE with homogeneous boundary data. Note that the trivial solution \( w(x, t) \equiv 0 \) satisfies (5.11)–(5.13), but not (5.14) for arbitrary choices of \( f(x) \). We want a solution to (5.11)–(5.13) that also satisfies initial data (5.14). Thus, whenever we encounter the trivial solution in what follows, we will “throw it out.”

The general idea of separation of variables is to assume the solution to (5.11) is of the form

\[
w(x, t) = \phi(x)T(t),
\]  

(5.15)

where \( \phi \) is twice differentiable and \( T \) is differentiable. Then

\[
\begin{align*}
    \frac{\partial^2 w}{\partial x^2} &= \phi''(x)T(t), \\
    \frac{\partial w}{\partial t} &= \phi(x)T'(t),
\end{align*}
\]

(5.16)–(5.17)

and substituting (5.16), (5.17) into (5.11) implies

\[
\phi''(x)T(t) = \frac{1}{\kappa} \phi(x)T'(t), \quad 0 < x < a, t > 0.
\]  

(5.18)
Dividing (5.18) by \( \phi(x)T(t) \), we find

\[
\frac{\phi''(x)}{\phi(x)} = \frac{1}{\kappa} \frac{T'(t)}{T(t)}, \quad 0 < x < a, t > 0.
\]  

(5.19)

Since the left-hand side of (5.19) is a function of \( x \) and the right-hand of (5.19) is a function of \( t \), the only way for (5.19) to hold is for the common value of these functions to be a constant, i.e.

\[
\frac{\phi''(x)}{\phi(x)} = p = \frac{1}{\kappa} \frac{T'(t)}{T(t)}, \quad 0 < x < a, t > 0.
\]

(5.20)

From (5.20), we get two ordinary differential equations:

\[
\phi''(x) = p\phi(x), \quad 0 < x < a,
\]

(5.21)

\[
T'(t) = p\kappa T(t), \quad t > 0.
\]

(5.22)

Also, (5.15) in (5.12) and (5.13) lead to the following conditions on \( \phi \) and \( T \):

\[
0 = w(a, t) = \phi(a)T(t), \quad t > 0,
\]

\[
0 = \frac{\partial w}{\partial x}(0, t) = \phi'(0)T(t), \quad t > 0.
\]

Therefore, either \( T(t) = 0 \) for all \( t > 0 \) or

\[
\phi(a) = 0 \text{ and } \phi'(0) = 0
\]

(5.23)

must hold. Since \( T(t) = 0 \) for all \( t > 0 \) implies that \( w(x, t) \equiv 0 \), we take (5.23) to hold.

Our problem of solving (5.11)–(5.14) has been reduced to solving the boundary value problem (BVP) (5.21), (5.23), solving the ODE (5.22), and imposing initial condition (5.14). The general solution to (5.21) depends on the sign of \( p \). For the cases \( p > 0 \) and \( p = 0 \), imposing boundary conditions (5.23) leads to the trivial solution \( w(x, t) \equiv 0 \). For the case \( p < 0 \), letting \( p = -\lambda^2 \) with \( \lambda > 0 \) for notational convenience, (5.21) has general solution

\[
\phi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).
\]

(5.24)

Imposing boundary conditions (5.23) in (5.24), we are led to \( c_2 = 0 \) and either \( c_1 = 0 \) or \( \cos(\lambda a) = 0 \). Since \( c_1 = 0 \) implies \( w(x, t) \equiv 0 \), we require

\[
\cos(\lambda a) = 0.
\]

(5.25)

For (5.25) to hold, we need \( \lambda \) to be odd multiples of \( \frac{\pi}{2a} \), i.e.

\[
\lambda = \frac{(2n - 1)\pi}{2a}, \quad n = 1, 2, 3, ...
\]

(5.26)

It follows that for each integer \( n \geq 1 \), there exists a solution to (5.21), (5.23) of the form

\[
\phi_n(x) = \cos(\lambda_n x),
\]

(5.27)
with $\lambda_n$ given by (5.26).

**Remark:** Any constant multiple of solutions of form (5.27) also solves the homogeneous problem (5.21), (5.23). This fact will be useful for when we impose initial condition (5.14).

Recall that we are looking for solutions (5.11)–(5.13) of the form $w(x, t) = \phi(x)T(t)$. For each $n \geq 1$ we have found a solution $\phi_n(x)$. Corresponding to $\phi_n(x)$ is a solution $T_n(t)$ which is found by solving (5.22) with $p = -\lambda_n^2$:

$$T_n(t) = ce^{-\lambda_n^2 kt}, \quad t > 0,$$

where $c \in \mathbb{R}$. As above, we will take $c = 1$ in (5.28).

