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Graph Cohomology

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Abstract

What is the cohomology of a graph? Cohomology is a topological invariant and encodes such information as genus and euler characteristic. Graphs are combinatorial objects which may not a priori admit a natural and isomorphism invariant cohomology ring. In this project, given any finite graph G, we constructively define a cohomology ring H*(G) of G. Our method uses graph associahedra and toric varieties. Given a graph, there is a canonically associated convex polytope, called the graph associahedron, constructed from G. In turn, a convex polytope uniquely determines a toric variety. We synthesize these results, and describe the cohomology of the associated variety directly in terms of the graph G itself.

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Chapter 1

Introduction

One of the most beautiful aspects of mathematics is the high degree to which different areas of study can impact one another. Mathematical study covers a vast range of topics, each individually rich with information. And when seemingly unrelated objects can be connected, even greater depth becomes accessible. This project is an effort to make such connections, specifically between graphs and toric varieties and their cohomologies.

Chapter 2 will discuss graph associahedra, which is the link from graphs to polytopes. Chapter 3 will be the other end of the chain, looking at toric varieties. Chapter 4 will explore the connections that have been made between the two as well as define our characterization of graph cohomology., and Chapter 5 will discuss the avenues for further investigation.

Chapter 2

Graph Associahedra

A graph associahedron is a polytope that is contructed from a simple graph. Much of the characterization and study of graph associahedra can be found in Carr and Devadoss (2006) and Devadoss (2009). This includes discussion about the properties of graph associahedra and their structural connections to graphs, but the key pieces of information for this project are the two methods of constructing graph associahedra which will be detailed in the following sections.

2.1 Tubes

Before going into the constructions, there are some important definitions to establish.

Definition 2.1.1. *For a graph* Γ*, a* tube *is a nonempty set of vertices whose induced subgraph is a proper, connected subgraph of* Γ*.*

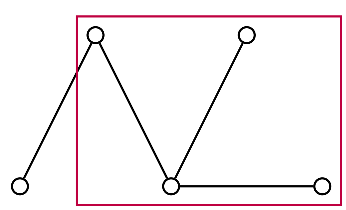
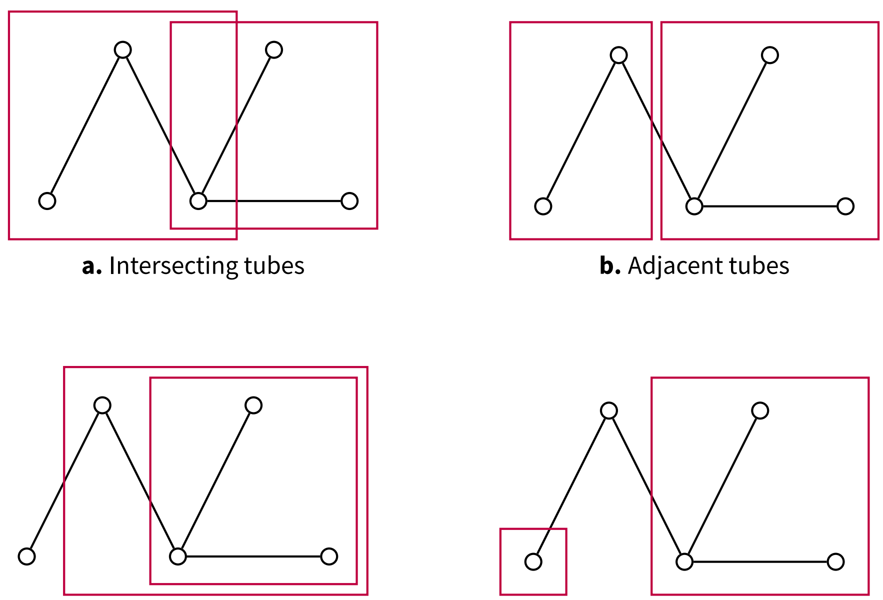


Figure 2.1 This set of vertices is a tube of the graph.

There is an immediate correspondence between tubes and proper subgraphs of Γ, but note that a tube refers strictly to the set of vertices and does not contain any information about the edge set.

Definition 2.1.2. *A pair of distinct tubes t_1 and t_2:*

- *are* nested *if $t_1 \subset t_2$ or $t_2 \subset t_1$.*
- intersect *if $t_1 \cap t_2 \neq \emptyset$ and t_1 and t_2 are not nested.*
- *are* adjacent *if $t_1 \cap t_2 = \emptyset$ and $t_1 \cup t_2$ has a connected induced subgraph in G.*
- *are* compatible *if they do not intersect and are not adjacent.*



a. Intersecting tubes

b. Adjacent tubes

c. Nested tubes (which are also compatible)

d. Compatible (and non-nested) tubes

Figure 2.2 Pairs of tubes.

In other words, intersecting tubes overlap one another whereas adjacent tubes do not overlap but can be traveled between via a single edge. That is, there is an edge which has an endpoint in each of the adjacent tubes. For a pair of compatible tubes, the induced subgraph of their union must either equal the induced subgraph of one of the original tubes ore be a disconnected graph consisting of the induced subgraphs of the original two tubes.

Definition 2.1.3. *A* tubing *is a set of valid tubes such that every pair of tubes is compatible. A k*-tubing *is a tubing consisting of k tubes.*

Notably, any single tube is a valid 1-tubing. In addition, any single vertex is a valid tube, and is therefore also a valid tubing. On the other hand, a tube cannot contain all vertices in the graph because that would not be a proper subset. If a set of tubes contained all vertices in the graph, there must be tubes that intersect or are adjacent or there must be a single tube containing all vertices in the graph. In any of these cases, the set is not a valid tubing.

Definition 2.1.4. *A* maximal tubing *is a tubing for which no other valid tubings can be created by adding tubes.*

Observe that a maximal tubings contains exactly $n-1$ tubes. Any maximal tubing can be created by sequentially picking $n-1$ vertices. If the selected vertex v is not adjacent to any tubes, add the tube containing just v. If the selected tube is adjacent to another previously created tube t, add the tube containing t and v. This creates a tubing of $n-1$ vertices. No more tubes can be added because it would have to contain the final vertex, and a valid tubing cannot contain all vertices in G.

2.2 Construction via Truncation

The first method of construction involves truncating faces of a simplex. Simply put, truncating a face of a polytope entails replacing the given face with facet (a face of codimension 1). For example, truncating a 1-dimensional face of a cube (a corner) replaces that corner with a triangle. This can be visualized as cutting off the corner of the cube leaving a triangle where it used to be.

For a graph Γ on n vertices, the graph associahedron can be constructed as follows:

1. Let Δ_Γ be the $n-1$ simplex.

2. Assign each vertex in Γ to one of the facets (faces of codimension 1) of Δ_Γ. This is possible because the $n-1$ simplex always contains n facets.

3. Starting with 1-dimensional faces of Δ_Γ, for each such face v, examine the set of vertices assigned to the facets containing v. If this set is a valid tube of Γ, truncate v.

