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Adinkras and Arithmetical Graphs

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Abstract

Adinkras and arithmetical graphs have divergent origins. In the spirit of Feynman diagrams, adinkras encode representations of supersymmetry algebras as graphs with additional structures. Arithmetical graphs, on the other hand, arise in algebraic geometry, and give an arithmetical structure to a graph. In this thesis, we will interpret adinkras as arithmetical graphs and see what can be learned.

Our work consists of three main strands. First, we investigate arithmetical structures on the underlying graph of an adinkra in the specific case where the underlying graph is a hypercube. We classify all such arithmetical structures and compute some of the corresponding volumes and linear ranks.

Second, we consider the case of a reduced arithmetical graph structure on the hypercube and explore the wealth of relationships that exist between its linear rank and several notions of genus that appear in the literature on graph theory and adinkras.

Third, we study modifications of the definition of an arithmetical graph that incorporate some of the properties of an adinkra, such as the vertex height assignment or the edge dashing. To this end, we introduce the directed arithmetical graph and the dashed arithmetical graph. We then explore properties of these modifications in an attempt to see if our definitions make sense, answering questions such as whether the volume is still an integer and whether there are still only finitely many arithmetical structures on a given graph.

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Chapter 1

Introduction

Adinkras encode representations of supersymmetry algebras as graphs with additional properties. In this way, they convert representation theoretic problems arising in physics to combinatorial problems. Arithmetical graphs arise in algebraic geometry. In this thesis, we aim to interpret adinkras as arithmetical graphs.

1.1 Background on Adinkras

We first give a definition of an adinkra. There are many equivalent definitions of an adinkra. Ours combines ideas from Naples (2009) and Zhang (2014) to create a definition suited to our purposes.

An adinkra has three levels of structure: its graph theoretic properties, its coloring properties, and its dashing and ranking. A graph with the graph theoretic properties of an adrinkra is called an (adinkra) topology. A topology with the coloring properties of an adinkra is called a chromotopology. A chromotopology with the dashing and ranking properties of an adinkra is called an adinkra. We elaborate in the following definitions.

Definition 1. *An* ***(adinkra) topology*** *G is graph with the following properties:*

- *G is finite.*
- *G is simple.*
- *G is connected.*
- *G is bipartite.*

- *G is regular, that is, there exists a number n such that each vertex is incident to exactly n edges. (In this case, G is called n-regular.)*

Definition 2. *A **chromotopology** is a topology G with the following additional structures:*

- *Suppose G is $n-$regular. The edges of G are colored such that each vertex is incident to an edge of each of the colors $1, \ldots, n$.*
- *Every pair of edge colors $\{i, j\}$ incident to a single vertex is part of a 4-cycle where the colors alternate between i and j.*

Definition 3. *An **adinkra** is a chromotopology G with the following additional structures:*

- *There exists a way to assign directions to the edges of G such than on the corresponding directed graph $G^{'}$ it is possible to define a height assignment $h : V(G^{'}) \to \mathbb{N}$ such that $h(i) = h(j) + 1$ if $ij \in E(G^{'})$. (Note that ij is the directed edge from i to j.)*
- *There is a dashing function $d : E(G) \to \{1, -1\}$. We called an edge ij where $d(ij) = 1$ a solid edge and an edge ij where $d(ij) = -1$ a dashed edge. The dashing function creates an odd dashing, which means that in every 2-colored 4-cycle, either 1 or 3 of the edges is dashed.*

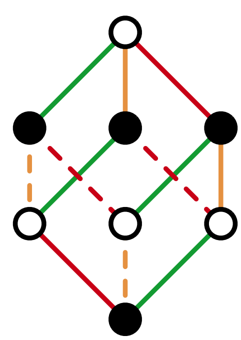

Figure 1.1 An example of an adinkra.

Given an adinkra topology, when is it possible to color it such that it becomes a chromotopology? Given a chromotopology, when is it possible to assign heights and dashing so that it becomes an adinkra? We will now describe the general theory. These questions are answered in terms of taking quotients of hypercubes by linear codes. We first define these terms. Definitions are adapted from Zhang (2014).

Definition 4. *A* ***n-bitstring*** *is a vector in $\mathbb{Z}_2^n$. The* ***weight*** *of a bitstring is the number of entries that are 1. An* ***(n,k)-linear binary code*** *(abbreviated here as* ***code****) is a $k-$dimensional subspace in $\mathbb{Z}_2^n$ of bitstrings. A code is* ***even*** *if each bitstring in the code has even weight and* ***doubly even*** *if each bitstring has weight divisible by 4.*

Definition 5. *The* ***n-dimensional hypercube*** *is the graph G where $V(G)$ is the set of n-bitstrings (that is, the numbers from 0 to 2^{n-1} written in binary) and $E(G)$ consists of those pairs of vertices that differ in exactly one bit.*

Note that the hypercube fulfills the graph theoretic properties necessary to be an adinkra topology. We call it the **n-cubical topology**, notated I^n. For use in future definitions and theorems, we consider a particular edge coloring of the hypercube that gives a chromotopology.

Definition 6. *The* ***n-cubical chromotopology*** *I_c^n is I^n with the following coloring: If two vertices differ at bit i, color the edge between them with the color i.*

Taking a quotient may result in a multigraph (that is, a graph with loops or multiple edges), so we must generalize our definitions in the following way before we can define quotients.

