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Fibonomial Tilings and Other Up-Down Tilings

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Abstract

The Fibonomial coefficients are a generalization of the binomial coefficients with a rather nice combinatorial interpretation. While the ordinary binomial coefficients count lattice paths in a grid, the Fibonomial coefficients count the number of ways to draw a lattice path in a grid and then tile the regions above and below the path with squares and dominoes.

We may forgo a literal tiling interpretation and, instead of the Fibonacci numbers, use an arbitrary function to count the number of ways to "tile" the regions of the grid delineated by the lattice path. When the function is a combinatorial sequence such as the Lucas numbers or the q-numbers, the total number of tilings is some multiple of a generalized binomial coefficient corresponding to the sequence chosen.

Acknowledgments

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Contents

Chapter 1

Introduction

1.1 The Fibonomial Coefficients

The Fibonomial Coefficients are defined by[1]

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{F_n \cdot F_{n-1} \cdots F_2 \cdot F_1}{(F_k \cdots F_1)(F_{n-k} \cdots F_1)} & 0 \le k \le n \\ \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

where F_i denotes the ith Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$, which counts the number of ways to tile a board of length $i - 1$ with squares and dominoes. In other words, $\begin{bmatrix} n \\ k \end{bmatrix}$ is the standard binomial coefficient $\binom{n}{k}$ with each number i in the numerator and denominator replaced with F_i. As shown in [Bruce E. Sagan (2009)], the Fibonomial coefficients count the number of ways to draw a lattice path from $(0, 0)$ to $(k, n-k)$ in a rectangular grid of width k and height $n-k$, and then tile the k vertical strips below the path and the $n-k$ horizontal strips to the left of the path with squares and dominoes such that each vertical tiling of nonzero length begins with a domino (in particular, no vertical tiling may have a height of 1). Figure 1.1 shows one such tiling for $n = 9, k = 5$.

There are a number of well-known identities for the binomial coefficients that also hold for the Fibonomial coefficients. For instance, the binomial coefficients are symmetric, in that $\binom{n}{k} = \binom{n}{n-k}$. This also holds for the Fibonomial coefficients: $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$. This is clear from the definition — one may

[1]If $k = 0$ or $k = n$, then one of the products in the denominator will be empty; by convention, empty products evaluate to 1, so $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ for all $n \ge 0$.

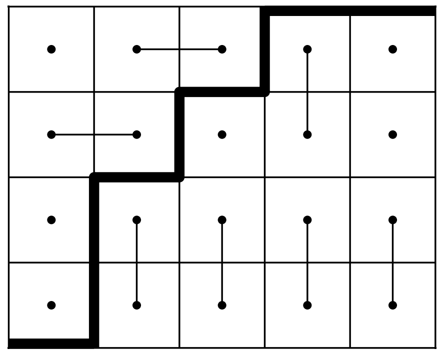

Figure 1.1 A Fibonomial tiling counted by $\begin{bmatrix} 9 \\ 5 \end{bmatrix}$.

cancel terms in the numerator and denominator — but a straightforward combinatorial proof of this simple fact is not known. There are many other Fibonomial identities for which analogous binomial identities, and their combinatorial proofs, are known, yet which have no known combinatorial proofs themselves. Of course, there are Fibonomial identities for which combinatorial proofs are known. We will review a number of Fibonomial identities in this paper.

1.2 The Tiling Coefficients

Choosing to tile the strips above and below the lattice path of a Fibonomial tiling with squares and dominoes, i.e., Fibonacci-ly, is somewhat arbitrary. Instead of using squares and dominoes, we might tile the strips in some other (possibly unspecified) manner. Then, there would not be F_i ways to tile a strip of length $i-1$, but rather $f(i)$ ways for some function f [Gerdemann (2015)].[2] With f arbitrary, we will not focus on any particular tiling of a strip (and by extension, the board); we are concerned only with the number of ways to tile a strip (and the board). We will call a tiling of board, in a manner counted by f, an *up-down tiling*, replacing the notation $\begin{bmatrix} n \\ k \end{bmatrix}$ with $T_f(n,k)$. We call the set of $T_f(n,k)$ the *tiling coefficients*. In this paper we investigate the tiling coefficients, proving identities for arbitrary choices of f as well as evaluating $T_f(n,k)$ for specific choices of f. Finally, we will

[2]Note that we are replacing F_i with $f(i)$ in the combinatorial interpretation of the Fibonomial coefficients, *not* in the definition. There are $f(i)$ ways to tile a strip of length $i-1$, but the total number of tilings of the board is not necessarily $\frac{f(n)f(n-1)\cdots f(2)f(1)}{(f(k)\cdots f(1))(f(n-k)\cdots f(1))}$.

generalize up-down tilings further and prove some theorems relating the Fibonacci numbers, the Lucas numbers, and these generalized tilings.

Chapter 2

Fibonomial Theorems and Identities

In this chapter, we present a number of theorems involving the Fibonomial coefficients. Given the similarity between the Fibonomial coefficients and binomial coefficients, it should come as no surprise that many binomial coefficient identities have analogs for the Fibonomial coefficients. In addition, the relationship between the Fibonacci numbers and the Lucas numbers[1] leads to some identities relating the Lucas numbers and Fibonomial coefficients.

2.1 Proofs Using Algebraic Manipulation

Below are a number of identities which can be proven through basic algebraic manipulation of the Fibonomial coefficients, i.e., by canceling the terms in the numerator and denominator. Despite their apparent simplicity, these identities have no known combinatorial proofs.[2]

[1]The Lucas numbers are the *companion Lucas sequence* of the Fibonacci numbers; they have the same recurrence, $L_n = L_{n-1} + L_{n-2}$, but the initial conditions are $L_0 = 2$ and $L_1 = 1$. While the Fibonacci numbers count tilings of a linear board with squares and dominoes, the Lucas numbers count tilings of a circular board (with a distinguished 0th edge joining the first and last squares of the board) with squares and dominoes, in which a domino is allowed to cover the distinguished edge.

[2]Actually, a combinatorial proof, given in [Bruce E. Sagan (2009)], is known for the identity $2^n \left[{n \atop k}\right] = 2^n \left[{n \atop n-k}\right]$, which is equivalent to 2.1; however the proof does not use Fibonomial tiling.

Theorem 2.1.

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix} \tag{2.1}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} k \\ j \end{bmatrix} = \begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} n-j \\ k-j \end{bmatrix} \tag{2.2}$$

$$F_k \begin{bmatrix} n \\ k \end{bmatrix} = F_n \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \tag{2.3}$$

$$F_k \begin{bmatrix} n \\ k \end{bmatrix} = F_{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix} \tag{2.4}$$

$$F_{n+1} \begin{bmatrix} n \\ k \end{bmatrix} = F_{n-k+1} \begin{bmatrix} n+1 \\ k \end{bmatrix} \tag{2.5}$$

Here is a more complex identity which can also be proven algebraically:

Theorem 2.2 (Gould (1969))**.**

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=k}^{n} \frac{F_j - F_{j-k}}{F_k} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix} \tag{2.6}$$

Proof. We manipulate the summand as follows:

$$\begin{aligned} \frac{F_j - F_{j-k}}{F_k} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix} &= \left(\frac{F_j}{F_k} - \frac{F_{j-k}}{F_k} \right) \frac{F_{j-1} \cdots F_1}{(F_{k-1} \cdots F_1)(F_{j-k} \cdots F_1)} \\ &= \begin{bmatrix} j \\ k \end{bmatrix} - \begin{bmatrix} j-1 \\ k \end{bmatrix} \end{aligned}$$

As $\left[\begin{smallmatrix} j \\ k \end{smallmatrix}\right]$ occurs positively in the jth term of the sum and negatively in the $(j+1)$th term of the sum, the sum telescopes and all that is left is $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] - \left[\begin{smallmatrix} k-1 \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$. □

The Fibonomial coefficients also play nicely with greatest common divisors, as is evident in the following two theorems.