4. Repeat the previous step for 2-dimensional faces of Δ_Γ and so on up through faces of dimension $n-2$.

5. The resulting polytope is the graph associahedron for Γ.

Through this construction, the graph associahedron for Γ will necessarily be a simple, convex, $(n-1)$-dimensional polytope. This is simply because the $n-1$ simplex is a simple, convex, $(n-1)$-dimensional polytope, and truncation cannot change any of these properties.

Because every subset of Γ is checked to determine if it forms a valid tube, this process creates a facet for every possible 1-tubing of Γ. Note that 1-tubings consisting of a single vertex are not checked in this construction, but as they are assigned to facets to begin with, they are indeed accounted for. In addition, the faces with codimension 2 will correspond to 2-tubings. A face with codimension 2 will be contained in exactly 2 facets, and as a result of the above construction, those facets will correspond to a pair of compatible tubes. There is then a natural correspondence between faces of dimension 2 and tubings containing the compatible tubes of the pair of facets.

Continuing this line of thinking, it is possible to glean some bits of the relationship between graphs and graph associahedra. The most fundamental connection is that the face poset of Δ_Γ is isomorphic to a poset of the valid tubings of Γ. In particular, the poset of valid tubings of Γ is that in which tubings T and T' are related by $T \prec T'$ if T can be obtained by adding tubes to T'.

Observe that for the graph of n disjoint vertices, that is, the graph on n vertices with no edges, the only valid tubes are those containing a single vertex, as the induced subgraph for any set with multiple vertices will necessarily be disconnected. Hence, no set containing more than 1 vertex is a valid tube, so no truncations will occur. Therefore, the graph associahedron for the graph of n disjoint vertices is the $n-1$ simplex.

2.3 Numerical Construction

Alternatively, Devadoss (2009) provides another construction for graph associahedra by determining a set of points in $\mathbb{R}^n$ and taking the convex hull of that set.

1. Let Γ be a graph on n vertices, and let M_Γ be the set of maximal tubings of Γ.

2. For each vertex v in Γ and maximal tubing $U \in M_\Gamma$, let $t_U(v)$ be the smallest tube in U containing v. If v is not contained in any tubes in U, let $t_U(v)$ be the set of all vertices in G.

3. For each vertex v and maximal tubing U, define f_U as follows. If v is contained in a size 1 tube (meaning that it is in a tube by itself), let $f_U(v) = 0$. For all other vertices, f_U must satisfy $\sum_{v_i \in t(v)} f_U(v_i) = 3^{|t(v)|-2}$.

   - Given the recursive nature of f_U, the values must be found in order. If $t_U(v) = k$, evaluating $f_U(v)$ will require that f_U already be evaluated for some vertices w where $t_U(w) < k$. Thus, the value for all vertices in tubes of size 1 must be assigned to 0 before all else. Then, the value for all other vertices in tubes of size 2 must be determined, and so on until all vertices have been assigned a value.

4. Given any ordering of the vertices of G, $(f_U(v_1), f_U(v_2), \ldots, f_U(v_n))$ is a point in $\mathbb{R}^n$ determined by maximal tubing U. Every maximal tubing can therefore be mapped to a point in $\mathbb{R}^n$ in this way.

5. The convex hull of all such points forms the graph associahedron for G.

This construction creates a particular form of the graph associahedron with specific lengths and angles whereas the previous construction is really only concerned with the face poset of the polytope. Because this construction works by finding real points rather than by truncating faces of a general polytope, it lends itself well to being performed by a computer. Figures 2.3, 2.4, and 2.5 are from a script in SageMath which takes a graph and produces the graph associahedron.

As mentioned, no maximal tubing can contain every vertex in the graph. As such, for any maximal tubing U of graph Γ, there is at least one vertex v which is not contained in any tube in U. Thus, $t_U(v)$ will be all vertices in Γ out of necessity. Then, because of the way f_U is defined,

$$\sum_{v_i \in t(v)} f_U(v_i) = 3^{|t(v)|-2},$$

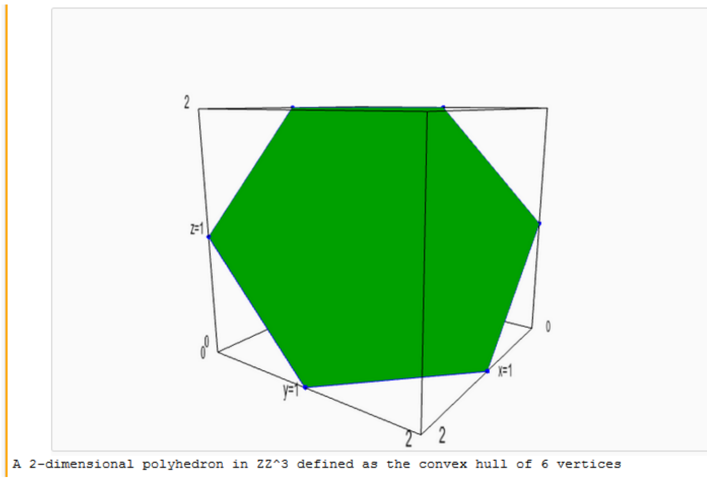


Figure 2.3 The graph associahedron for the complete graph on 3 vertices, K_3, is a hexagon. A hexagon can also be thought of as the 2-dimensional permutohedron, and in general K_n will have the $(n-1)$-dimensional permutohedron as its graph associahedron.

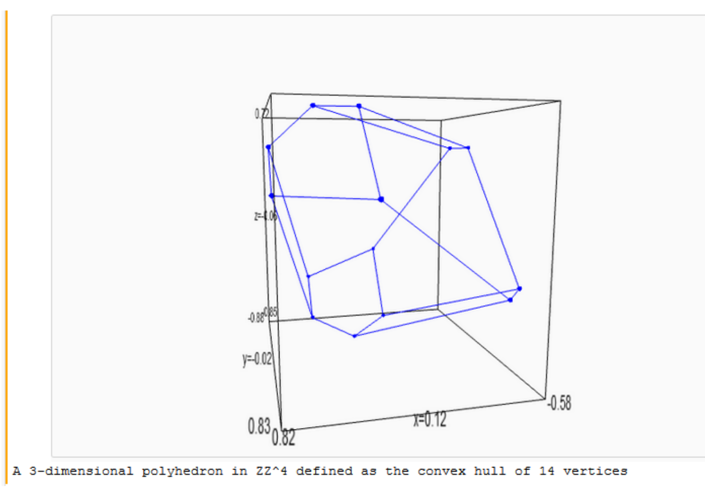


Figure 2.4 The graph associahedron for the path on 4 vertices, P_4, is the 3-dimensional associahedron. In general the graph associahedron for P_n will be the $(n-1)$ dimensional associahedron.