Definition 7. *A* ***pretopology*** *is a n-regular finite connected multigraph. A* ***prechromotopology*** *is a generalization of a chromotopology where the corresponding graph can be a pretopology rather than just a topology.*

We now explain what it means to take the graph quotient of I^n by the code L. This material comes from (Zhang, 2014: Section 4.2).

Let $L \subset \mathbb{Z}_2^n$ be a linear code. Then the quotient $\mathbb{Z}_2^n/L$ is a $\mathbb{Z}_2$-subspace. We define a map p_L which sends I_c^n to a prechromotopology which we call the graph quotient I_c^n/L: Label the vertices of I_c^n/L by the equivalence classes of $\mathbb{Z}_2^n/L$. Define $p_L(v)$ to be the image of a vertex $v \in V(I_c^n)$ under the quotient $\mathbb{Z}_2^n/L$. Let there be an edge in I_c^n/L of color i between $p_L(v)$ and $p_L(w)$ in I^n/L if there is at least one edge with color i of with endpoints $v' \in p_L^{-1}(v)$ and $w' \in p_L^{-1}(w)$.

The following three theorems provide a complete answer to our questions about which structures underlie adinkras.

Theorem 1. *(Zhang, 2014: Theorem 4.3) A structure is a prechromotopology if and only if it is a quotient I_c^n/L for some code L.*

Theorem 2. *(Zhang, 2014: Theorem 4.4) A structure is a chromotopology if and only if it is a quotient I_c^n/L for some even code L with no bitstring of weight 2.*

Theorem 3. *(Doran et al., 2008: Theorem 4.1) It is possible to give the structure of an adinkra to a chromotopology if and only if it is a quotient I_c^n/L for some doubly even code L.*

1.2 Background on Arithmetical Graphs

The notion of an arithmetical graph was defined by Dino Lorenzini in 1989. The concept arose in the study of degenerating curves, and the underlying algebraic geometry motivates the definition.

We now give a definition of an arithmetical graph and of linear rank and volume, two characteristics of an arithmetical graph. We also state a few theorems that are essential for our study of adinkras as arithmetical graphs.

Definition 8. *Lorenzini (1989) Let G be a connected graph with n vertices. Let A be the adjacency matrix of G. Let D be a diagonal matrix where the diagonal entries d_i are positive integers. Let $M = D - A$. Let R be a vector $R^T = [r_1, \dots, r_n]$ such that the r_i are positive integers and $\gcd(r_1, \dots, r_n) = 1$ and $MR = 0$. We call (G, M, R) an* ***arithmetical graph****, and say that (M, R) defines an* ***arithmetical structure*** *on G.*

Lorenzini defines the volume of an arithmetical graph, and proves that it is an integer, though a priori it is only a rational number as a vertex may have degree one.

Definition 9. *Lorenzini (1989) Let (G, M, R) be an arithmetical graph, where v_i are the vertices of G and $\deg(v_i)$ is the degree of v_i. The volume v of (G, M, R) is defined as $v = \prod_{i=1}^{n} r_i^{\deg(v_i)-2}$.*

Theorem 4. *(Lorenzini, 1989: Theorem 4.10) The volume of an arithmetical graph is an integer.*

Another important property of an arithmetical graph is linear rank. Linear rank serves as an analogue of genus, and is closely related to the first Betti number of a graph. Lorenzini shows that the linear rank is always an integer. Lorenzini also shows that the linear rank of a graph is always greater than or equal to its first Betti number.

Definition 10. *Let G be a graph. The **first Betti number** of G is defined by*

$$\beta = |E(G)| - |V(G)| + 1.$$

Definition 11. *Lorenzini (1989) The **linear rank** g_0 of an arithmetical graph (G, M, R) is defined by $2g_0 - 2 = \sum_{i=1}^{n} r_i(\deg(v_i) - 2)$.*

Lemma 1. *Lorenzini (1989) The linear rank of an arithmetical graph is an integer.*

Proof. Let $c_{i,j}$ be an entry in the adjacency matrix and r_i an entry in the vector R. The linear rank g_0 is defined by $2g_0 - 2 = \sum_{q=1}^{n} r_q(c_{q,q} - 2)$, so it will be an integer if $\sum_{q=1}^{n} r_q c_{q,q}$ is even. A number is equivalent mod 2 to its square, so $\sum_{i=1}^{n} c_{i,i} r_i^2 \equiv \sum_{i=1}^{n} c_{i,i} r_i \mod 2$. Since $MR = 0$, we have $(R^T)MR = 0$, so $\sum_{i=1}^{n} c_{i,i} r_i^2 = \sum_{i \neq j} c_{i,j} r_i r_j$. Since M is symmetric, $\sum_{i \neq j} c_{i,j} r_i r_j = 2 \sum_{i<j} c_{i,j} r_i r_j$. Thus $\sum_{i=1}^{n} c_{i,i} r_i^2$ is even, so $\sum_{i=1}^{n} c_{i,i} r_i$ is even. Therefore the linear rank is an integer. □

Theorem 5. *Lorenzini (1989) For any arithmetical graph, we have $g_0 \geq \beta$.*

The fact that $g_0 \geq \beta$ is stated explicitly in the introduction of Lorenzini (1989), but assembling a proof requires combining various theorems throughout the paper.