Theorem 2.3 (Gould and Schlesinger (1993))**.**

$$F_n \left(\begin{bmatrix} n \\ k \end{bmatrix}, \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right) = \begin{bmatrix} n \\ k \end{bmatrix} (F_n, F_k) \tag{2.7}$$

$$F_{n+1-k} \left(\begin{bmatrix} n \\ k \end{bmatrix}, \begin{bmatrix} n \\ k-1 \end{bmatrix} \right) = \begin{bmatrix} n \\ k \end{bmatrix} (F_{n+1-k}, F_k) \tag{2.8}$$

where (a, b) denotes the greatest common divisor of a and b.

Proof. For the first identity,

$$\begin{aligned} F_n\left(\begin{bmatrix} n \\ k \end{bmatrix}, \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right) &= \left(F_n\begin{bmatrix} n \\ k \end{bmatrix}, F_n\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right) \\ &= \left(F_n\begin{bmatrix} n \\ k \end{bmatrix}, F_k\begin{bmatrix} n \\ k \end{bmatrix}\right) \\ &= \begin{bmatrix} n \\ k \end{bmatrix}(F_n, F_k) \end{aligned}$$

For the second,

$$\begin{aligned} F_{n+1-k}\left(\begin{bmatrix} n \\ k \end{bmatrix}, \begin{bmatrix} n \\ k-1 \end{bmatrix}\right) &= \left(F_{n-1+k}\begin{bmatrix} n \\ k \end{bmatrix}, F_{n-1+k}\begin{bmatrix} n \\ k-1 \end{bmatrix}\right) \\ &= \left(F_{n-1+k}\begin{bmatrix} n \\ k \end{bmatrix}, F_k\begin{bmatrix} n \\ k \end{bmatrix}\right) \\ &= \begin{bmatrix} n \\ k \end{bmatrix}(F_{n-1+k}, F_k) \end{aligned}$$

□

2.2 Relating the Fibonomial Coefficients and Lucas Numbers

There are countless identities relating the Fibonacci numbers and Lucas numbers, so perhaps it is not surprising that there is at least one relating the Fibonomial coefficients and Lucas numbers.

Theorem 2.4 (Kilic et al. (2012))**.**

$$\sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_F = \prod_{k=1}^{n} L_{2k} \tag{2.9}$$

Proof. We prove the following theorem, which we will show is a generalization of Theorem 2.4.

Theorem 2.5.

$$\sum_{k=0}^{n}(-q)^{mk(k-2n-1)/2}\begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} = \begin{cases} (-1)^{\binom{n+1}{2}}q^{-m\binom{n+1}{2}}(-q^{2m};q^{2m})_n & m \text{ odd} \\ q^{-m\binom{n+1}{2}}(-q^m;q^m)_n^2 & m \text{ even} \end{cases} \tag{2.10}$$

where we define

$$(x; y)_n := (1-x)(1-xy)(1-xy^2)\cdots(1-xy^{(n-1)}) \tag{2.11}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^m} := \frac{(q^m; q^m)_n}{(q^m; q^m)_k (q^m; q^m)_{n-k}}. \tag{2.12}$$

To prove Theorem 2.5, we will use the following lemma.

Lemma 2.1. *For $n \in \mathbb{N}$, if a is such that $a(k) = a(2n+1-k)$, then*

$$\sum_{k=0}^{2n+1} a(k) = (1+i) \sum_{k=0}^{2n+1} i^{-k^2} a(k) \tag{2.13}$$

where $i^2 = -1$.

Proof. We see that

$$\begin{aligned}\sum_{k=0}^{2n+1} i^{-k^2} a(k) &= \frac{1}{2} \sum_{k=0}^{2n+1} (i^{-k^2} + i^{-(2n+1-k)^2} a(k)) \\ &= \frac{1}{2} \sum_{k=0}^{2n+1} (i^{-k^2} + i^{-(1-k)^2}) a(k) \\ &= \frac{1-i}{2} \sum_{k=0}^{2n+1} a(k)\end{aligned}$$

In the last step above, we have used the fact that one of k^2 and $(1-k)^2$ must be even and the other must be odd. All even squares are equivalent to 0 mod 4 and all odd squares are equivalent to 1 mod 4. Thus one of $-k^2$ and $-(1-k)^2$ must be equivalent 0 mod 4 and the other must be equivalent to $-1 \mod 4$. Since $i^0 = 1$ and $i^{-1} = -i$, the last step follows.

Rearranging,

$$\sum_{k=0}^{2n+1} a(k) = \frac{2}{1-i} \sum_{k=0}^{2n+1} i^{-k^2} a(k) = (1+i) \sum_{k=0}^{2n+1} i^{-k^2} a(k)$$

and the lemma has been proved. □

We also require the Cauchy Binomial Theorem, which we prove below.

Theorem 2.6 (Cauchy Binomial Theorem)**.**

$$\prod_{j=1}^{n}(1+yq^{mj}) = \sum_{k=0}^{n} y^k q^{m\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \tag{2.14}$$

Proof. Each choice of k terms of the product will contribute to the coefficient of y^k in the sum, as we may pick out the yq^{mj} term from each of the k chosen terms and then multiply by 1 from each of the rest of the $n-k$ terms. The coefficient of y in the jth term of the product is $(q^m)^j$. Therefore, the coefficient of the y^k term is

$$\begin{aligned}
[y^k]\prod_{j=1}^{n}(1+yq^{mj}) &= \sum_{1\le j_1<\cdots<j_k\le n} (q^m)^{\sum_{i=1}^{k} j_i} \\
&= \sum_{1\le j_1<\cdots<j_k\le n} (q^m)^{\sum_{i=1}^{k} i \,+\, \sum_{i=i}^{k}(j_i-i)} \\
&= \sum_{1\le j_1<\cdots<j_k\le n} (q^m)^{\binom{k+1}{2}+\sum_{i=1}^{k}(j_i-i)} \\
&= (q^m)^{\binom{k+1}{2}} \sum_{1\le j_1<\cdots<j_k\le n} (q^m)^{\sum_{i=1}^{k}(j_i-i)} \\
&= (q^m)^{\binom{k+1}{2}} \sum_{0\le j_1'\le\cdots\le j_k'\le n-k} (q^m)^{\sum_{i=1}^{k} j_i'}
\end{aligned}$$

