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Interval Graphs

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Abstract

We examine the problem of counting interval graphs. We answer the question posed by Hanlon, of whether the generating function of i_n, the number of interval graphs on n vertices, has a positive radius of convergence. We have found that it is zero. We have also found that the exponential generating function of i_n has a radius of convergence greater than or equal to one half. We have obtained a lower bound and an upper bound on i_n. We also study the application of interval graphs to the dynamic storage allocation problem. Dynamic storage allocation has been shown to be NP-complete by Stockmeyer. Coloring interval graphs on-line has applications to dynamic storage allocation. The most colors used by Kierstead's algorithm is $3\omega - 2$, where ω is the size of the largest clique in the graph. We determine a lower bound on the colors used. One such lower bound is $2\omega - 1$.

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Chapter 1

Introduction

1.1 Definition

An undirected graph G is an interval graph if there is a one-to-one correspondence between the vertices of G and a set of intervals of real numbers such that the intervals corresponding to x and y overlap if and only if $x \sim y$.

Hajos (1957) defined interval graphs as a study of intersection graphs. Benzer (1959) independently defined interval graphs.

Interval graphs are precisely the graphs that have no induced cycle of size larger than 3 and that have no asteroidal induced subgraph. (Lekkerkerker and Boland, 1962)

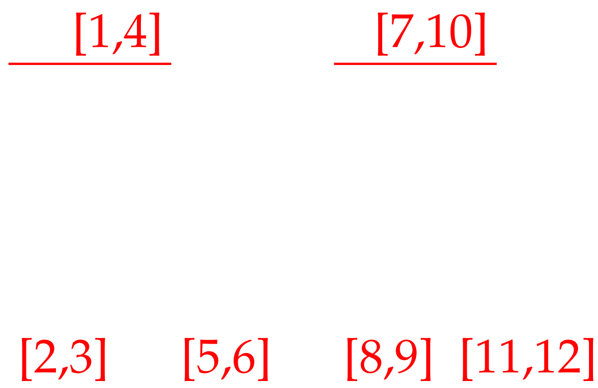

Figure 1.1 Interval graph with $n = 6$

1.2 Two problems

In the field of graph theory the question of counting interval graphs is an interesting combinatorial topic. Hanlon describes the generating function of i_n, the number of interval graphs on n vertices, as an implicit function (Hanlon, 1982). Hanlon posed the question of whether the generating function

$$I(x) = \sum_{n \geq 1} i_n x^n$$

of the interval graphs has a positive radius of convergence; that is, whether $i_n \leq C^n$ for some constant C. We shall show that the radius of convergence is zero.

We also study the application of interval graphs to the dynamic storage allocation problem. Dynamic storage allocation has been shown to be NP-complete by Stockmeyer. Coloring interval graphs on-line has applications to dynamic storage allocation. Kierstead (1991) presented a polynomial-time algorithm for on-line coloring of interval graphs. The most colors used by Kierstead's algorithm is $3\omega - 2$, where ω is the size of the largest clique in the graph. We shall show that in some cases Kierstead's algorithm uses at least $2\omega - 1$ colors.

Chapter 2

Counting Interval Graphs

2.1 Background

Define the generating function

$$I(x) = \sum_{n\geq 1} i_n x^n.$$

Its first 11 coefficients are

n	1	2	3	4	5	6	7	8	9	10	11
i_n	1	2	4	10	92	369	1,807	10, 344	67, 659	491,347	3, 894,446

(Hanlon, 1982)

2.2 Results

Theorem 1. *$i_n \leq (2n-1)!!$*

Proof. Let the endpoints of the interval representation of a graph be the integers $1, 2 \ldots, 2n$. Because intervals cannot share the same endpoints,there are $2n-1$, then $2n-3$, then $2n-5$, and so on, choices for the next interval's endpoints. Thus i_n satisfies $i_n \leq (2n-1)!!$ where $(2n-1)!!$ denotes the product $(2n-1)(2n-3)(2n-5)\ldots(3)(1)$. □

Theorem 2. *$i_n \geq \frac{(\frac{n}{3})!}{3^n}$ for $n = 3m$, where $m \geq 1$ is an integer.*

Proof. To obtain the lower bound, we counted colored graphs of the following type. For every value of m, we can construct the graph that is associated with a permutation of $[1, 2, \ldots m]$, and that has $n = 3m$ intervals total. Let $n = 3m$, where $m \geq 1$. Let π be a permutation of $[1, 2, \ldots m]$For each $0 \leq i \leq m$, consider the intervals

$$r_i = (3i - 1, 3i),$$

$$b_i = (2n - 3i + 1, 2n - 3i + 2),$$

and

$$w_i = (3i - 2, 2n + 3 - 3\pi(i))$$

Construct a graph G_π from the w_i, r_i, and b_i for $i \leq m$, and color the w_i white, the r_i red, and the b_i blue. Then there are $m!$ colored graphs, one for each permutation. There are m red indicator intervals on the left which correspond to the things being permuted, and m blue indicator intervals on the right which are the permutations. There are m white intervals which all overlap with different numbers of blue intervals. We can determine the permutation π from the graph G_π. Call the number of red adjacent intervals the red degree. Call the number of blue adjacent intervals the blue degree. Then $\pi(i)$ is the blue degree of the white interval of red degree i. There are at most 3^n colorings, so the i_n satisfies