Lastly, we note that there are only finitely many arithmetical graph structures on any given graph.

Theorem 6. *(Lemma 1.6 of Lorenzini, 1989) Given a connected graph G, there exist only finitely many M, R such that (G, M, R) is an arithmetical graph.*

Chapter 2

The Hypercube as an Arithmetical Graph

We aim to interpret adinkras as arithmetical graphs, and learn what properties adinkras have as arithmetical graphs, such as what claims can be made about the volume or linear rank of an adinkra. In Chapter 4, we discuss how to incorporate some of the extra-graphical properties of an adinkra, such as the dashing of the edges or the heights assigned to the vertices, into the data of an arithmetical graph. For now, we consider only the underlying graph of an adinkra. We restrict to the case where the underlying graph is an n-dimensional hypercube for some n, noting that as previously described in Theorems 1,2, and 3, all adinkras can be obtained as a graph quotient of a hypercube by a code.

Our aim in this section is to classify all arithmetical graphs on the n-dimensional hypercube. We start by explicitly proving a classification of all arithmetical graphs on the 2-dimensional hypercube. We then present an algorithm that constructs a classification of all arithmetical graphs of the n-dimensional hypercube for general n. We also compute the volumes and linear ranks of adinkras with hypercubes as underlying graphs.

2.1 2-Dimensional Hypercube

Theorem 7. *The following is a complete list of vectors R that form arithmetical graph structures on the 2-dimensional hypercube with adjacency matrix*

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

[1,1,1,1][1,1,1,2]	[1,1,2,1][1,1,2,3]	[1,1,3,2][1,2,1,1]
[1,2,1,3][1,2,2,1]	[1,2,3,5][1,2,4,3]	[1,3,1,2][1,3,2,5]
[1,3,3,2][1,4,2,3]	[2,1,1,1][2,1,1,2]	[2,1,3,1][2,1,3,4]
[2,1,5,3][2,3,1,1]	[2,3,1,4][2,3,3,1]	[2,5,1,3][3,1,2,1]
[3,1,2,3][3,1,5,2]	[3,2,1,1][3,2,1,3]	[3,2,4,1][3,4,2,1]
[3,5,1,2][4,1,3,2]	[4,3,1,2][5,2,3,1]	[5,3,2,1]

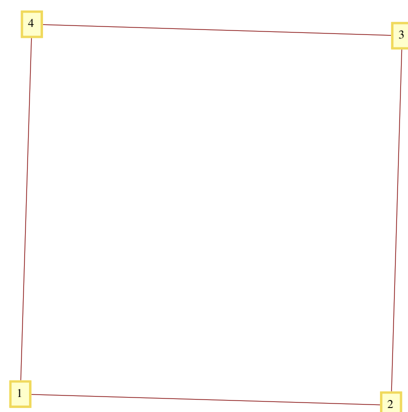

Figure 2.1 The graph for Theorem 7.

Proof. Let G be the 2-dimensional hypercube. We find M, R that define an arithmetical structure on G. Since $MR = 0$, we have

$$\begin{pmatrix} d_1 & -1 & 0 & -1 \\ -1 & d_2 & -1 & 0 \\ 0 & -1 & d_3 & -1 \\ -1 & 0 & -1 & d_4 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} d_1r_1 - r_2 - r_4 \\ -r_1 + d_2r_2 - r_3 \\ -r_2 + d_3r_3 - r_4 \\ -r_1 - r_3 + d_4r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus:

$$d_1 = \frac{r_2 + r_4}{r_1} \quad d_2 = \frac{r_1 + r_3}{r_2} \quad d_3 = \frac{r_2 + r_4}{r_3} \quad d_4 = \frac{r_1 + r_3}{r_4}.$$

This set of d_i and r_i define a valid arithmetical structure on G if d_i and r_i are positive integers for all i and $\gcd(r_1, r_2, r_3, r_4) = 1$. Note that given the r_i, the d_i are determined. The vector R will define an arithmetical structure on G if and only if $\gcd(r_1, r_2, r_3, r_4) = 1$ and

$$r_1 | r_2 + r_4 \qquad r_2 | r_1 + r_3 \qquad r_3 | r_2 + r_4 \qquad r_4 | r_1 + r_3$$

so that the d_i are integers.

Consider a tuple (a, b, c, d) with $a \le b \le c \le d$, such that some matching of the variables a, b, c, d to r_1, r_2, r_3, r_4 satisfies the divisibility statements. Then $d \le b + c$. Since $b \le c \le d$, d is greater than or equal to the average of b and c. So either $a + b = d$, $b + c = d$, or $a + c = d$, or $a + b = 2d$, $b + c = 2d$, or $a + c = 2d$. In the cases that sum to $2d$, we get $a = b = c = d$, so since $\gcd(a, b, c, d) = 1$, we have the vector $(1, 1, 1, 1)$. Thus we can restrict to the cases $(a, b, c, b + c), (a, b, c, a + c), (a, b, c, a + b)$. (Note that once we know all possible (a, b, c), we thus know all possible (a, b, c, d).)