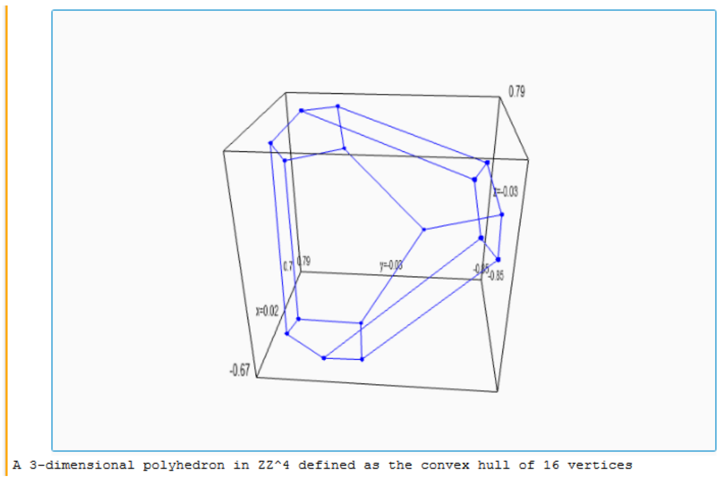


Figure 2.5 The graph associahedron for the star graph on 4 vertices.

and $t(v)$ is the set of all vertices in Γ, so equivalently

$$\sum_{v_i \in \Gamma} f_U(v_i) = 3^{n-2}.$$

These are also the values in the coordinates for the point in $\mathbb{R}^n$ corresponding to maximal tubing U, so the sum of those coordinates will also equal 3^{n-2}. This holds for every maximal tubing, so every point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ found by the described construction will be a point at which

$$x_1 + x_2 + \cdots + x_n = 3^{n-2}.$$

This explains why the graph associahedra for graphs on n vertices are always $n-1$-dimensional polytopes despite the construction selecting points existing in ambient n-dimensional space. The above equation characterizes a hyperplane in $\mathbb{R}^n$; all of the found points must satisfy it and are thus all contained in that hyperplane. The convex hull of this set of points will also be contained within the hyperplane, and will therefore be 1 dimension lower than the ambient space.

Chapter 3

Toric Varieties

This chapter introduces toric varieties and presents the information about them that is relevant to this project. This information can be found in Hartshorne (1977).

3.1 Varieties

First, let me provide some brief descriptions of affine and projective varieties.

Definition 3.1.1. *In general, given a set of polynomials, a* variety *is the set of points in some affine or projective space for which all polynomials in the set evaluate to* 0*.*

Definition 3.1.2. *For an ideal* $I \subseteq S = \mathbb{C}[x_1, \ldots, x_n]$*,* $V(I)$ *is the set of points in* $\mathbb{C}^n$ *that are zero for all polynomials in* I*. On the flipside of that, for an affine variety* $V \subseteq \mathbb{C}^n$*,* $I(V)$ *is the set of all polynomials in* n *variables that evaluate to* 0 *at all points in* V*. For an affine variety* V*,* $\mathbb{C}[V] = S/I(V)$*. A way of thinking of this is that they are* $\mathbb{C}$*-valued polynomials functions on* V*.*

As an illustrative example, consider the following.

Example 3.1.3. *Let* I *be* $(y) \subseteq \mathbb{C}[x, y]$*. That is, it's the ideal generated by* y *on the polynomial ring in variables* x *and* y*. Every polynomial in this ideal is a polynomial multiplied by* y*, so whenever* $y = 0$*, the polynomial will evaluate to* 0*, regardless of what* x *is. On the other hand, if* $y \neq 0$*, then the polynomial* $y \in I$ *will not evaluate to* 0*, so any points where* $y \neq 0$ *do not evaluate to* 0 *for all polynomials in* I*. In other words, the points in* $\mathbb{C}^2$ *such that all polynomials in* I *evaluate to* 0 *are exactly the ones where* $y = 0$, *so any point of the form* $(x, 0)$ *for* $x \in \mathbb{C}$. *Thus,* $V(I) = \{(x, 0) | x \in \mathbb{C}\}$

Let V *be* $\{(x, 0) | x \in \mathbb{C}\} \subseteq \mathbb{C}^2$. *In other words, it is what can be imagined as the x-axis. If a polynomial contains a factor of* y, *then it will necessarily evaluate to* 0 *at all points in* V *because* $y = 0$ *for all points in* V. *On the other hand, if it does not contain a multiplicative factor of* y, *then there is some term that contains no* y, *which is either a constant or a multiple of* x. *Either way, at the point* $(1, 0)$, *such terms will not go to* 0, *so the overall polynomial will not go to* 0. *Hence, a polynomial evaluates to* 0 *exactly if it is a multiple of* y. *This means that* $I(V) = (y)$.

For the same V, $\mathbb{C}[V] = S/I(V) = \mathbb{C}[x, y]/(y)$. *This is all polynomials in variables* x *and* y *modded out by* y. *Equivalently, it is any polynomial where* y *is set to* 0. *This matches up with* $\mathbb{C}$*-valued polynomial functions on* V *because on* V, $y = 0$, *and this leaves only* x *as a variable to create polynomial functions with.*

Projective varieties are similar, but they can only be defined for homogeneous polynomials, or forms, and not for any set of polynomials. This is to account for the fact that in projective space, two points $(x_1 : x_2 : \ldots : x_n)$ and $(x'_1 : x'_2 : \ldots : x'_n)$ are equivalent if and only if there is some scalar k such that $(x_1 : x_2 : \ldots : x_n) = (kx'_1 : kx'_2 : \ldots : kx'_n)$. Nonhomogeneous polynomials do not have well-defined zero sets..

Example 3.1.4. *In* $\mathbb{CP}^2$, $x^2 - z$ *does not have a well-defined zero set.* $(2 : 0 : 4)$ *is in the zero set of* $x^2 - z$ *because* $2^2 - 4 = 0$. *In projective space,* $(2 : 0 : 4)$ *is equivalent to* $\frac{1}{2}(2 : 0 : 4) = (1 : 0 : 2)$. *Because these points are equivalent, if the zero set of* $x^2 - z$ *were well-defined, it would also contain* $(1 : 0 : 2)$. *However,* $1^2 - 2 = -1 \neq 0$.

In $\mathbb{CP}^2$, $x^2 - yz$ *does have a well-defined zero set. For example,* $(2 : 1 : 4)$ *is in the zero set because* $2^2 - (1)(4) = 0$. $(1 : \frac{1}{2} : 2)$ *is also in the set as desired because* $1^2 - \left(\frac{1}{2}\right)(2) = 0$. *In general, if* $(a : b : c)$, *is in the zero set, implying that* $a^2 - bc = 0$, *then* $(ka : kb : kc)$ *will also be in the zero set because* $(ka)^2 - (kb)(kc) = k^2(a^2 - bc) = k^2(0)$.

From this example, the reason that homogeneous polynomials are needed for well-defined zero sets becomes more clear. By making sure that each term is of the same degree, multiplying points in the zero set by scalars will have the same effect on each term ensuring that the point remains in the zero set as it should.

In addition, note that homogeneous polynomials will not be well-defined for all of projective space because while the scaling does not cause any problems when multiplying by 0, if the polynomial evaluates to a nonzero value, scaling will change the value. To be well-defined over all of projective space, the function must be a rational polynomial where the top and bottom are homogeneous and of the same degree.