Interpreting this summation is straightforward: the coefficient of $(q^m)^p$ is the number of solutions to $j_1'+\cdots+j_k'=p$ subject to $0\le j_1'\le\cdots\le j_k'\le n-k$. There is a one-to-one correspondence between these solutions and lattice paths in a $k\times(n-k)$ grid, with j_i' giving the number of vertical steps preceding the ith horizontal step. The area of the grid under the ith horizontal step is the number of preceding vertical steps, j_i', so the total area under the lattice path is $j_1'+\cdots+j_k'=p$. The coefficient of $(q^m)^p$ must then be the number of lattice paths with area under the path equal to p. Because $\begin{bmatrix} n \\ k \end{bmatrix}_{q^m}$ is the generating function (in q^m) for the number of lattice paths in a $k\times(n-k)$ grid, indexed by area under the path, it follows that

$$\sum_{0\le j_1'\le\cdots\le j_k'\le n-k} (q^m)^{\sum_{i=1}^{k} j_i} = \begin{bmatrix} n \\ k \end{bmatrix}_{q^m}.$$

Hence

$$[y^k]\prod_{j=1}^{n}(1+yq^{mj}) = (q^m)^{\binom{k+1}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_{q^m}$$

and thus

$$\prod_{j=1}^{n}(1+yq^{mj}) = \sum_{k=0}^{n} y^k (q^m)^{\binom{k+1}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_{q^m}$$

□

Now we prove Theorem 2.5. First suppose that m is odd. Note that

$$2\sum_{k=0}^{n}(-q)^{mk(k-2n-1)/2}\begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} = \sum_{k=0}^{2n+1}(-q)^{mk(k-2n-1)/2}\begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m}.$$

Because the summand is symmetric under the substitution $k \to 2n+1-k$, we apply Lemma 2.1 to obtain

$$\sum_{k=0}^{2n+1}(-q)^{mk(k-2n-1)/2}\begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m} = (1+i)\sum_{k=0}^{2n+1} i^{-k^2}(-q)^{mk(k-2n-1)/2}\begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{q^m}.$$

Now, because m is odd, $(-1)^m = -1$, hence

$$\begin{aligned}(-q)^{mk(k-2n-1)/2} &= (-q^m)^{k(k-2n-1)/2} \\ &= i^{k(k-2n-1)}(q^m)^{k(k-2n-1)/2} \\ &= i^{k^2}(i^{-(2n+1)}q^{-m(n+1)})^k q^{m\binom{k+1}{2}}\end{aligned}$$

Thus

$$
\begin{aligned}
&(1+i)\sum_{k=0}^{2n+1} i^{-k^2}(-q^m)^{k(k-2n-1)/2}\begin{bmatrix}2n+1\\ k\end{bmatrix}_{q^m}\\
&= (1+i)\sum_{k=0}^{2n+1}(i^{-(2n+1)}q^{-m(n+1)})^k q^{m\binom{k+1}{2}}\begin{bmatrix}2n+1\\ k\end{bmatrix}_{q^m}\\
&= (1+i)\prod_{k=1}^{2n+1}(1+(i^{-(2n+1)}q^{-m(n+1)})q^{mk}) \qquad \text{(Thm. (2.6))}\\
&= (1+i)\prod_{k=1}^{2n+1}(1+i^{-(2n+1)}q^{m(k-n-1)})
\end{aligned}
$$

Now, remove the $k = n+1$ term, $1 + i^{-(2n+1)}$, and note that as k goes from 1 to $2n+1$ (skipping $k = n+1$), $k-n-1$ takes on values $-n, -(n-1), \ldots, -2, -1, 1, 2, \ldots, n-1, n$. Group opposite pairs of multiplicands (i.e., pairs with $k = i, 2n+2-i$ for $i = 1, \ldots, n$) to obtain the following:

$$
\begin{aligned}
&= (1+i)(1+i^{-(2n+1)})\prod_{k=1}^{n}(1+i^{-(2n+1)}q^{-mk})(1+i^{-(2n+1)}q^{mk})\\
&= (1+i+i^{-(2n+1)}+i^{-2n})\prod_{k=1}^{n} 1+i^{-(2n+1)}(q^{-mk}+q^{mk})+i^{-2(2n+1)}\\
&= (1+i+i^{-(2n+1)}+i^{-2n})\prod_{k=1}^{n} i^{-(2n+1)}(q^{-mk}+q^{mk})
\end{aligned}
$$

If n is even, then $i^{-(2n+1)}+i^{-2n} = 1-i$, and if n is odd, then $i^{-(2n+1)}+i^{-2n} = i-1$. Hence $1+i+i^{-(2n+1)}+i^{-2n} = 2$ if n is even, and $2i$ if n is odd. This is equivalent to $1+i+i^{-(2n+1)}+i^{-2n} = 2i^{n^2}$ always, as $i^{n^2} = 1$ when n is even and $i^{n^2} = i$

when n is odd, as discussed in the proof of Lemma 2.1. Hence we have

$$
\begin{aligned}
&= 2i^{n^2}\prod_{k=1}^{n} i^{-(2n+1)}(q^{-mk}+q^{mk}) \\
&= 2i^{n^2} i^{-n(2n+1)}\prod_{k=1}^{n} q^{-mk}(1+q^{2mk}) \\
&= 2i^{-n(n+1)}q^{-m\binom{n+1}{2}}\prod_{k=1}^{n}(1+q^{2mk}) \\
&= 2i^{-2\binom{n+1}{2}}q^{-m\binom{n+1}{2}}\prod_{k=1}^{n}(1+q^{2mk}) \\
&= 2(-1)^{\binom{n+1}{2}}q^{m\binom{n+1}{2}}(-q^{2m};q^{2m})_n
\end{aligned}
$$

Dividing the first and last lines by 2 proves (2.10) when m is odd.

When m is even, then because $k(k-2n-1)$ is necessarily even, we have that $mk(k-2n-1)/2$ is also even, and thus

$$
\begin{aligned}
2\sum_{k=0}^{n}(-q)^{mk(k-2n-1)/2}\begin{bmatrix}2n+1\\ k\end{bmatrix}_{q^m} &= \sum_{k=0}^{2n+1} q^{mk(k-2n-1)/2}\begin{bmatrix}2n+1\\ k\end{bmatrix}_{q^m} \\
&= \sum_{k=0}^{2n+1} q^{m\binom{k+1}{2}}q^{-mk(n+1)}\begin{bmatrix}2n+1\\ k\end{bmatrix}_{q^m} \\
&= \prod_{k=1}^{2n+1}(1+q^{m(k-n-1)})
\end{aligned}
$$

Now, use the same trick as above: pull out the $k=n+1$ term (equal to 2), and then pair opposite terms of the product to obtain

$$
\begin{aligned}
&= 2\prod_{k=1}^{n}(1+q^{-mk})(1+q^{mk}) \\
&= 2\prod_{k=1}^{n} q^{-mk}(1+q^{mk})^2 \\
&= 2q^{-m\binom{n+1}{2}}(-q^m;q^m)_n^2
\end{aligned}
$$

Dividing by 2 proves (2.10) when m is even. Thus, the proof of Theorem 2.5 is complete.