Consider the case $(a, b, c, b+c)$. We have $a|b+c$, $b|a+b+c$, and $c|a+b+c$, or equivalently, $a|b+c$, $b|a+c$, and $c|a+b$. Since $a \le b \le c$, we have $c = a+b$ or $c = \frac{a+b}{2}$. The latter just reduces to (a, a, a). Consider the former. We have $a|a + 2b$ and $b|2a + b$, so $a|2b$ and $b|2a$. So for some integers m, n, we have $2b = na$ and $2a = mb$. Then $4a = m2b = mna$. So $mn = 4$. Considering all possible positive integer pairs (m, n) such that $mn = 1$, we get $(a, 2a, 3a)$ and $(a, a, 2a)$.

Now consider the case $(a, b, c, a + c)$. We have $a|a + b + c$, $b|a + c$, and $c|a + b + c$, which reduces to the same conditions as above.

The case $(a, b, c, a+b)$ similarly reduces to the same conditions as above.

Thus our list is exactly the tuples of the form $(a, a, a), (a, a, 2a), (a, 2a, 3a)$, with the fourth coordinate as specified above. Each coordinate is a multiple of a, and one coordinate is a, so the gcd is a, so a must be 1.

The above enumerates all possibilities. It is easy to show when a given vector R satisfies the specified divisibility conditions. □

2.2 N-Dimensional Hypercube

Theorem 8. *There exists an effective algorithm to characterize the complete list of vectors R and the corresponding matrices M that form arithmetical graph structures on the n-dimensional hypercube.*

Proof. Let G be the n-dimensional hypercube. We exhibit an algorithm to find M, R that define an arithmetical structure on G. For a fixed binary number i such that $0 \leq i \leq 2^n - 1$, let A_i be the set of binary numbers in $[0, 2^n - 1]$ of Hamming distance 1 from i. Solving the equation $MR = 0$ yields equations for $d_0, \ldots, d_{2^n-1}$ in terms of $r_0, \ldots, r_{2^n-1}$ of the form $d_i = \frac{\sum_{j=0}^{2^n-1} r_j \chi_{A_i}}{r_i}$ where χ_{A_i} is the characteristic function of A_i. Let $(a_1, \ldots, a_n)$ be a tuple with $a_1 \leq \cdots \leq a_n$ such that some matching of the variables $a_1, \ldots, a_n$ to $r_0, \ldots, r_{2^n-1}$ satisfies the divisibility statements. Since a_n is greater than or equal to the average of any set of these terms, but a_n divides the sum of some subset of these terms of size n, then there exists some i, j, k such that the following is true: $ma_n = a_i + a_j + a_k$ for some integer m such that $1 \leq m \leq n$. Thus we can reduce our problem to divisibility statements involving only the first $2^n - 1$ entries of the tuple. We perform such a reduction in $2^n - 2$ steps, reducing until we need only to consider tuples of the form $(a_1, a_2, s_1a_1 + s_2a_2)$ for some rational numbers s_1, s_2. Then our divisibility statements will yield $a_1 = \frac{p_1}{q_1}a_2$ and $a_2 = \frac{p_2}{q_2}b_2$ where $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are rational numbers in reduced form. Substitution yields $a_1 = \frac{p_1p_2}{q_1q_2}a_1$, so $p_1p_2 = q_1q_2$. There are only finitely many integers p_1, p_2, q_1, q_2 that satisfy these conditions. Thus for a given a_1, there are only finitely many a_2 that satisfy the conditions. Since the variable a_k can only be made from finitely many linear combinations of the variables $a_1, \ldots, a_{k-1}$, we have determined that for a fixed a_1, there are finitely many tuples $(c_1a_1, \ldots, c_na_n)$ that satisfy the condition, and all can be characterized as we have described above. Since there are only finitely many arithmetical graph structures on any connected graph, there are only finitely many a_1 which determine such a tuple. Thus we can determine all R that give arithmetical structures on G. Note that our equations give the entries of D in terms of the entries of R. Then $M = D - A$ where A is the adjacency matrix of G. $\square$

In addition to determining theoretically a classification of all arithmetical structures on the n-dimensional hypercube, we implemented a computer program to generate all such arithmetical structures for $n = 2$ and $n = 3$. The code is included in Appendix A. The complete results for $n = 2$ are given above. For $n = 3$, the computation was too large to run in its entirety. Instead, we ran a computation that considered only those vectors R with maximum entry less than or equal to some convenient bound. While it was impractical to store every such vector R that is part of an arithmetical structure, we stored the sum and product of the entries in each vector. This information allows us to compute the volumes and linear ranks for these arithmetical structures on the 3-dimensional hypercube. For $n = 2$ every vertex has degree 2, so it is immediate from the definitions of volume and linear rank that the volume is 1 and the linear rank is 0 for every arithmetical structure on the 2-d hypercube. For $n = 3$, we display a partial list of volumes and linear ranks below.

Remark 1. *The following is a list of the volumes of the arithmetical graphs* (G, M, R) *where G is the* 3*-d hypercube and R has maximum entry less than or equal to* 6.
1, 3, 4, 8, 9, 12, 16, 24, 27, 32, 36, 48, 64, 72, 80, 81, 96, 128, 144, 160, 180, 192, 216, 225, 240, 288, 320, 324, 360, 375, 384, 400, 480, 576, 648, 720, 800, 900, 1125, 1152, 1200, 1296, 1600, 1728, 1800, 2000, 2304, 3456, 4500, 5184, 6000, 8100, 10368, 32400, 162000.