3.2 Definition of a Toric Variety

The following information about toric varieties is drawn from Fulton (1993).

First, note that through the deformation retraction

$$f(z,t) = \frac{z}{1 + t(|z| - 1)},$$

$\mathbb{C}^*$ can be deformation retracted onto a circle in the complex plane. Thus, as a parallel to the fact that an n-dimensional torus is $(S^1)^n$, an n-dimensional complex torus is $(\mathbb{C}^*)^n$. Toric varieties are varieties which demonstrate a deep connection to complex tori.

Definition 3.2.1. *For a variety V to be* toric, *there must be a torus, $T = (\mathbb{C}^*)^k$ for some k, such that T is an open dense subset (in the Zariski topology) of V. In addition, there must be an action of T on V which behaves like coordinatewise multiplication for points in T.*

Consider the following illustrative example.

Example 3.2.2. *Consider all of $\mathbb{CP}^2$. Note that this is indeed a projective variety, as it is $V(\{0\})$. Let $T = (\mathbb{C}^*)^2$. Consider the map $f : T \to \mathbb{CP}^2$ where $f(t_1, t_2) = (1 : t_1 : t_2)$.*

$\mathbb{CP}^2 \setminus T$ is the set of all points in $\mathbb{CP}^2$ that are not mapped to by f. f always maps to points in $\mathbb{CP}^2$ that have a 1 in the first coordinate, so it is impossible to get any points that are 0 in the first coordinate. In addition, $\mathbb{C}^*$ does not contain 0, so if $(t_1, t_2) \in T$, then both t_1 and t_2 must be nonzero. Thus, any point in $\mathbb{CP}^2$ that has a 0 in any coordinate cannot be mapped to by f. However, for any point $(x : y : z) \in \mathbb{CP}^2$ for nonzero x, y, and z can be mapped to by f as follows.

$$f\left(\frac{y}{x}, \frac{z}{x}\right) = \left(1 : \frac{y}{x} : \frac{z}{x}\right) = x\left(1 : \frac{y}{x} : \frac{z}{x}\right) = (x : y : z)$$

Note that x, y, and z being nonzero implies that $(\frac{y}{x}, \frac{z}{x}) \in T$. Also, this uses the fact that points are equivalent in $\mathbb{CP}^2$ if they are scaled by a constant.

Hence, a point is in $\mathbb{CP}^2 \setminus T$ if and only if it has a zero in at least one coordinate, so $\mathbb{CP}^2 \setminus T = \{(1:0:0), (0:1:0), (0:0:1), (0:y:z), (x:0:z), (x:y:0)\}$ where x, y, z are all nonzero.

Observe that this is a set of 3 *points and* 3 *lines in* $\mathbb{CP}^2$ *each of these is a subvariety of* $\mathbb{CP}^2$*, so in the Zariski topoplogy, they are closed sets. The finite union of these closed sets,* $\mathbb{CP}^2 \setminus T$*, must also be closed. Thus,* T *is an open subset of* $\mathbb{CP}^2$*. Nonempty open sets in the Zariski topology are dense, so* T *is an open dense subset of* $\mathbb{CP}^2$*.*

Let the group action of T *on* $\mathbb{CP}^2$ *be defined as follows. For* $(t_1, t_2) \in T$ *and* $(x : y : z) \in \mathbb{CP}^2$*,*

$$(t_1, t_2) \cdot (x : y : z) = (x : t_1 y : t_2 z).$$

For points on the torus, this behaves like coordinatewise multiplication. For $(t_1, t_2), (t_3, t_4) \in T$*,*

$$(t_1, t_2) \cdot f(t_3, t_4) = (t_1, t_2) \cdot (1 : t_3 : t_4) = (1 : t_1 t_3 : t_2 t_4) = f(t_1 t_3, t_2 t_4).$$

Thus, there is a complex torus that is an open dense subset of $\mathbb{CP}^2$ *and for which a satisfactory group action can be defined, so* $\mathbb{CP}^2$ *is a toric variety.*

3.3 Fans

In addition, the geometric information about a toric variety is encoded in the combinatorial properties of a fan.

Definition 3.3.1. *In a lattice, such as* $\mathbb{Z}^n$*, let* $\{v_1, \ldots, v_k\}$ *be a set of vectors in the lattice. A* cone σ *is the set in* $\mathbb{R}^n$ *of all vectors* $a_1 v_1 + \cdots + a_k v_k$ *such that* $a_1, \ldots, a_k \in \mathbb{R}$ *and all* $a_i \geq 0$*. In addition, to be a valid cone, the intersection of* σ *and* $-\sigma$ *is* $\{0\}$*.*

This can be thought of as starting with a set of vectors and finding everything in the positive span of that set of vectors.

Definition 3.3.2. *A* fan Σ *is a set of cones such that*

- *every face of every cone is itself a cone in the fan.*
- *the intersection of every pair of cones is exactly a face of each of the cones. Note that the intersection of any pair of cones must then be a cone itself as per the above condition.*

With this definition in place, the logical next question is how the fan can be constructed from a toric variety. Having shown in Example 3.2.2 that $\mathbb{CP}^2$ is a toric variety, it's fan can be constructed as follows.

Example 3.3.3. *Begin by considering the lattice* $\mathrm{Hom}(\mathbb{C}^*, T)$. *Because* $T = (\mathbb{C}^*)^2$, $\mathrm{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^2$. *This is because every homomorphism from* $\mathbb{C}^* \to (\mathbb{C}^*)^2$ *maps*

$$t \mapsto (t^a, t^b)$$

where a and b are integers. So every homomorphism can be identified by $(a, b) \in \mathbb{Z}^2$. *In general,* $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n) \cong \mathbb{Z}^n$.

The fan of the toric variety is made of cones which can be found by finding $Z_\psi = \overline{T \cdot \lim_{t\to 0} f(\psi(t))}$, *which I will henceforth refer to as the orbit closure for* ψ, *for all* $\psi \in \mathrm{Hom}(\mathbb{C}^*, T)$ *and where* f *is as defined previously. These* Z_ψ *depend on a and b for* ψ. *For example, if* $a > 0$ *and* $b > 0$, *then* $\lim_{t\to 0} f(\psi(t)) = \lim_{t\to 0}(1, t^a, t^b)$. *Because a and b are positive,* t^a *and* t^b *go to* 0 *as* $t \to 0$. *This yields the point* $(1, 0, 0) \in \mathbb{CP}^2$. *Note that in this case, the orbit closure of* $\{(1, 0, 0)\}$ *is the set itself because the group action of any point in the torus on* $(1, 0, 0)$ *simply results in* $(1, 0, 0)$ *again. This is a T-invariant subvariety of* $\mathbb{CP}^2$, *and in fact all of the orbit closures will also be T-invariant.*

This is the orbit closure for any ψ *where a and b are both positive, so we create an equivalence class of all such* ψ. *The set of all points in* $\mathbb{Z}^2$ *corresponding to this equivalence class is all points* $(a, b) \in \mathbb{Z}^2$ *where a and b are positive which is the first quadrant. The convex hull of all of these points then is still the first quadrant, and this whole first quadrant is then a cone in the fan of* $\mathbb{CP}^2$.