Now, let $\{U_n\}$ and $\{V_n\}$ be companion Lucas sequences defined by $U_0 = 0, U_1 = 1, U_n = pU_{n-1} + U_{n-2}$ and $V_0 = 2, V_1 = p, V_n = pV_{n-1} + V_{n-2}$. for indeterminate variable p. Define

$$\begin{bmatrix} n \\ k \end{bmatrix}_{U;m} = \frac{U_m U_{2m} \cdots U_{nm}}{(U_m U_{2m} \cdots U_{km})(U_m U_{2m} \cdots U_{(n-k)m})}. \tag{2.15}$$

Let $\alpha = (p + \sqrt{p^2+4})/2$ and $\beta = (p - \sqrt{p^2+4})/2$ be the solutions to the characteristic equation $x^2 - px - 1 = 0$ of $\{U_n\}$ and $\{V_n\}$, and let $\gamma = \beta/\alpha$. Then $U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}(\frac{1-\gamma^n}{1-\gamma})$ and $V_n = \alpha^n + \beta^n = \alpha^n(1+\gamma^n)$. Note that $-\alpha^2\gamma = -\alpha\beta = 1$. Then,

$$\begin{aligned}
\sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{U;m} &= \sum_{k=0}^{n} \frac{U_m \cdots U_{(2n+1)m}}{(U_m \cdots U_{km})(U_m \cdots U_{(2n+1-k)m})} \\
&= \sum_{k=0}^{n} \frac{\prod_{i=1}^{2n+1} \alpha^{mi-1}(\frac{1-\gamma^{mi}}{1-\gamma})}{\prod_{i=1}^{k} \alpha^{mi-1}(\frac{1-\gamma^{mi}}{1-\gamma}) \prod_{i=1}^{2n+1-k} \alpha^{mi-1}(\frac{1-\gamma^{mi}}{1-\gamma})} \\
&= \sum_{k=0}^{n} \frac{\alpha^{m\binom{2n+2}{2}}}{\alpha^{m\binom{k+1}{2}}\alpha^{m\binom{2n+2-k}{2}}} \cdot \frac{(\gamma^m;\gamma^m)_{2n+1}}{(\gamma^m;\gamma^m)_k(\gamma^m;\gamma^m)_{2n+1-k}} \\
&= \sum_{k=0}^{n} \alpha^{-mk(k-2n-1)} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{\gamma^m} \\
&= \sum_{k=0}^{n} \left(-\alpha^2\gamma\right)^{mk(k-2n-1)/2} \alpha^{-mk(k-2n-1)} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{\gamma^m} \\
&= \sum_{k=0}^{n} (-\gamma)^{mk(k-2n-1)/2} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{\gamma^m}
\end{aligned}$$

If m is odd, then

$$\begin{aligned}
\prod_{k=1}^{n} V_{2mk} &= \prod_{k=1}^{n} \alpha^{2mk}(1+\gamma^{2mk}) \\
&= \alpha^{2m\binom{n+1}{2}}(-\gamma^{2m};\gamma^{2m})_n \\
&= \left(-\alpha^2\gamma\right)^{-m\binom{n+1}{2}} \alpha^{2m\binom{n+1}{2}}(-\gamma^{2m};\gamma^{2m})_n \\
&= (-1)^{\binom{n+1}{2}}\gamma^{-m\binom{n+1}{2}}(-\gamma^{2m};\gamma^{2m})_n
\end{aligned}$$

If m is even, then

$$\begin{aligned}\prod_{k=1}^{n} V_{mk}^2 &= \prod_{k=1}^{n} \alpha^{2mk}(1+\gamma^{mk})^2 \\ &= \alpha^{2m\binom{n+1}{2}}(-\gamma^m;\gamma^m)_n^2 \\ &= \left(-\alpha^2\gamma\right)^{-m\binom{n+1}{2}} \alpha^{2m\binom{n+1}{2}}(-\gamma^m;\gamma^m)_n^2 \\ &= \gamma^{-m\binom{n+1}{2}}(-\gamma^m;\gamma^m)_n^2\end{aligned}$$

Letting $q \to \gamma$ and applying (2.10) proves that

$$\sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{U;m} = \begin{cases} \prod_{k=1}^{n} V_{2mk} & m \text{ odd} \\ \prod_{k=1}^{n} V_{mk}^2 & m \text{ even} \end{cases}$$

If $m = p = 1$, then $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{L_n\}$, and $\begin{bmatrix} 2n+1 \\ k \end{bmatrix}_{U;m} = \begin{bmatrix} 2n+1 \\ k \end{bmatrix}$. Thus we obtain (2.9),

$$\sum_{k=0}^{n} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_F = \prod_{k=1}^{n} L_{2k}$$

as desired. □

Chapter 3

Up-Down Tilings

In this chapter, we investigate the tiling coefficients, which generalize the Fibonomial coefficients. While the Fibonomial coefficients count tilings of a $k \times (n-k)$ board such that there are F_i ways to tile a strip of length $i-1$, the tiling coefficients count tilings with $f(i)$ ways to tile a strip of length $i-1$ for some function f.

3.1 Generalizing the Fibonomial Coefficients

Let us look more closely at how the individual strips above and below the lattice path of a Fibonomial tiling contribute to the Fibonomial coefficient. A vertical step of the lattice path preceded by h horizontal steps will produce a horizontal strip to its left of length h, which may be tiled in F_{h+1} ways. A horizontal step of the lattice path preceded by v vertical steps will produce a vertical strip below it of length v, which may be tiled in F_{v-1} ways (recall that vertical tilings of nonzero length must begin with a domino).

An *up-down* tiling is constructed the same way as a Fibonomial tiling, except we replace F_i with $f(i)$ for an arbitrary function f defined on $\{-1, 0, 1, 2, 3, \ldots\}$ (we will see why -1 is needed shortly). With f arbitrary, the manner in which the strips of the board are tiled remains unspecified. However, as we are concerned only with the total *number* of tilings and not any *particular* tiling, this is not a problem; we may count the number of tilings with knowledge of f alone, without reference to explicit arrangements of tiles. Even though we will not explicitly tile anything, we will still use the words *tiling*, *tiled*, etc. to mean the process of choosing a lattice path in a grid and "tiling" the resulting strips in an abstract manner counted by f.

As with Fibonomial tilings, a horizontal strip of length h may be tiled in $f(h+1)$ ways and a vertical strip of length v may be tiled in $f(v-1)$ ways. Hence the name "up-down tiling"; the horizontal strips increment their argument, while the vertical strips decrement their argument.

There is a small complication that we must handle before progressing. In the combinatorial interpretation of the Fibonomial coefficients, all *nonempty* vertical strips must begin with a domino, and so a vertical strip of length $v \geq 1$ may be tiled in F_{v-1} ways.[1] Ideally, there would be F_{v-1} tilings of a vertical strip of length $v = 0$ as well, so that there would be F_{v-1} ways to tile vertical strip of any length v. Defining F_{-1} according to the Fibonacci recurrence,[2] we see that $F_{-1} = 1$. Since there is, in fact, one way to tile an empty vertical strip with squares and dominos, a vertical strip of nonnegative length v—in other words, any vertical strip—may be tiled in F_{v-1} ways. Therefore, in an up-down tiling with f arbitrary, we no longer need to worry about the requirement that vertical strips begin with a domino; we maintain consistency with Fibonomial tilings simply by specifying that *any* vertical strip of length v may be tiled in $f(v-1)$ ways (which is we require that the domain of f include -1).

For a function f, which we shall call a *tiling function*, denote by $T_f(n,k)$ the number of ways to tile a $k \times (n-k)$ board with f counting tilings of the strips. We will refer to the set of $T_f(n,k)$ as the *tiling coefficients*. As with the binomial coefficients, we set $T_f(0,0) = 1$ and $T_f(n,k) = 0$ when $k < 0$ or $k > n$. We aim to investigate these tiling coefficients, both for arbitrary f and for specific choices of f.