Remark 2. *The following is a list of the linear ranks of the arithmetical graphs* (G, M, R) *where G is the* 3*-d hypercube and R has maximum entry less than or equal to* 20.
5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 49.

Chapter 3

Linear Rank and Genus

Lorenzini defines linear rank as an analogue of genus for an arithmetical graph. We now explore the relationship between linear rank and several notions related to genus considered in the study of graphs and adinkras. In particular, we compare these notions in the case of a reduced arithmetical graph structure on a hypercube.

Consider the linear rank of a reduced arithmetical graph. Let $n = |V(G)|$ and $m = |E(G)|$. Since $\sum_{i=1}^{n} \deg(v_i) = 2m$, we have the following:

$$2g_0 - 2 = \sum_{i=1}^{n} (\deg(v_i) - 2)$$

$$g_0 = m - n + 1.$$

Note that the hypercube Q_d has 2^d vertices and $d2^{d-1}$ edges. Thus the linear rank of the reduced arithmetical graph on Q_d is

$$g_0 = (4d - 8)2^{d-3} + 1.$$

3.1 Relations to Genus, Circuit Rank, and First Betti Number of Graph

Note the following definition of the genus of graph.

Definition 12. *König (1950) The **genus** of a graph is the minimal integer n such that the graph can be drawn without crossing itself on an oriented surface of genus n.*

By this definition, we see that a planar graph has genus 0. In general, the problem of finding the graph genus is NP-hard [Thomassen (1989)]. It is known that the hypercube graph $Q_d (d > 1)$ is planar if and only if $d \leq 3$. For $d \geq 4$, the formula $(d-4)2^{d-3} + 1$ gives the genus [Beineke and Harary (1965)].

We note the following relationship between the linear rank and the genus of the reduced arithmetical graph on a hypercube.

Remark 3. *The linear rank of the reduced arithmetical graph structure on Q_d equals the genus of Q_{d+2}.*

We explore this relationship further in an attempt to understand its origins. The linear rank of a reduced arithmetical graph, $g_0 = |E(G)| - |V(G)| + 1$, contains purely graph theoretic quantities. So the relationship between the linear rank of Q_d and the genus of Q_{d+2} reveals the following alternate graph theoretic characterization of the formula given by Beineke and Harary for the genus of the hypercube graph.

Remark 4. *The genus of Q_d for $d \geq 4$ is*

$$g = |E(Q_{d-2})| - |V(Q_{d-2})| + 1.$$

The graph theory literature about the genus of the hypercube graph does not seem to explore its relationship with the quantity $|E(G)| - |V(G)| + 1$. The quantity $|E(G)| - |V(G)| + 1$ does have significance in graph theory, as described in the following definition.

Definition 13. *The **circuit rank** r (or **cyclomatic number** or **nullity**) of an undirected graph G is the minimum number of edges that must be removed to make the resulting graph acyclic.*

An alternate, equivalent characterization is that circuit rank is the number of independent cycles in a graph. Equivalently, this is the graph's first Betti number. In fact, Matthew Baker and Serguei Norine refer to the circuit rank as the genus in Baker and Norine (2007).

The circuit rank of a graph can be computed in terms of its number of edges, vertices, and connected components.

Proposition 1. *Let c be the number of connected components of a graph G. Then the circuit rank r of G is given by*

$$r = |E(G)| - |V(G)| + c$$

.

Thus for a connected graph, we have

$$r = |E(G)| - |V(G)| + 1.$$

As the hypercube graph is connected, we can make two further claims. First, we note the connection between the linear rank of the reduced arithmetical graph on Q_d and the circuit rank of Q_d.

Remark 5. *The linear rank of the reduced arithmetical graph on Q_d equals the circuit rank of Q_d.*

Second, we can restate Remark 4 as follows.

Remark 6. *The genus of Q_d for $d \geq 4$ equals the circuit rank of Q_{d-2}.*

Charles Doran, Kevin Iga, Greg Landweber and Stefan Méndez-Diez have explored ways to situate adinkras in a geometric context. In Doran et al. (2013), they give a canonical way to associate an adinkra chromotopology to a Riemann surface. This Riemann surface has a genus, which can then be thought of as a notion of genus for the chromotopology. In the case of a hypercube Q_d for $d \geq 2$, the genus of the associated Riemann surface is $(d-4)2^{d-3}+1$. In further comments, we will refer to the genus of the Riemann surface canonically associated to a hypercube as the **geometrization genus**, denoted g' to avoid confusion with the notion of genus traditionally used in graph theory, as defined in Definition 12.

3.2 Summary of Relations Between Linear Rank and Various Notions of Genus

We have noted in the previous sections the connections that exist between the linear rank of a reduced arithmetical graph on a hypercube and various notions of genus defined for graphs and adinkras. We summarize those relations here.

Let Q_d be the hypercube graph of dimension d. Let $f(d) = (4d-8)2^{d-3}+1$.