$$\frac{(\frac{n}{3})!}{3^n} \leq i_n$$

□

The number of interval graphs on n vertices satisfies $\frac{(\frac{n}{3})!}{3^n} \leq i_n \leq (2n - 1)!!$ This means that the number of interval graphs grows faster than any power of e^n for n large enough. Thus, we have shown that the radius of convergence is zero. Define the exponential generating function

$$J(x) = \sum_{n \geq 1} \frac{i_n x^n}{n!}.$$

Our upper bound shows that $J(x)$ has radius of convergence at least $\frac{1}{2}$. It is an open question whether this radius of convergence is infinite.

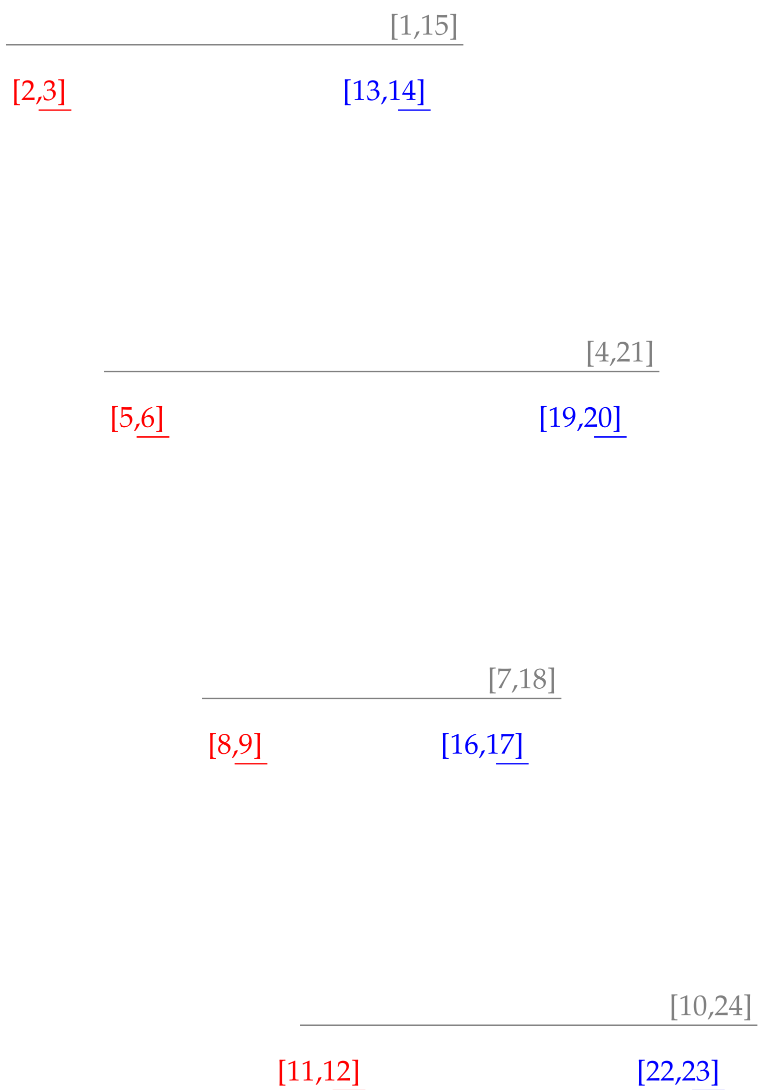

Figure 2.1 Permutation interval graph for $n = 12$

Chapter 3

Coloring Interval Graphs

3.1 Background

We have made some explorations of coloring interval graphs on-line.

In particular, we have found a worst case scenario for the minimum number of colors used by Kierstead's algorithm for on-line coloring.

As a review, the following definitions will be useful. A graph coloring is a way of attaching colors to the vertices of a graph such that no two adjacent vertices share a color. An on-line algorithm is one which does not know its input in advance–an example is a greedy algorithm. The chromatic number χ of a graph is the smallest number of colors needed to color a graph. Because it is used to find the minimum number of colors, vertex coloring has use in network-related problems and resource allocation problems. A perfect graph is a graph for which for every induced subgraph $\chi = \omega$ where ω is the size of the largest clique.

An example of a perfect graph is a bipartite graph on $2n$ vertices. The clique number will be 2. The chromatic number will be 2. The strong perfect graph theorem states that a graph is perfect if and only if neither it nor its complement has an induced cycle of size $2n + 1$ where $n \geq 2$. (Chudnovsky et al., 2006)

Next, we will cover an application of on-line graph coloring in computer science, dynamic storage allocation. On a machine, it is often important to store hundreds of thousands of variables in linear programming, in applications to physics and chemistry, for example. This idea is important in Dynamic Storage Allocation. Stockmeyer formulated dynamic storage allocation and showed that it is NP-complete (Garey and Johnson, 1979).