Before continuing, we present an example of how to compute $T_f(n,k)$ by counting the number of tilings per lattice path and then summing over all lattice paths. If $f(i) = i$, then $T_f(4,2) = 6$, as shown in the following table. A vertical step, denoted by V, produces a horizontal strip which may be tiled in $f(h+1)$ ways, where h is the number of preceding (read left to right) H steps. A horizontal step, denoted by H, produces a vertical strip which may be tiled in $f(v-1)$ ways, where v is the number of preceding V steps.

[1]A strip of length 1 that must begin with a domino leaves a strip of length -1, which, as expected, may be tiled with squares and dominoes in $F_0 = 0$ ways.

[2]We extend sequences $\{a_n\}_{n\geq 0}$ with $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ to the negative numbers according to the same recurrence, so that, for instance, $a_{-1} = (a_1 - c_1 a_0)/c_2$. This means that arithmetic sequences are defined on all of $\mathbb{Z}$.

Path	Number of ways to tile strips	Product ($f(i) = i$)
VVHH	$f(0+1), f(0+1), f(2-1), f(2-1)$	$1 \cdot 1 \cdot 1 \cdot 1 = 1$
VHVH	$f(0+1), f(1-1), f(1+1), f(2-1)$	$1 \cdot 0 \cdot 2 \cdot 1 = 0$
VHHV	$f(0+1), f(1-1), f(1-1), f(2+1)$	$1 \cdot 0 \cdot 0 \cdot 3 = 0$
HVVH	$f(0-1), f(1+1), f(1+1), f(2-1)$	$-1 \cdot 2 \cdot 2 \cdot 1 = -4$
HVHV	$f(0-1), f(1+1), f(1-1), f(1+1)$	$-1 \cdot 2 \cdot 0 \cdot 2 = 0$
HHVV	$f(0-1), f(0-1), f(2+1), f(2+1)$	$-1 \cdot -1 \cdot 3 \cdot 3 = 9$
		$\sum = 6$

As we will see, it is not merely a coincidence that $T_f(4,2) = \binom{4}{2}$ for this choice of f.

3.2 Identities

As a generalization of Fibonomial tilings, up-down tilings share many of the same identities. For instance, the following versions of the hockey-stick identity hold:

Theorem 3.1 (Hockey Stick Identity 1)**.**

$$T_f(n,k) = \sum_{i=k}^{n} T_f(i-1,k-i)f(i-k-1)f(k+1)^{n-i} \tag{3.1}$$

Theorem 3.2 (Hockey Stick Identity 2)**.**

$$T_f(n,k) = \sum_{i=n-k}^{n} T_f(i-1,i-(n-k))f(i-(n-k)+1)f(n-k-1)^{n-i} \tag{3.2}$$

Proof. We prove (3.1) first. Consider the location of the last horizontal step of the lattice path. If it is the ith step, then the first $i-1$ steps consist of $k-1$ horizontal steps and $i-k$ vertical steps. Therefore, the portion of the board before the last horizontal step may be tiled in $T_f(i-1,k-1)$ ways and the last vertical strip may be tiled in $f(i-k-1)$ ways. The remaining $n-i$ steps must all be vertical and they are each preceded by k horizontal steps, so together they may be tiled in $f(k+1)^{n-i}$ ways. Therefore, there are $T_f(i-1,k-i)f(i-k-1)f(k+1)^{n-i}$ tilings whose last horizontal step occurs at step i. As the position of the last horizontal step may range from k to n, $T_f(n,k)$ is the sum of this value for $i = k, \ldots, n$, as desired.

The proof of (3.2) is identical, except that we consider the location of the last vertical step of the lattice path. □

Another theorem that holds for the tiling coefficients is this version of Pascal's identity:

Theorem 3.3 (Pascal's Identity)**.**

$$T_f(n,k) = f(k+1)T_f(n-1,k) + f(n-k-1)T(n-1,k-1) \tag{3.3}$$

Proof. We condition on the last step of the lattice path. If the last step is vertical, then the horizontal strip to its left may be tiled in $f(k+1)$ ways, and there are $T_f(n-1,k)$ ways to tile the remainder of the board. If the last step is horizontal, then the vertical strip below it may be tiled in $f(n-k-1)$ ways, and there are $T_f(n-1,k-1)$ ways to tile the remainder of the board. □

Pascal's identity is a powerful theorem because its simple recursive nature is conducive to the use of induction. Below, we use induction to prove three identities for the tiling coefficients for different tiling functions.

Theorem 3.4. *The q-numbers, or the q-analogs of the integers, are defined by* $[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$ *for indeterminate variable q. Let* $T_q(n,k)$ *be the number of up-down tilings, counted by* $T_f(n,k)$*, with* $f(i) = [i]_q$*. Then*

$$T_q(n,k) = \frac{1}{(-q)^k}\binom{n}{k}_q \tag{3.4}$$

Proof. The proof is by induction. Clearly the statement holds for $n = k = 0$ (both sides equal 1) and for $k < 0$ or $k > n$ (both sides equal 0). Now suppose that the statement holds for $T_q(n-1,k)$ and $T_q(n-1,k-1)$. By Pascal's identity,

$$T_q(n,k) = [k+1]_q T(n-1,k) + [n-k-1]_q T(n-1,k-1)$$

Substituting according to the inductive hypothesis,

$$\begin{aligned}
T_q(n,k) &= \frac{1-q^{k+1}}{1-q}\frac{1}{(-q)^k}\binom{n-1}{k}_q + \frac{1-q^{n-k-1}}{1-q}\frac{1}{(-q)^{k-1}}\binom{n-1}{k-1}_q \\
&= \frac{1}{(1-q)(-q)^k}\left[(1-q^{k+1})\binom{n-1}{k}_q - q(1-q^{n-k-1})\binom{n-1}{k-1}_q\right] \\
&= \frac{1}{(1-q)(-q)^k}\Bigg[\left(\binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q\right) - \\
&\qquad q\left(q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q\right)\Bigg]
\end{aligned}$$

Now, the identities $\binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \binom{n}{k}_q$ **[?]** let us reach the desired conclusion:

$$T_q(n,k) = \frac{1}{(1-q)(-q)^k}\left[\binom{n}{k}_q - q\binom{n}{k}_q\right]$$
$$= \frac{1}{(-q)^k}\binom{n}{k}_q.$$

□

Corollary 3.1. *If $f(i) = i$, then*

$$T_f(n,k) = (-1)^k\binom{n}{k} \tag{3.5}$$

In particular, $T_f(4,2) = \binom{4}{2} = 6$, as shown in the table above.

Proof. When $q = 1$, then $[n]_q = n$ which means $\binom{n}{k}_q = \binom{n}{k}$. Then

$$T_q(n,k) = \frac{1}{(-1)^k}\binom{n}{k} = (-1)^k\binom{n}{k}.$$

□

Theorem 3.5. *Let $\{L_n\}$ be the Lucas numbers, defined by $L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Then*

$$T_L(n,k) = (-1)^k\begin{bmatrix} n \\ k \end{bmatrix} \tag{3.6}$$

Proof. Again, the proof is by induction, but first we prove a lemma.