- The linear rank of the reduced arithmetical graph on Q_d for $d \geq 2$ is

$$g_0 = f(d).$$

- The genus of the graph Q_d for $d \geq 4$ is

$$g = f(d-2).$$

- The circuit rank of the graph Q_d for $d \geq 2$ is
$$r = f(d).$$
- The first Betti number of the graph Q_d for $d \geq 2$ is
$$\beta = f(d).$$
- The geometrization genus of the Riemann surface associated to Q_d for $d \geq 4$ is
$$g^{'} = f(d-2).$$

Chapter 4

Modifications of an Arithmetical Graph

The arithmetical graph structure is defined for graphs. An adinkra, in addition to its graph theoretic properties, has an edge-coloring, an edge-dashing, and a vertex height assignment. We explore the question of whether it is possible to incorporate some of these features of an adinkra into the structure of an arithmetical graph. To do so, we test various modifications of the definition of an arithmetical graph. We define the directed arithmetical graph in an attempt to incorporate the height assignment of the vertices. Separately, we define the dashed arithmetical graph in an attempt to incorporate the dashing of the edges.

We explore what properties of an arithmetical graph the directed arithmetical graph and dashed arithmetical graph retain. Are the volume and linear rank still integers? Are there still only finitely many arithmetical graph structures on a given graph? By answering these questions, we explore whether our new definitions make sense.

4.1 Directed Arithmetical Graph

The arithmetical graph structure is defined only for undirected graphs. To encode the height assignment of the vertices of an adinkra into its representation as an arithmetical graph, we consider the adinkra as a directed graph. To do so, we replace the adjacency matrix in the definition of an arithmetical graph with the adjacency matrix of an directed graph. The rest of the definition is the same as that of an ordinary arithmetical graph.

Definition 14. *Let G be a connected, directed graph with n vertices. Let A be the adjacency matrix of G; that is, let $a_{ij} = 1$ if there is an edge from i to j and let $a_{ij} = 0$ otherwise. Let D be a diagonal matrix where the diagonal entries d_i are positive integers. Let $M = D - A$. Let R be a vector $R^T = [r_1, \dots, r_n]$ such that the r_i are positive integers and $\gcd(r_1, \dots, r_n) = 1$ and $MR = 0$. We call (G, M, R) a **directed arithmetical graph**.*

In an arithmetical graph, when R is the vector of all $1's$, for $MR = 0$ we must have that $d_i = \deg(v_i)$. In a directed arithmetical graph, when R is the vector of all $1's$, for $MR = 0$ we must have that d_i is the outdegree of v_i. The outdegree can thus be considered an analogue of degree for a directed arithmetical graph, and we will use it in our modified definitions of volume and linear rank. We now define directed degree, directed volume, and directed linear rank.

Definition 15. *The **directed degree** of a vertex v_i in a directed arithmetical graph, denoted $\deg^+(v_i)$, is the outdegree of v_i, that is, the number of edges from v_i to other vertices.*

Definition 16. *Let (G, M, R) be a directed arithmetical graph, where v_i are the vertices of G. The **directed volume** v of (G, M, R) is defined as $v = \prod_{i=1}^{n} r_i^{\deg^+(v_i)-2}$.*

Definition 17. *The **directed linear rank** g_0 of an arithmetical graph (G, M, R) is defined by $2g_0 - 2 = \sum_{i=1}^{n} r_i(\deg^+(v_i) - 2)$.*

We show that in modifying the definitions in this way, we lose two properties of an arithmetical graph. First, the directed volume of a directed arithmetical graph need not be an integer. Second, there can be infinitely many directed arithmetical graph structures on some graphs. We demonstrate examples of each of these scenarios. In both cases, the underlying graph is the 2-dimensional hypercube.

Example 1. *We give an example where the directed volume of a directed arithmetical graph is not an integer. Consider the directed 2-dimensional hypercube with the adjacency matrix*

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\deg^+(v_1) = 2, \deg^+(v_2) = 1, \deg^+(v_3) = 1,$ and $\deg^+(v_4) = 0$. Solving the equation $MR = 0$ yields the following equations:

$$d_1 = \frac{r_2 + r_3}{r_1} \qquad d_2 = \frac{r_4}{r_2} \qquad d_3 = \frac{r_4}{r_3} \qquad d_4 = 0.$$

Let $R = [1,1,1,2]^T$. We note that R satisfies the divisibility conditions required to make the d_i be integers. The directed volume is

$$v = \prod_{i=1}^{4} r_i^{deg^+(v_i)-2} = \frac{1}{4},$$

which is not an integer.

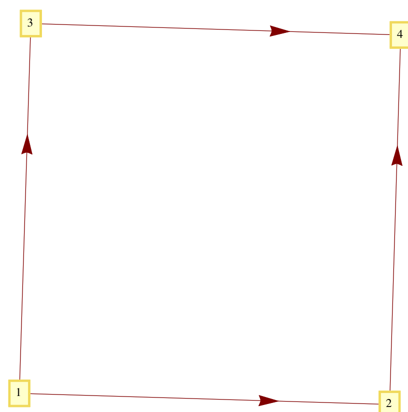

Figure 4.1 The graph used in Example 1.

Example 2. *Consider the same graph as in Example 1. Note that $R = [1,1,1,n]^T$ for any positive integer n will give integer values of d_i. Thus we can define infinitely many directed arithmetical graph structures on the directed 2-dimensional hypercube.*

To further explore the arithmetical features of a directed graph, it would be interesting to consider a notion of a "directed genus." A literature search suggests that such a notion is defined only for directed graphs that are Eulerian. Due to the nature of the height assignment of vertices, the directed graph associated to an adinkra is never Eulerian. Thus in our study of adinkras, we do not consider a notion of directed genus.