Therefore, for each $n \geq 1$, we have a solution to (5.11)–(5.13)

$$w_n(x, t) = \phi_n(x)T_n(t) = \cos(\lambda_n x)e^{-\lambda_n^2 kt},$$

where $\lambda_n = \frac{(2n - 1)\pi}{2a}$.

We still need a solution to (5.11)–(5.14). Recalling the Principle of Superposition which says that linear combinations of solutions to a linear homogeneous ODE are also solutions, let’s suppose a solution to (5.11)–(5.14) is of the form:

$$w(x, t) = \sum_{n=1}^{\infty} b_n w_n(x, t) = \sum_{n=1}^{\infty} b_n \cos(\lambda_n x)e^{-\lambda_n^2 kt}$$

Formally $w(x, t)$ in form (5.30) satisfies (5.11)–(5.13). Imposing (5.14), we find

$$g(x) = w(x, 0) = \sum_{n=1}^{\infty} b_n \cos(\lambda_n x)$$

Problem (5.31) is a Fourier series problem. To find $b_n$, we use the formula

$$b_n = \frac{2}{a} \int_{0}^{a} g(x) \cos \left( \frac{(2n - 1)\pi}{2a} x \right) dx,$$  

with $g(x)$ given by (5.14). If $g(x)$ is “nice” (for example sectionally smooth), the Fourier series of $g(x)$, (5.31), with coefficients given by (5.32) will converge to $g(x)$ and it can be shown that $w(x, t)$ satisfies (5.11)–(5.14), [16].

Now that we’ve solved the transient problem, we can solve our original problem. Using (5.10), the solution to (5.1)–(5.4) is given by:

$$h(x, t) = H(x) + w(x, t)$$

with $w(x, t)$ given by (5.30) and (5.32) and $H(x)$ given by (5.9):

$$h(x, t) = H_1 + \sum_{n=1}^{\infty} b_n \cos \left( \frac{(2n - 1)\pi}{2a} x \right) e^{-\left( \frac{\alpha \pi}{a} \right)^2 kt}.$$  

(5.33)
Appendix 2: Revised IVBVP

If we add a recharge term of the form \( R(x, t) = R \) for constant \( R \) to (5.1)–(5.4), we get the following revised IVBVP:

\[
\frac{\partial^2 h}{\partial x^2} = \frac{1}{k} \frac{\partial h}{\partial t} - \frac{R}{K}, \quad \text{for } 0 < x < a, t > 0, \tag{5.34}
\]

\[ h(a, t) = H_1, \quad \text{for } t > 0. \tag{5.35} \]

\[ \frac{\partial}{\partial x} h(0, t) = 0, \quad \text{for } t > 0, \tag{5.36} \]

\[ h(x, 0) = f(x), \quad \text{for } 0 < x < a. \tag{5.37} \]

With one slight modification, namely a revised steady-state problem, the same separation of variables argument used in Appendix 1: Solving the IVBVP to solve (5.1)–(5.4), can be used to solve (5.34)–(5.37). In this case, the steady-state problem (5.6)–(5.8) becomes

\[
\frac{d^2 H}{dx^2} = -\frac{R}{K}, \quad 0 < x < a, \tag{5.38}
\]

\[ H(a) = H_1, \tag{5.39} \]

\[ H'(0) = 0. \tag{5.40} \]

Integrating (5.38) twice and imposing conditions (5.39) and (5.40), one finds that the steady-state problem (5.38)–(5.40) has solution

\[ H(x) = \frac{a^2 R + 2H_1 K - Rx^2}{2K}. \tag{5.41} \]

Starting with equation (5.9) in the argument in Appendix 1: Solving the IVBVP, if we replace \( H(x) \equiv H_1 \) with \( H(x) \) given by (5.41), we find that the solution to (5.34)–(5.37) is

\[
h(x, t) = \frac{a^2 R + 2H_1 K - Rx^2}{2K} + \sum_{n=1}^{\infty} b_n \cos \left( \frac{(2n - 1)\pi}{2a} x \right) e^{-\left(\frac{\pi}{a}\right)^2 k t},
\]

with

\[ b_n = \frac{2}{a} \int_0^a \left( f(x) - \frac{a^2 R + 2H_1 K - Rx^2}{2K} \right) \cos \left( \frac{(2n - 1)\pi}{2a} x \right) dx. \]

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References