I won't repeat all of the computation here, but the idea is that for each equivalence class in $\mathrm{Hom}(\mathbb{C}^*, T)$, *as characterized by having the same orbit closure, the convex hull of all points (as plotted in* $\mathbb{Z}^2$*) forms a cone in the fan. The cones and orbit closures are as follows.*

$a, b, > 0$	$(1 : 0 : 0)$
$a < 0$ *and* $b > a$	$(0 : 1 : 0)$
$b < 0$ *and* $a > b$	$(0 : 0 : 1)$
$a = 0$ *and* $b > 0$	$(x : y : 0)$
$b = 0$ *and* $a > 0$	$(x : 0 : z)$
$a = b < 0$	$(0 : x : y)$
$a = b = c = 0$	$(x : y : z)$

In the case of $\mathbb{CP}^2$, *there are* 7 *cones*: 3 *of which are* 2*-dimensional,* 3 *of which are* 1*-dimensional, and* 1 *of which is* 0*-dimensional.*

The general construction for fans given a toric variety is as in Example 3.3.3. For every homomorphism in the lattice, find the orbit closure for the homomorphism. The convex hull of all homomorphisms that have the same orbit closure will form a cone, and the fan is the set of all such cones.

As a further illustration, here is the fan of toric variety $\mathbb{CP}^3$.

Example 3.3.4. *To do so, first, identify how $\mathbb{CP}^3$ is a toric variety. Somewhat unsurprisingly, it contains $T = (\mathbb{C}^*)^3$ via the map $f : T \to \mathbb{CP}^2$ where $f(t_1, t_2, t_3) = (1, t_1, t_2, t_3)$. Again similar to with $\mathbb{CP}^2$, the lattice $\mathrm{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^3$, and every $\psi \in \mathrm{Hom}(\mathbb{C}^*, T)$ is of the form $\psi(t) = (t^a, t^b, t^c)$.*

We want to find the orbit closures for the various $\psi \in \mathrm{Hom}(\mathbb{C}^*, T)$, and they can be found to be as follows.

$a, b, c > 0$	$(1 : 0 : 0 : 0)$
$a < 0$ *and* $b, c > a$	$(0 : 1 : 0 : 0)$
$b < 0$ *and* $a, c > b$	$(0 : 0 : 1 : 0)$
$c < 0$ *and* $a, b > c$	$(0 : 0 : 0 : 1)$
$a = 0$ *and* $b, c > 0$	$(x : y : 0 : 0)$
$b = 0$ *and* $a, c > 0$	$(x : 0 : z : 0)$
$c = 0$ *and* $a, b > 0$	$(x : 0 : 0 : w)$
$a = b = 0$ *and* $c > a$	$(0 : y : z : 0)$
$a = c = 0$ *and* $b > a$	$(0 : y : 0 : w)$
$b = c = 0$ *and* $a > b$	$(0 : 0 : z : w)$
$c > 0$ *and* $a = b = 0$	$(x : y : z : 0)$
$b > 0$ *and* $a = c = 0$	$(x : y : 0 : w)$
$a > 0$ *and* $b = c = 0$	$(x : 0 : z : w)$
$a = b = c < 0$	$(0 : x : y : z)$
$a = b = c = 0$	$(x : y : z : w)$

This gives a 1*-skeleton, defined to be the set of all* 1*-dimensional cones, which is the positive half of each axis in $\mathbb{R}^3$ along with the positive multiples of the vector* $(-1, -1, -1)$*. In addition, $\mathbb{CP}^3$ is complete meaning that any proper subset of the* 1*-skeleton will span a cone in the fan of $\mathbb{CP}^3$. This is also true of $\mathbb{CP}^2$ and its* 1*-skeleton of size* 3 *in $\mathbb{R}^2$*

A notable characteristic of the fan of a toric variety is that there is a natural correspondence between fans and polytopes. For the fan of $\mathbb{CP}^2$, taking lines perpendicular to each of the 1-dimensional cones creates the outline of a right triangle, as drawn in Figure 3.1. The 1-dimensional cones are drawn in red, and the blue lines are perpendicular to them.

For $\mathbb{CP}^3$, the associated polyhedron is a tetrahedron which can be found by taking planes perpendicular to each piece of the 1-skeleton. By cutting them off where they intersect, a tetrahedron will be formed which looks like the corner of a cube. This is the analog of the right triangle that is the polygon associated with the fan of $\mathbb{CP}^2$. Another way of seeing that the polyhedron is a tetrahedron is to create a graph of the inclusions of the cones in the fan. That is, create a graph with each vertex being the highest

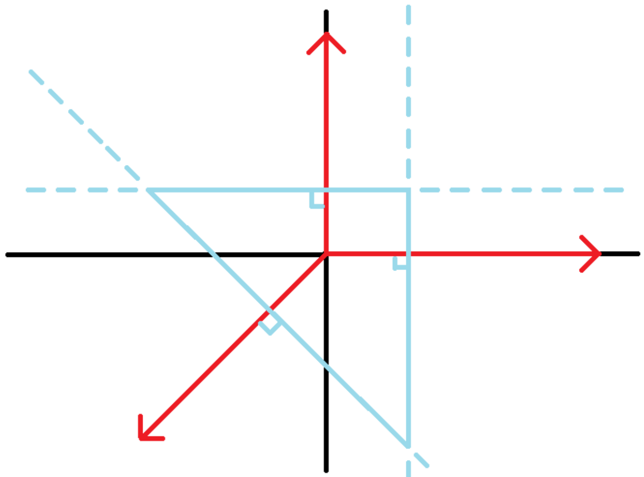


Figure 3.1 The 1-dimensional cones of $\mathbb{CP}^2$ and the corresponding triangle.

dimension cones. Then have the two share an edge if they intersect. This graph will an embedding of the polyhedron, in this case a tetrahedron. This makes sense because of the inclusion reversing nature of the fan and the polyhedron.

Having established the manner of constructing the fan of a toric variety, it is also important consider the reverse direction, how to find the toric variety given a fan. Following the pattern from the previous examples, consider the fan of $\mathbb{CP}^2$.