4.2 Dashed Arithmetical Graph

We define the dashed arithmetical graph to incorporate the edge dashing of an adinkra into the structure of an arithmetical graph. We modify the adjacency matrix so that solid edges are represented by 1 and dashed edges are represented by -1. The rest of the definition remains unchanged.

Definition 18. *Let G be a connected graph with n vertices. Let every edge in $E(G)$ be either dashed or solid. Let A be a modified adjacency matrix of G: Let $a_{ij} = 1$ if $ij \in E(G)$ and ij is solid, let $a_{ij} = -1$ if $ij \in E(G)$ and ij is dashed, and let $a_{ij} = 0$ otherwise. Let D be a diagonal matrix where the diagonal entries d_i are positive integers. Let $M = D - A$. Let R be a vector $R^T = [r_1, \ldots, r_n]$ such that the r_i are positive integers and $\gcd(r_1, \ldots, r_n) = 1$ and $MR = 0$. We call (G, M, R) a* ***dashed arithmetical graph****.*

As for a directed arithmetical graph, we consider a new concept of degree when studying dashed arithmetical graphs. If R is the vector of all 1's, then to have $MR = 0$ we must have that d_i is the number of solid edges incident to v_i minus the number of dashed edges incident to v_i. This motivates the definition of dashed degree. We use the dashed degree to define dashed volume and dashed linear rank.

Definition 19. *The* ***dashed degree*** *of a vertex v_i in a dashed arithmetical graph, denoted $\deg^*(v_i)$ is the number of solid edges incident to v_i minus the number of dashed edges incident to v_i.*

Definition 20. *Let (G, M, R) be a dashed arithmetical graph, where v_i are the vertices of G. The* ***dashed volume*** *v of (G, M, R) is defined as $v = \prod_{i=1}^{n} r_i^{\deg^*(v_i)-2}$.*

Definition 21. *The* ***dashed linear rank*** *g_0 of an arithmetical graph (G, M, R) is defined by $2g_0 - 2 = \sum_{i=1}^{n} r_i(\deg^*(v_i) - 2)$.*

We now discuss what properties we lose when we modify the definition of an arithmetical graph in this way. We show that the linear rank can be negative, and it is possible to have $\beta > g_0$, even though neither of these situations is possible for the original definition of an arithmetical graph. We give an example where the dashed volume is a noninteger. We also show that it is possible for there to be infinitely many dashed arithmetical graph structures on a given graph.

Example 3. *We show an example where the dashed linear rank is negative. Consider the 2-dimensional hypercube where three edges are solid and one is dashed. Incorporating dashing, the adjacency matrix is*

$$A = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Solving the equation $MR = 0$ yields the following equations:

$$d_1 = \frac{r_4 - r_2}{r_1} \quad d_2 = \frac{r_3 - r_1}{r_2} \quad d_3 = \frac{r_2 + r_4}{r_3} \quad d_4 = \frac{r_1 + r_3}{r_4}.$$

Then two vertices have dashed degree 0 *and two vertices have dashed degree* 2*. Let $R = [1,1,1,1]^T$. Then $\beta = g_0$. Then $2\beta - 2 = \sum_{i=1}^{n}(d_i - 2) = (0+0+2+2) - 2(4) = -4$, so $\beta = g_0 = -1$. Note that the dashed linear rank is negative.*

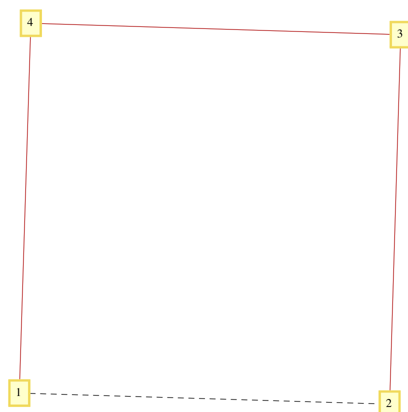

Figure 4.2 The graph used in Example 3.

Example 4. *We find an R for which the dashed four cycle has $\beta > g_0$. Let $R = [4,1,1,1]^T$. Then $\sum_{i=1}^{n}(d_i - 2) = (0+0+2+2) - 2(4) = -4 > \sum_{i=1}^{n} r_i(d_i - 2) = 4(-2) + 0 - 2 - 2 = -12$, so $\beta > g_0$.*

Example 5. *We see that for $R = [4,1,1,1]^T$ for the above graph, the dashed volume is not an integer. The dashed volume is*

$$v = \prod_{i=1}^{4} r_i^{\deg^*(v_i) - 2} = \frac{1}{{r_1}^2 {r_2}^2} = \frac{1}{16},$$

which is not an integer.