The question then arises, are there ways to approximate the minimum storage space in Dynamic Storage Allocation? To answer this question we consider on-line coloring of interval graphs.

It can be shown that interval graphs are perfect, by greedy coloring according to order of left end-points.

We have been studying an algorithm by Kierstead. This algorithm is for coloring interval graphs on-line, though the process can possibly label vertices that are adjacent with the same label (Kierstead, 1991). The algorithm by Kierstead uses at most $3\omega - 2$ colors (Kierstead, 1991). In the algorithm by Kierstead, the method used is to conduct a preliminary and a final examination of all of the vertices. Each vertex is examined for overlaps with previous vertices. This algorithm will not yield the clique number in general.

One may wonder, what on-line colorings are produced for various graphs? We consider some examples.

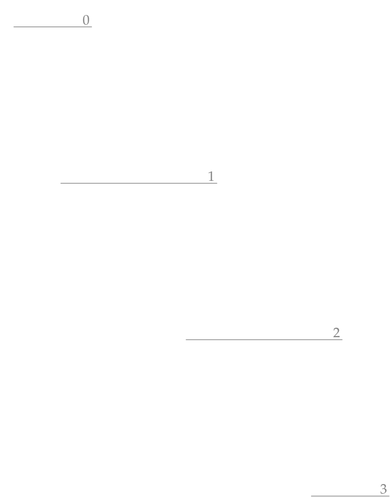

Figure 3.1 Interval graph labeled from left to right

The graph is labeled from left to right and uses Kierstead's algorithm with 2 colors.

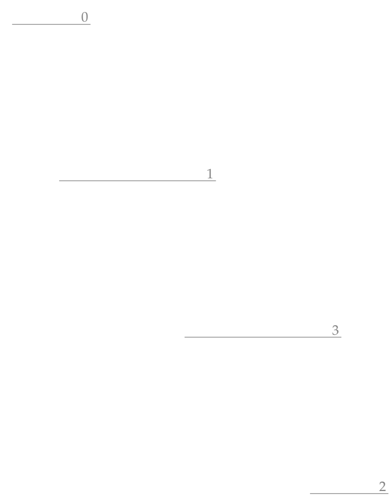

Figure 3.2 Interval graph labeled from outside to inside

The graph is labeled from outside to inside. The graph uses 3 colors.

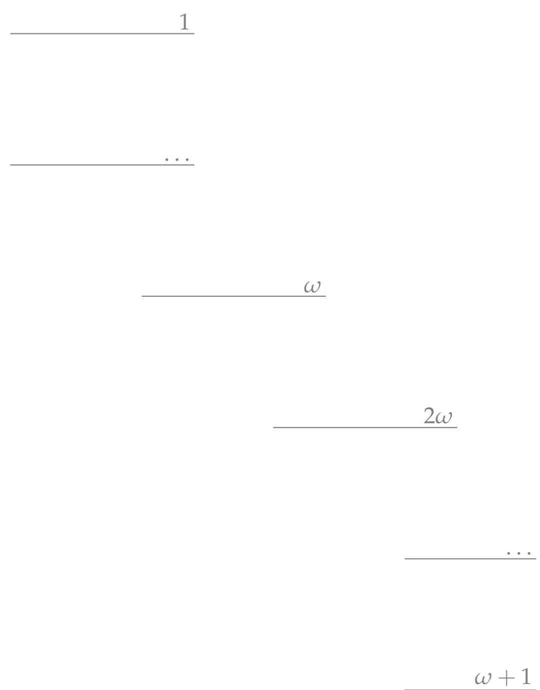

Figure 3.3 Interval graph labeled outside to inside

This graph is labeled with the ωth and 2ωth lines overlapping. Kierstead's algorithm uses $\omega + 1$ colors. Note that the number of colors used is larger than the clique size, ω.

3.2 Results

There is a way of enumerating a graph of clique size ω that uses $2\omega - 1$ colors.

The staircases get progressively smaller until they reach 4 lines, so the total number of colors that Kierstead's algorithm uses is $2\omega - 1$.

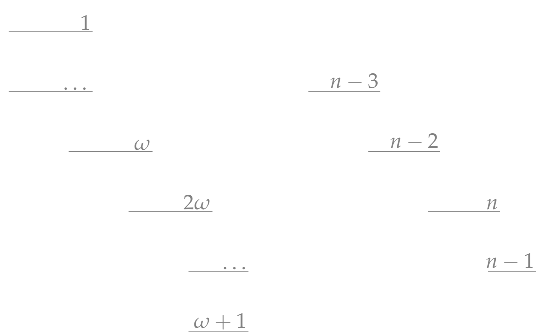

Figure 3.4 Interval graph labeled outside to inside

Chapter 4

Future Work

First, it is an open question whether $(\frac{n}{3})!$ can be improved to $(\frac{n}{2})!$ or $(n)!$

Second, we found that the exponential generating function

$$J(x) = \sum_{n \geq 1} \frac{i_n x^n}{n!}$$

has radius of convergence $\geq \frac{1}{2}$. It is an open question whether this radius of convergence is infinite.

It is an open question whether the bound can be improved from $2\omega - 1$ to $3\omega - 2$.

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