Example 3.3.5. *What really matters here are the three* 1*-dimensional cones. The primitive generators for these cones are* $(-1,-1)$*,* $(1,0)$*, and* $(0,1)$*. These will respectively be* v_1*,* v_2*, and* v_3*, and to each* 1*-dimensional cone/primitive generator let* x_i *be a coordinate corresponding to it. As mentioned, every proper subset of this set of* 1*-dimensional cones spans a cone of* $\mathbb{CP}^2$*. Thus, the only subset of the* 1*-skeleton of* $\mathbb{CP}^2$ *which does not span a cone is the entire* 1*-skeleton.*

For a set of 1*-dimensional cones* S*,* $V(S)$ *be the linear subspace where* x_i *is set to* 0 *if* v_i *is one of the cones in* S*. For a fan* Σ*, we let* $Z(\Sigma)$ *be the union of all* $V(S)$ *for all* S *that do not span a cone of* Σ*.*

In this case, the only S *that does not span cone of the fan* Σ *is the entire* 1*-skeleton of the fan, so the only* $V(S)$ *in* $Z(\Sigma)$ *is* $(0,0,0)$*. That is,* $Z(\Sigma) = \{(0,0,0)\}$*.*

We define a group G that is the kernel of $\phi : (\mathbb{C}^*)^3 \to (\mathbb{C}^*)^2$ *defined by*

$$\begin{aligned}\phi(t_1, t_2, t_3) &= \left(\prod_{j=1}^{3} t_j^{v_{j1}}, \prod_{j=1}^{3} t_j^{v_{j2}}\right) \\ &= \left(t_1^{v_{11}} t_2^{v_{21}} t_3^{v_{31}}, t_1^{v_{12}} t_2^{v_{22}} t_3^{v_{32}}\right) \\ &= \left(t_1^{-1} t_2^{1} t_3^{0}, t_1^{-1} t_2^{0} t_3^{1}\right) \\ &= \left(t_1^{-1} t_2, t_1^{-1} t_3\right).\end{aligned}$$

The kernel of ϕ *then is any* (t_1, t_2, t_3)*,* $t_1, t_2, t_3 \in \mathbb{C}^*$ *, where* $t_1^{-1}t_2 = 1$ *and* $t_1^{-1}t_3 = 1$*. That is,* $t_1 = t_2$ *and* $t_1 = t_3$*. Thus, the kernel is* $\{(t, t, t) | t \in \mathbb{C}^*\}$*.*

The toric variety from the fan Σ *with n elements in the* 1*-skeleton is defined to be* $(\mathbb{C}^n - Z(\Sigma))/G$ *where* $Z(\Sigma)$ *and G are as defined. Thus, in the case of the fan for* $\mathbb{CP}^2$ *(though technically we are only given the fan and are only now about to show that it is* $\mathbb{CP}^2$*), the toric variety is*

$$(\mathbb{C}^3 - \{(0, 0, 0)\})/\{(t, t, t) | t \in \mathbb{C}^*\}.$$

This is exactly $\mathbb{CP}^2$ *as we would hope. Each coordinate must be in* $\mathbb{C}$*, and at least one of them must be nonzero because* $(0, 0, 0)$ *is removed. In addition, points are equivalent if they can be scaled by some* (t, t, t) *for* $t \in \mathbb{C}^*$ *which is the condition on points in* $\mathbb{CP}^2$*.*

This example illustrates the process of finding a toric variety from its fan. For the most part, the general process can be intuited fairly naturally from this example. Simply generalize to using the 1-skeleton of whatever fan rather than this specific fan. One point which should be made more clear is that in general, G is the kernel of $\phi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^r$ defined by

$$\phi(t_1, \ldots, t_n) = \left(\prod_{j=1}^{n} t_j^{v_{j1}}, \ldots, \prod_{j=1}^{n} t_j^{v_{jr}}\right),$$

which is why the product notation was used in Example 3.3.5 despite only being a product of two t_j.

This process given the fan of $\mathbb{CP}^3$ plays out in an all but identical fashion, but for the sake of additional illustrative examples, it is also presented here.

Example 3.3.6. *The fan of* $\mathbb{CP}^3$ *was found earlier, but as with* $\mathbb{CP}^2$*, the only cones that need to be explicitly identified are the* 1*-dimensional cones. The primitive generators for the four* 1*-dimensional cones are* $(-1, -1, -1)$*,* $(1, 0, 0)$*,* $(0, 1, 0)$*,* *and* $(0,0,1)$. *These will respectively be* v_1, v_2, v_3*and* v_4, *and to each* 1-*dimensional cone/primitive generator let* x_i *be a coordinate corresponding to it. As discussed, every proper subset of this set of* 1-*dimensional cones spans a cone of* $\mathbb{CP}^3$, *so the only subset* S *of the* 1-*skeleton of* $\mathbb{CP}^3$ *which does not span a cone is the entire* 1-*skeleton. This yields* $V(S)$ *where every coordinate is set to* 0. *Thus, the only* $V(S)$ *in* $Z(\Sigma)$ *is* $(0,0,0,0)$, *and* $Z(\Sigma) = \{(0,0,0,0)\}$.

G is the kernel of $\phi : (\mathbb{C}^*)^4 \to (\mathbb{C}^*)^3$ *defined by*

$$\begin{aligned}\phi(t_1,t_2,t_3,t_4) &= \left(\prod_{j=1}^{4} t_j^{v_{j1}}, \prod_{j=1}^{4} t_j^{v_{j2}}, \prod_{j=1}^{4} t_j^{v_{j3}}\right)\\ &= \left(t_1^{v_{11}}t_2^{v_{21}}t_3^{v_{31}}t_3^{v_{41}}, t_1^{v_{12}}t_2^{v_{22}}t_3^{v_{32}}t_3^{v_{42}}, t_1^{v_{13}}t_2^{v_{23}}t_3^{v_{33}}t_3^{v_{43}}\right)\\ &= \left(t_1^{-1}t_2^{1}t_3^{0}t_4^{0}, t_1^{-1}t_2^{0}t_3^{1}t_4^{0}, t_1^{-1}t_2^{0}t_3^{0}t_4^{1}\right)\\ &= \left(t_1^{-1}t_2, t_1^{-1}t_3, t_1^{-1}t_4\right).\end{aligned}$$

The kernel of ϕ *then is any* (t_1,t_2,t_3,t_4), $t_1,t_2,t_3,t_4 \in \mathbb{C}^*$, *where* $t_1^{-1}t_2 = 1$, $t_1^{-1}t_3 = 1$, *and* $t_1^{-1}t_4 = 1$. *That is,* $t_1 = t_2$, $t_1 = t_3$, *and* $t_1 = t_4$. *Thus, they must all be equal, so* $G = \{(t,t,t,t) | t \in \mathbb{C}^*\}$.

The toric variety is

$$(\mathbb{C}^4 - \{(0,0,0,0)\})/\{(t,t,t,t)|t \in \mathbb{C}^*\}.$$

This is exactly $\mathbb{CP}^3$ *as desired. Each coordinate must be in* $\mathbb{C}$, *and at least one of them must be nonzero because* $(0,0,0,0)$ *is removed. In addition, points are equivalent if they can be scaled by some* (t,t,t,t) *for* $t \in \mathbb{C}^*$ *which is the condition on points in* $\mathbb{CP}^3$.

3.4 Subdividing

These constructions can be used for any valid fan, but because I am currently only working with toric varieties corresponding to graph associahedra, not all fans need to be considered. For graph associahedra, the only fans which need to be considered are those for $\mathbb{CP}^n$ and those which can be formed by subdividing cones in the fans for some $\mathbb{CP}^n$.