Proposition 2. *There exists a graph with infinitely many dashed arithmetical graph structures.*

Proof. We use the 2-dimensional hypercube with one edge dashed and three solid, and adjacency matrix as given above in Example 3. Consider the infinite class of tuples given by $(n, 1, n+2, n+1)$ for $n \geq 1$. We have

$$d_1 = \frac{r_4 - r_2}{r_1} = \frac{n+1-1}{n} = 1 \qquad d_2 = \frac{r_3 - r_1}{r_2} = \frac{n+2-n}{1} = 2$$
$$d_3 = \frac{r_2 + r_4}{r_3} = \frac{n+1+1}{n+2} = 1 \qquad d_4 = \frac{r_1 + r_3}{r_4} = \frac{2n+2}{n+1} = 2.$$

Note that $\gcd(n, 1, n+2, n+1) = 1$. Thus this infinite set of tuples give vectors R that define an infinite set of arithmetical structures on G. □

Chapter 5

Conclusion

In this thesis, we have studied adinkras in the context of arithmetical graphs. We have tried to see how connections can be established between these two seemingly disparate concepts in hopes of bringing new tools to the study of adinkras.

When we consider an adinkra in terms only of its underlying graph, an adinkra lends itself naturally to the structure of an arithmetical graph. Using this approach, we were able to classify all possible arithmetical structures on hypercubes of any dimension and to compute some of the corresponding volumes and linear ranks.

Interesting questions also arose in an examination of the linear rank of a reduced arithmetical graph structure on a hypercube. The linear rank of a reduced arithmetical graph can be stated in purely graph theoretic terms. This expression can be compared to several other notions of genus. In particular, we explored connections to the genus as traditionally defined in graph theory, as well as the first Betti number of a graph, and the genus of the Riemann surface canonically associated to a chromotopology on a hypercube. We found that these notions are indeed quite related.

Though we can learn some things by defining an arithmetical structure on just the underlying graph of an adinkra, an adinkra has so much more structure than just that of a graph. So one of the goals for the project was to incorporate some of the other properties of an adinkra when defining an arithmetical graph structure on an adinkra. To this end, we defined the directed arithmetical graph and the dashed arithmetical graph. These definitions introduced certain pathologies, such as noninteger volume and the existence of infinitely many arithmetical structures corresponding to a single graph. We did not conduct this project with a predetermined use intended for the definition of an arithmetical graph structure on an adinkra. As such, we cannot say whether or not these pathologies are enough to invalidate the definitions. They do speak, however, to the difficulty of combining the structures of an adinkra and an arithmetical graph while holding true to the meaning of both.

We attempted to connect adinkras and arithmetical graphs. We learned it can be difficult to simultaneously consider all of the properties of both structures. However, we also learned that it is possible to find interesting relationships through certain properties, such as those related to linear rank and genus. The full potential of the enterprise of relating adinkras and arithmetical graphs remains unknown.

Appendix A

Code

This is the Python code we used to generate the volumes and linear ranks of arithmetical graph structures on the 3-dimensional hypercube.

```
import time
import random
import fractions

def checkvector(vec):
#We call this function in cube.
output=[0,0]
if reduce(fractions.gcd,vec)==1:
        sumvertexvector=vec[0]+vec[1]+vec[2]+vec[3]+
    vec[4]+vec[5]+vec[6]+vec[7]
        productvertexvector=vec[0]*vec[1]*vec[2]*vec[3]*vec[4]
    *vec[5]*vec[6]*vec[7]
        output[0]=sumvertexvector
        output[1]=productvertexvector
return output

def cube(n):
#For n, input the maximum entry of a vector R that you wish to consider.
    listofsums=[]
    listofproducts=[]
#These 8 for loops go through the elements of
#the 8-fold Cartesian product of the integers in [1,n].
#Let's call a given element of the Cartesian product a "vertex vector."
#It assigns a number to each of the
```

```
#eight vertices of the 3-D hypercube.
    for a000 in range(1,n+1):
        for a001 in range(1,n+1):
            for a010 in range(1,n+1):
                for a011 in range(1,n+1):
                    for a100 in range(1,n+1):
                        for a101 in range(1,n+1):
                            for a110 in range(1,n+1):
                                for a111 in range(1,n+1):
#These 8 "if statements" check divisibility properties
#of a given vertex vector.
#A vertex vector is "almost good" if the number assigned
#to each vertex divides the sum of the number assigned
#to its neighbors.
#Note that the neighbors of a vertex of the N-D hypercube
#are the vertices that differ from it in exactly one binary digit.
#The 8 if statements below are all nested
#within the innermost for loop and within each other.
#They are shifted left for purposes of display.
if (a011+a101+a110)%a111==0:
        if (a010+a100+a111)%a110==0:
                if (a001+a111+a100)%a101==0:
                        if (a000+a110+a101)%a100==0:
if (a111+a001+a010)%a011==0:
        if (a110+a000+a011)%a010==0:
                if (a101+a011+a000)%a001==0:
                        if (a100+a010+a001)%a000==0:
vertexvector=[a000,a001,a010,a011,a100,a101,a110,a111]
#The function checkvector checks if the gcd of a vector is 1.
#If so, it outputs the sum and product of the entries of the vector.
#Otherwise, it outputs [0,0].
outputcheckvector=checkvector(vec)
sumvertexvector=outputcheckvector[0]
productvertexvector=outputcheckvector[1]
if not sumvertexvector in listofsums:
        listofsums+=[sumvertexvector]
if not productvertexvector in listofproducts:
        listofproducts+=[productvertexvector]
#The sums and products are natural numbers.
#We order them from smallest to largest.
```

```
listofsums.sort()
listofproducts.sort()
print listofsums
print listofproducts
return
```

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