Lemma 3.1.

$$L_{k+1}F_{n-k} + F_kL_{n-k-1} = F_n \tag{3.7}$$

Proof. The following two identities may be found in [Weisstein (2000)]:

$$F_{-n} = (-1)^{n+1}F_n \tag{3.8}$$

$$F_mL_n = F_{m+n} + (-1)^nF_{m-n} \tag{3.9}$$

Then,

$$\begin{aligned}
L_{k+1}F_{n-k} - F_k L_{n-k-1} &= \Big(F_{n+1} + (-1)^{k+1}F_{n-2k-1}\Big) - \\
&\qquad \Big(F_{n-1} + (-1)^{n-k-1}F_{-(n-2k-1)}\Big)\Big) \\
&= \Big(F_{n+1} - F_{n-1}\Big) + \\
&\qquad \Big((-1)^{k+1}F_{n-2k-1} - (-1)^{n-k-1}(-1)^{n-2k}F_{n-2k-1}\Big) \\
&= F_n + ((-1)^{k+1} - (-1)^{2n-3k-1})F_{n-2k-1} \\
&= F_n
\end{aligned}$$

In the last step, we have used the fact that $k + 1 \equiv 2n - 3k - 1 \pmod 2$. □

Now we may prove Theorem (3.5). The base cases for the induction are the same as in Theorem (3.4). Using Pascal's identity and applying the inductive hypothesis,

$$\begin{aligned}
T_L(n,k) &= L_{k+1}T_L(n-1,k) + L_{n-k-1}T_L(n-1,k-1) \\
&= (-1)^k L_{k+1}\begin{bmatrix} n-1 \\ k \end{bmatrix} + (-1)^{k-1}L_{n-k-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\
&= (-1)^k \left(L_{k+1}\frac{F_{n-k}}{F_n}\begin{bmatrix} n \\ k \end{bmatrix} - L_{n-k-1}\frac{F_k}{F_n}\begin{bmatrix} n \\ k \end{bmatrix}\right) \\
&= (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}\left(\frac{L_{k+1}F_{n-k} - L_{n-k-1}F_k}{F_n}\right) \\
&= (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \qquad \text{(Lemma 3.1)}
\end{aligned}$$

□

Chapter 4

Extending the Tiling Numbers Further

The fact that the definition of the tiling coefficients includes incrementing or decrementing the argument of f by 1 is due to their origins in the Fibonomial tilings, in particular the fact that the Fibonacci numbers are off by one — there are F_{i+1} ways to tile a board of length i. This is a somewhat arbitrary decision. Instead, we could insist that there be $f(h+d)$ ways to tile a horizontal strip of length h and $f(v-d)$ ways to tile a vertical strip of length v for some fixed integer d.[1] Let $T_f^d(n,k)$ denote the tiling numbers with this modification; i.e., $T_f^d(n,k)$ is the number of ways to choose a lattice path in a $k \times (n-k)$ grid, tile (in the abstract sense) each resulting horizontal strip such that a strip of length h may be tiled in $f(h+d)$ ways, and tile each vertical strip such that a strip of length v may be tiled in $f(v-d)$ ways. Thus far, we have only examined $T_f^1(n,k)$.

In [Bruce E. Sagan (2009)], Sagan and Savage show that

$$T_L^0(n,k) = 2^n \begin{bmatrix} n \\ k \end{bmatrix} \tag{4.1}$$

Recall Theorem 3.5, which states that

$$T_L^1(n,k) = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \tag{4.2}$$

[1]To do so, the domain of f must contain $\{-d, -d+1, \cdots, -1, 0, 1, 2, 3, \ldots\}$. If f is an arithmetic sequence, then its domain is already all of $\mathbb{Z}$, as stated in Footnote 2 of Section 3.1.

and Theorem 3.4, which states that

$$T_q^1(n,k) = \left(-\frac{1}{q}\right)^k \binom{n}{k}_q \tag{4.3}$$

and of course, the definition of the Fibonomial coefficients themselves:

$$T_F^1(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} \tag{4.4}$$

These are special cases of the following three theorems.

Theorem 4.1.

$$T_F^d(n,k) = (-1)^{k(d+1)} F_d^n \begin{bmatrix} n \\ k \end{bmatrix} \tag{4.5}$$

Proof. By an identical argument as in the proof of Theorem 3.3, Pascal's Identity for T_f^d is the following:

$$T_f^d(n,k) = f(k+d)T_f^d(n-1,k) + f(n-k-d)T_f^d(n-1,k-1) \tag{4.6}$$

Now we use induction to prove (4.1). Fix $d \in \mathbb{N}$. If $n = k = 0$ then both sides equal 1. If $n > 0$ and $k = 0$, then the only lattice path is the straight vertical line from $(0,0)$ to $(0,n)$, which leads to F_d^n tilings. Since $(-1)^{k(d+1)} F_d^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = F_d^n$ when $k = 0$, the base cases hold.

Now, assume the statement holds up to $T_F^d(n-1,k)$. Using Pascal's Identity and then the inductive hypothesis,

$$\begin{aligned} T_F^d(n,k) &= F_{k+d}T_F^d(n-1,k) + F_{n-k-d}T_F^d(n-1,k-1) \\ &= F_{k+d}(-1)^{k(d+1)}F_d^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix} + F_{n-k-d}(-1)^{(k-1)(d+1)}F_d^{n-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ &= (-1)^{k(d+1)}F_d^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}\left(F_{k+d}\frac{F_{n-k}}{F_n} + (-1)^{d+1}F_{n-k-d}\frac{F_k}{F_n}\right) \\ &= (-1)^{k(d+1)}F_d^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}\left(\frac{F_{k+d}F_{n-k} + (-1)^{d+1}F_kF_{n-k-d}}{F_n}\right) \end{aligned}$$

[Weisstein (2000)] has the following theorem:

$$F_mF_n = \tfrac{1}{5}(L_{m+n} - (-1)^n L_{m-n}) \tag{4.7}$$

From which we obtain

$$F_{k+d}F_{n-k} = \frac{1}{5}(L_{n+d} - (-1)^{n-k}L_{2k+d-n})$$

and

$$\begin{aligned}(-1)^{d+1}F_kF_{n-k-d} &= \tfrac{1}{5}(-1)^{d+1}(L_{n-d} - (-1)^{n-k-d}L_{2k+d-n}) \\ &= \tfrac{1}{5}(-(-1)^{d}L_{n-d} + (-1)^{n-k}L_{2k+d-n})\end{aligned}$$

Therefore,

$$\begin{aligned}F_{k+d}F_{n-k} + (-1)^{d+1}F_kF_{n-k-d} &= \tfrac{1}{5}(L_{n+d} - (-1)^{n-k}L_{2k+d-n}) + \\ &\quad \tfrac{1}{5}(-(-1)^{d}L_{n-d} + (-1)^{n-k}L_{2k+d-n}) \\ &= \tfrac{1}{5}(L_{n+d} - (-1)^{d}L_{n-d}) \\ &= F_nF_d\end{aligned}$$

And thus

$$\begin{aligned}&(-1)^{k(d+1)}F_d^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}\left(\frac{F_{k+d}F_{n-k} + (-1)^{d+1}F_kF_{n-k-d}}{F_n}\right) \\ &= (-1)^{k(d+1)}F_d^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}\left(\frac{F_nF_d}{F_n}\right) \\ &= (-1)^{k(d+1)}F_d^{n}\begin{bmatrix} n \\ k \end{bmatrix}\end{aligned}$$

as desired. □

Theorem 4.2.

$$T_L^d(n,k) = (-1)^{kd}L_d^n\begin{bmatrix} n \\ k \end{bmatrix} \tag{4.8}$$

Proof. Again, the proof is by induction. Fix $d \in \mathbb{N}$. If $n = k = 0$ then both sides equal 1. If $n > 0$ and $k = 0$, then the only lattice path is the straight vertical line from $(0,0)$ to $(0,n)$, which leads to L_d^n tilings. Since $(-1)^{kd}L_d^n\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = L_d^n$ when $k = 0$, the base cases hold.