3.4.1 Previous Method

Definition 3.4.1. *Let* σ *be a cone with dimension* $k > 1$ *in fan* Σ. σ *has dimension* $k > 1$, *so it is spanned by* k 1-*dimensional cones. Let* $\{v_1, \ldots, v_k\}$ *be the set* *of* 1*-dimensional cones which spans* σ*. To* subdivide σ*, remove* σ *from* Σ*. In its place, add the* 1*-dimensional cone* $v = v_1 + \cdots + v_k$ *to* Σ*, and for each* v_i*, add the* k*-dimensional cone* σ_i *which is spanned by the set of* 1*-dimensional cones* $\{v, v_1 \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\}$*. That is, add in a cone spanned by all of the* v *and all of the previously mentioned* 1*-dimensional cones except for* v_i*.*

The following example finds the toric variety for the fan which is formed by subdividing the 2-dimensional cone that is the first quadrant in the fan of $\mathbb{CP}^2$.

Example 3.4.2. *To subdivide the specified* 2*-dimensional cone, a new* 1*-dimensional cone must be added which has generator* $(1, 1)$*. Note that the primitive generator must be as such because subdividing necessarily requires that the new primitive generator be the sum of the primitive generators that span the cone being subdivided. In addition, the first quadrant is now split into two* 2*-dimensional cones. Let* $v_0, v_1, v_2, v_{1,2}$ *be* $(-1, -1), (1, 0), (0, 1), (1, 1)$ *respectively. These are the primitive generators of the* 1*-skeleton. With that, the subsets that do not span a cone are the sets of* 1*-dimensional cones corresponding to* $\{v_1, v_2\}, \{v_0, v_{1,2}\}$*, any set of* 3 *cones, and the singular set of all* 4 *cones. Setting the corresponding coordinates to* 0 *results in subspaces*

$$\{(x_0, 0, 0, x_{1,2}), (0, x_1, x_2, 0), (0, 0, 0, x_{1,2}), \ldots, (x_0, 0, 0, 0), (0, 0, 0, 0)\}.$$

These subspaces should be subtracted from $\mathbb{C}^4$ *to get the set of points that are in the toric variety. Let* C *be the union of those subspaces, leaving* $\mathbb{C}^4 - C$*. Based on the subspaces removed from* $\mathbb{C}^4$*, every point in* $\mathbb{C}^4 - C$ *has at most* 2 *coordinates equal to* 0*, and if there are exactly* 2 *coordinates equal to* 0*, then the pair of* 0 *coordinates must not be* $\{x_0, x_{1,2}\}$ *or* $\{x_1, x_2\}$*.*

G is the kernel of $\phi : (\mathbb{C}^*)^4 \to (\mathbb{C}^*)^2$ *defined by*

$$\begin{aligned}\phi(t_0, t_1, t_2, t_{1,2}) &= \left(\prod t_j^{v_{j1}}, \prod t_j^{v_{j2}}\right) \\ &= \left(t_0^{v_{-1}} t_1^1 t_2^0 t_{1,2}^1, t_0^{v_{-1}} t_1^0 t_2^1 t_{1,2}^1\right) \\ &= \left(t_0^{-1} t_1 t_{1,2}, t_0^{-1} t_2 t_{1,2}\right).\end{aligned}$$

The kernel of ϕ *then is any* $(t_0, t_1, t_2, t_{1,2})$*,* $t_0, t_1, t_2, t_{1,2} \in \mathbb{C}^*$ *, where* $t_0^{-1} t_1 t_{1,2} = 1$ *and* $t_0^{-1} t_2 t_{1,2} = 1$*. From these equations, it can be deduced that* $t_1 = t_2$ *and* $t_0 = t_1 t_{1,2}$*. Then the toric variety is* $(\mathbb{C}^4 - C)/G$*. This is all points in* $\mathbb{C}^4$ *which satisfy the previously found conditions, and in addition, taking the quotient by G means that points* $(t_0, t_1, t_2, t_{1,2})$ *and* $(t'_0, t'_1, t'_2, t'_{1,2})$ *in* $\mathbb{C}^4 - C$ *are equivalent if and only if there exist some* $c_1, c_2 \in \mathbb{C}^*$ *such that* $t_0 = c_1 c_2 t'_0$*,* $t_1 = c_1 t'_1$*,* $t_2 = c_1 t'_2$*, and* $t_{1,2} = c_2 t'_{1.2}$*.*

While it may seem somewhat convoluted, this is a full description of the toric variety defined by the complete fan with 1-skeleton $\{(-1,-1,-1), (1,0,0),(0,1,0),(0,0,1)\}$.

3.4.2 Blowups

As seen in Example 3.4.2, while the previous method works, it can create presentations of the toric variety which are fairly opaque. Blowups present another perspective which can produce a more transparent presentation.

Example 3.4.3. *The blowup of $\mathbb{C}^2$ at $(0,0)$ entails adding a copy of $\mathbb{P}^1$ at the origin. This is*

$$\mathrm{Bl}_{(0,0)}\,\mathbb{C}^2 = \widetilde{\mathbb{C}^2} := \{((z_1,z_2),(x_1:x_2)) \in \mathbb{C}^2 \times \mathbb{P}^1 | z_1x_2 = z_2x_1\}.$$

By defining it this way, the preimage of the projection map from $\widetilde{\mathbb{C}^2}$ onto $\mathbb{C}$ for any point other than the origin will consist of exactly a single point, specifically $((z_1,z_2),(z_1:z_2))$. However, the preimage of the origin will be an entire copy of $\mathbb{P}^1$. Hence why it is called the blowup at $(0,0)$. $\mathbb{P}^1$ can kind of be thought of as just corresponding to the possible directions in $\mathbb{A}^2$ of the corresponding affine space, and blowing up a point in $\mathbb{C}^2$ effectively keeps track of the direction that you go to that point from.

It is possible to blow up more than a single point, and in general, any linear subspace can be blown up.

Definition 3.4.4. *Let $\{z_1,\ldots,z_n\}$ be the ordered coordinates for $\mathbb{C}^n$. A coordinate subspace is the set of all points in $\mathbb{C}^n$ for which coordinates $\{z_{k+1},\ldots,z_n\}$ are all equal to 0 for some k. The blowup of this coordinate subspace is*

$$\widetilde{\mathbb{C}^n} = \{(z,l) \in \mathbb{C}^n \times \mathbb{P}^{n-(k+1)} \,|\, z_il_j = z_jl_i \,\forall\, k+1 \leq i,j \leq n\},$$

where points in $\mathbb{P}^{n-(k+1)}$ are of the form $(l_{k+1}:\ldots:l_n)$.