Now assume the statement holds up to $T_L^d(n-1,k)$. Using Pascal's identity and then the inductive hypothesis,

$$\begin{aligned} T_L^d(n,k) &= L_{k+d}T_L^d(n-1,k) + L_{n-k-d}T_L^d(n-1,k-1) \\ &= L_{k+d}(-1)^{kd}L_d^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix} + L_{n-k-d}(-1)^{(k-1)d}L_d^{n-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ &= (-1)^{kd}L_d^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}\left(L_{k+d}\frac{F_{n-k}}{F_n} + (-1)^d L_{n-k-d}\frac{F_k}{F_n}\right) \\ &= (-1)^{kd}L_d^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}\left(\frac{F_{n-k}L_{k+d} + (-1)^d F_k L_{n-k-d}}{F_n}\right) \end{aligned}$$

Now, [Weisstein (2000)] has the following theorem:

$$F_m L_n = F_{m+n} + (-1)^n F_{m-n} \tag{4.9}$$

Therefore,

$$F_{n-k}L_{k+d} = F_{n+d} + (-1)^{k+d}F_{n-2k-d}$$

and

$$\begin{aligned} (-1)^d F_k L_{n-k-d} &= (-1)^d F_{n-d} + (-1)^d(-1)^{n-k-d}F_{-(n-2k-d)} \\ &= (-1)^d F_{n-d} + (-1)^{n-k}(-1)^{n-2k-d+1}F_{n-2k-d} \\ &= (-1)^d F_{n-d} - (-1)^{k+d}F_{n-2k-d} \end{aligned}$$

Hence

$$\begin{aligned} F_{n-k}L_{k+d} + (-1)^d F_k L_{n-k-d} &= \left(F_{n+d} + (-1)^{k+d}F_{n-2k-d}\right) + \\ &\quad \left((-1)^d F_{n-d} - (-1)^{k+d}F_{n-2k-d}\right) \\ &= F_{n+d} + (-1)^d F_{n-d} \\ &= F_n L_d \end{aligned}$$

Therefore,

$$\begin{aligned} (-1)^{kd}L_d^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}\left(\frac{F_{n-k}L_{k+d} + (-1)^d F_k L_{n-k-d}}{F_n}\right) &= (-1)^{kd}L_d^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}\left(\frac{F_n L_d}{F_n}\right) \\ &= (-1)^{kd}L_d^{n}\begin{bmatrix} n \\ k \end{bmatrix} \end{aligned}$$

as desired. □

Theorem 4.3.

$$T_q^d(n,k) = \left(-\frac{1}{q^d}\right)^k [d]_q^n \binom{n}{k}_q \tag{4.10}$$

Proof. The proof is by induction. Fix $d \in \mathbb{N}$. If $n = k = 0$ then both sides equal 1. If $n > 0$ and $k = 0$, then the only lattice path is the straight vertical line from $(0,0)$ to $(0,n)$, which leads to $[d]_q^n$ tilings. Since $\left(-\frac{1}{q^d}\right)^k [d]_q^n \binom{n}{k}_q = [d]_q^n$ when $k = 0$, the base case holds.

Now assume the statement holds up to $T_q^d(n-1,k)$. Using Pascal's identity and the inductive hypothesis,

$$\begin{aligned}
T_q^d(n,k) &= [k+d]_q T_q^d(n-1,k) + [n-k-d]_q T_q^d(n-1,k-1) \\
&= [k+d]_q \left(-\frac{1}{q^d}\right)^k [d]_q^{n-1} \binom{n-1}{k}_q + \\
&\qquad [n-k-d]_q \left(-\frac{1}{q^d}\right)^{k-1} [d]_q^{n-1} \binom{n-1}{k-1}_q \\
&= \left(-\frac{1}{q^d}\right)^k [d]_q^{n-1} \left([k+d]_q \binom{n-1}{k}_q - q^d [n-k-d]_q \binom{n-1}{k-1}_q \right) \\
&= \left(-\frac{1}{q^d}\right)^k [d]_q^{n-1} \binom{n}{k}_q \left([k+d]_q \frac{[n-k]_q}{[n]_q} - q^d [n-k-d]_q \frac{[k]_q}{[n]_q} \right)
\end{aligned}$$

Now,

$$\begin{aligned}
&[k+d]_q [n-k]_q - q^d [n-k-d]_q [k]_q \\
&= \frac{(1-q^{k+d})(1-q^{n-k})}{(1-q)^2} - q^d \frac{(1-q^{n-k-d})(1-q^k)}{(1-q)^2} \\
&= \frac{1 - q^{k+d} - q^{n-k} + q^{n+d}}{(1-q)^2} - \frac{q^d - q^{n-k} - q^{k+d} + q^n}{(1-q)^2} \\
&= \frac{1 - q^d - q^n + q^{n+d}}{(1-q)^2} \\
&= \frac{(1-q^d)(1-q^n)}{(1-q)^2} \\
&= [d]_q [n]_q
\end{aligned}$$

Therefore,

$$\begin{aligned} T_q^d(n,k) &= \left(-\frac{1}{q^d}\right)^k [d]_q^{n-1} \binom{n}{k}_q \left(\frac{[d]_q[n]_q}{[n]_q}\right) \\ &= \left(-\frac{1}{q^d}\right)^k [d]_q^{n} \binom{n}{k}_q \end{aligned}$$

as desired. □

Appendix A

Combinatorial Identities Via the Manipulation of Generating Functions

This chapter contains a few theorems proved during this research, but unrelated to the Fibonomial coefficients. This chapter is about how one may obtain combinatorial identities by manipulating the variables in generating functions. Comparing $[x^n]$, the coefficient of x^n, on both sides of some manipulated equation can yield an identity.

Theorem A.1.

$$F_n = \sum_{k=0}^{n} \binom{k}{n-k} \tag{A.1}$$

Proof. Let $f(x) = \frac{x}{1-x-x^2}$ be the generating function for the Fibonacci numbers. Then

$$\begin{aligned} f(x) &= \frac{x}{1-(x+x^2)} \\ &= x\sum_{k\geq 0}(x+x^2)^k \\ &= x\sum_{k\geq 0} x^k \sum_{j=0}^{k} \binom{k}{j} x^j \end{aligned}$$

Comparing coefficients on the LHS and RHS, we find that $[x^{n+1}]f(x) = F_{n+1}$, and also

$$\begin{aligned}[x^{n+1}]f(x) &= [x^{n+1}]x\sum_{k\geq 0}x^k\sum_{j=0}^{k}\binom{k}{j}x^j\\ &= \sum_{k=0}^{n}[x^{n-k}]\sum_{j=0}^{k}\binom{k}{j}x^j\\ &= \sum_{k=0}^{n}\binom{k}{n-k}\end{aligned}$$

Hence $F_n = \sum_{k=0}^{n}\binom{k}{n-k}$. □

Instead of simply regrouping the x variables, we can replace the original indeterminate variable of a generating function with a new indeterminate variable. This technique gives us the following theorems.

Theorem A.2.

$$\sum_{k=1}^{n} C_k \left(\!\!\binom{2k}{n-k}\!\!\right)(-1)^{n-k} = \begin{cases}1 & \textit{if } n=1\\ 0 & \textit{if } n\geq 2\end{cases} \tag{A.2}$$

Proof. Let $c(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{2}{1+\sqrt{1-4x}}$ be the generating function for the Catalan numbers[1]. If we let $y = \frac{1-\sqrt{1-4x}}{1+\sqrt{1-4x}}$, then $y = c(x) - 1$ and $x = \frac{y}{(1+y)^2}$. Substituting, we have

$$y = c\left(\frac{y}{(1+y)^2}\right) - 1 = \sum_{k\geq 1} C_k\left(\frac{y}{(1+y)^2}\right)^k$$

Now we have an identity solely in terms of y, which we may use similarly to the previous example. The coefficient of y^n on the LHS of the equation

[1]The Catalan numbers, denoted by C_n, count the the number of lattice paths in an $n \times n$ grid that do not cross (but perhaps touch) the line $y = x$. These are also known as Dyck paths.

is simply 1 if $n = 1$ and 0 otherwise. On the RHS,

$$\begin{aligned}
[y^n]\sum_{k\geq 1} C_k\left(\frac{y}{(1+y)^2}\right)^k &= \sum_{k\geq 1} C_k[y^{n-k}]\frac{1}{(1+y)^{2k}} \\
&= \sum_{k\geq 1} C_k[y^{n-k}]\sum_{j\geq 0}\left(\!\!\binom{2k}{j}\!\!\right)(-y)^j \\
&= \sum_{k=1}^{n} C_k\left(\!\!\binom{2k}{n-k}\!\!\right)(-1)^{n-k}
\end{aligned}$$

Thus we have

$$\sum_{k=1}^{n} C_k\left(\!\!\binom{2k}{n-k}\!\!\right)(-1)^{n-k} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

□

Theorem A.3.

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{k}{2k-n}(-1)^{n-k} = 2^n \tag{A.3}$$

Proof. Let $n(x) = \frac{1}{\sqrt{1-4x}}$ be the generating function for the central binomial coefficients $\binom{2n}{n}$. Let $y = 1 - \sqrt{1-4x}$. Then $x = \frac{y(2-y)}{4}$ and $n(x) = \frac{1}{1-y}$. Therefore,

$$\begin{aligned}
\frac{1}{1-y} &= n(x) \\
&= \sum_{k\geq 0}\binom{2k}{k}x^k \\
&= \sum_{k\geq 0}\binom{2k}{k}\left(\frac{y(2-y)}{4}\right)^k
\end{aligned}$$

Again we compare the coefficients of y^n on both sides. On the LHS it is clearly 1. On the RHS, we have

$$\begin{aligned}[y^n]\sum_{k\geq 0}\binom{2k}{k}\left(\frac{y(2-y)}{4}\right)^k &= \sum_{k\geq 0}\binom{2k}{k}\frac{1}{4^k}[y^n]y^k(2-y)^k \\ &= \sum_{k\geq 0}\binom{2k}{k}\frac{1}{2^{2k}}[y^{n-k}]\sum_{j=0}^{k}\binom{k}{j}2^j(-y)^{k-j} \\ &= \sum_{k\geq 0}\binom{2k}{k}\frac{1}{2^{2k}}\binom{k}{2k-n}2^{2k-n}(-1)^{n-k} \\ &= 2^{-n}\sum_{k\geq 0}\binom{2k}{k}\binom{k}{2k-n}(-1)^{n-k}\end{aligned}$$

When $k > n$, then $2k-n > k$ and hence $\binom{k}{2k-n} = 0$, so we may end the summation at $k = n$. Multiplying both sides by 2^n, we obtain

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{k}{2k-n}(-1)^{n-k} = 2^n$$

□

Now, we will use the same two generating functions in conjunction to prove two more theorems.

Theorem A.4. *Let $C_n^{(m)}$ denote the number of ways to draw m distinctly colored (i.e., distinguishable) Dyck paths with $2n$ steps in total (so that $C_n^{(1)} = C_n$). Then*

$$\sum_{k=0}^{n}(-1)^k F_{k+1} C_{n-k}^{(k)} = (-1)^n \tag{A.4}$$

Proof. Let $g(x) = \frac{1}{1-x-x^2}$ be the generating function for the tiling numbers $\{f_n\}$ (defined by $f_n = F_{n+1}$, as there are F_{n+1} ways to tile a board of length n with squares and dominoes) and let $c(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{2}{1+\sqrt{1-4x}}$ be the generating function for the Catalan numbers. Let $y = -x - x^2$. Then $x^2 + x + y = 0$, so $x = \frac{-1\pm\sqrt{1-4y}}{2}$. First, we consider the positive square root $x = \frac{-1+\sqrt{1-4y}}{2} = -y \cdot c(y)$. Now we have

$$\begin{aligned}g(-y\cdot c(y)) &= \frac{1}{1+y} \\ \sum_{k\geq 0} f_k \cdot (-y\cdot c(y))^k &= \sum_{k\geq 0}(-y)^k\end{aligned}$$

On the RHS the coefficient of y^n is clearly $(-1)^n$. On the LHS,

$$[y^n]\sum_{k\geq 0} f_k \cdot (-y \cdot c(y))^k = \sum_{k=0}^{n} (-1)^k f_k \cdot [y^{n-k}](c(y))^k$$

Note that $(c(y))^k$ is the generating function for the number of ways to draw k distinctly colored Dyck paths, indexed by total length, and hence $[y^{n-k}](c(y))^k$ is the number of ways to draw k distinctly colored Dyck paths with total length $2(n-k)$. By definition, this is $C_{n-k}^{(k)}$. Therefore,

$$\sum_{k=0}^{n} (-1)^k F_{k+1} C_{n-k}^{(k)} = (-1)^n$$

□

Theorem A.5.

$$\sum_{k=0}^{n} F_{k+1} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} C_{n-j}^{(j)} = (-1)^n \tag{A.5}$$

Proof. We use the same $g(x)$ and $c(x)$ as above, but now we take the negative root $x = \frac{-1-\sqrt{1-4y}}{2} = y \cdot c(y) - 1$. We use the same procedure as above.

$$g(x) = \sum_{k\geq 0} f_k \cdot (y \cdot c(y) - 1)^k = \frac{1}{1+y}$$

Again, the coefficient of y^n on the RHS is $(-1)^n$. For the middle expression,

$$\begin{aligned}[y^n]\sum_{k\geq 0} f_k \cdot (y \cdot c(y) - 1)^k &= \sum_{k\geq 0} f_k \cdot [y^n] \sum_{j=0}^{k} \binom{k}{j} (y \cdot c(y) - 1)^j (-1)^{k-j} \\ &= \sum_{k=0}^{n} f_k \cdot \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} [y^{n-j}](c(y))^j \\ &= \sum_{k=0}^{n} f_k \cdot \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} C_{n-j}^{(j)}\end{aligned}$$

Hence

$$\sum_{k=0}^{n} F_{k+1} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} C_{n-j}^{(j)} = (-1)^n.$$

This identity is particularly nice, as it combines the Fibonacci numbers, the binomial coefficients, *and* the Catalan numbers. □

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