Note that this definition only applies to blowups of coordinate subspaces, but through linear transformations, it is not much trouble to transform a blowup of some coordinate subspace into a blowup of an arbitrary linear subspace. It is in fact possible to blow up any variety, not just linear subspaces, but again, as will be discussed later, for the purposes of this project, only linear subspaces will need to be examined. Returning to the example,

Example 3.4.5. *The key point is that subdividing a cone in a fan corresponds to a blowup in the toric variety at the point corresponding to the subdivided cone. Thus, subdividing as was done in the earlier example corresponds to blowing up* $\mathbb{CP}^2$ *at point* $(1:0:0)$ *because that corresponds to the quadrant* 1 *cone. This initially seems daunting, but note that* $\mathbb{CP}^2$ *contains* 3 *copies of* $\mathbb{C}^2$*. One where each of the coordinates is nonzero. The point* $(1:0:0)$ *is only in* U_0 *which is where the first (zeroth) coordinate is nonzero. It does not exist in* U_1 *or* U_2 *because* x_1 *and* x_2 *are both zero. Blowing up is a local property, so only this copy of* $\mathbb{C}^2$ *is impacted by blowing up this point, and because it is a copy of* $\mathbb{C}^2$*, the blowup works the same way as it does in the example (though the coordinates must be shifted). That is, the toric variety is* $\{((z_0:z_1:z_2),(x_1:x_2)) \in \mathbb{CP}^2 \times \mathbb{P}^1 | z_1x_2 = z_2x_1\}$.

3.4.3 Showing Equivalence

In general, the different presentations of the toric variety can be shown to be isomorphic. This section will only do so for the specific example being worked through here, but the process will be more or less the same.

The goal is to find an isomorphic map between $\{((z_0:z_1:z_2),(x_1:x_2)) \in \mathbb{CP}^2 \times \mathbb{P}^1 | z_1x_2 = z_2x_1\}$ and the previously defined $(\mathbb{C}^4 - C)/G$. Consider the map $f : (\mathbb{C}^4 - C)/G \to \{((z_0:z_1:z_2),(x_1:x_2)) \in \mathbb{CP}^2 \times \mathbb{P}^1 | z_1x_2 = z_2x_1\}$ defined by

$$f(t_0,t_1,t_2,t_{1,2}) = ((t_0 : t_1t_{1,2} : t_2t_{1,2}),(t_1 : t_2)).$$

First of all, observe that this is a valid mapping. As mentioned, a valid point in $(\mathbb{C}^4 - C)/G$ either has at most 2 of its coordinates equal to 0, and if exactly 2 of the coordinates equal 0, it must not be the case that $\{t_0, t_{1,2}\}$ are both 0 and it must not be the case that $\{t_1, t_2\}$ are both 0. These restrictions prevent $((t_0 : t_1t_{1,2} : t_2t_{1,2}),(t_1 : t_2))$ from being an invalid point of $\mathbb{CP}^2 \times \mathbb{P}^1$, but just to confirm:

- if none of the coordinates are 0, then none of the coordinates in $((t_0 : t_1t_{1,2} : t_2t_{1,2}),(t_1 : t_2))$ will be 0, which is fine.
- if only $t_0 = 0$, then $((t_0 : t_1t_{1,2} : t_2t_{1,2}),(t_1 : t_2)) = ((0 : t_1t_{1,2} : t_2t_{1,2}),(t_1 : t_2))$ which is fine.
- if only $t_1 = 0$, then $((t_0 : t_1t_{1,2} : t_2t_{1,2}),(t_1 : t_2)) = ((t_0 : 0 : t_2t_{1,2}),(0 : t_2))$ which is fine.
- if only $t_2 = 0$, then $((t_0 : t_1t_{1,2} : t_2t_{1,2}),(t_1 : t_2)) = ((t_0 : t_1t_{1,2} : 0),(t_1 : 0))$ which is fine.

- if only $t_{1,2} = 0$, then $((t_0 : t_1t_{1,2} : t_2t_{1,2}), (t_1 : t_2)) = ((t_0 : 0 : 0), (t_1 : t_2))$ which is fine.
- if only $t_0 = t_1 = 0$, then $((t_0 : t_1t_{1,2} : t_2t_{1,2}), (t_1 : t_2)) = ((0 : 0 : t_2t_{1,2}), (0 : t_2))$ which is fine.
- if only $t_0 = t_2 = 0$, then $((t_0 : t_1t_{1,2} : t_2t_{1,2}), (t_1 : t_2)) = ((0 : t_1t_{1,2} : 0), (t_1 : 0))$ which is fine.
- if only $t_1 = t_{1,2} = 0$, then $((t_0 : t_1t_{1,2} : t_2t_{1,2}), (t_1 : t_2)) = ((t_0 : 0 : 0), (0 : t_2))$ which is fine.
- if only $t_2 = t_{1,2} = 0$, then $((t_0 : t_1t_{1,2} : t_2t_{1,2}), (t_1 : t_2)) = ((t_0 : 0 : 0), (t_1 : 0))$ which is fine.

This accounts for all possible points in $(\mathbb{C}^4 - C)/G$, as every other point has an invalid pair of coordinates valued at 0 or contains more than 2 coordinates valued at 0. Note that in some cases, the resulting points could be rewritten in a nicer way, for example by noting that $(t_0 : 0 : 0)$ is equivalent to $(1 : 0 : 0)$, but that this does not disrupt the argument. Basically, the removed subspaces contain all the points which would map to points where at least one of $((t_0 : t_1t_{1,2} : t_2t_{1,2})$ or $(t_1 : t_2))$ have all 0 coordinates, which cannot happen.

Next, observe that f is well-defined. If two points $(t_0, t_1, t_2, t_{1,2})$ and $(t'_0, t'_1, t'_2, t'_{1,2})$ in $(\mathbb{C}^4 - C)/G$ are equivalent, then from Example 3.4.2, it must be true that $(t'_0, t'_1, t'_2, t'_{1,2}) = (c_1c_2t_0, c_1t_1, c_1t_2, c_2t_{1,2})$ for some $c_1, c_2 \in \mathbb{C}^*$. Thus, mapping $(t'_0, t'_1, t'_2, t'_{1,2})$, we have

$$\begin{aligned} f(t'_0, t'_1, t'_2, t'_{1,2}) &= ((t'_0 : t'_1t'_{1,2} : t'_2t'_{1,2}), (t'_1 : t'_2)) \\ &= ((c_1c_2t_0 : c_1t_1c_2t_{1,2} : c_1t_2c_2t_{1,2}), (c_1t_1 : c_1t_2)) \\ &= (c_1c_2(t_0 : t_1t_{1,2} : t_2t_{1,2}), c_1(t_1 : t_2)). \end{aligned}$$

This is equivalent to $((t_0 : t_1t_{1,2} : t_2t_{1,2}), (t_1 : t_2))$ in $\mathbb{CP}^2 \times \mathbb{P}^1$ because it is just a scaling of $(t_0 : t_1t_{1,2} : t_2t_{1,2})$ and $(t_1 : t_2)$ by scalars c_1c_2 and c_1, respectively, in $\mathbb{C}^*$. Because points in the blowup of $\mathbb{CP}^2$ are points in $\mathbb{CP}^2 \times \mathbb{P}^1$, they are equivalent when the points in $\mathbb{CP}^2$ or $\mathbb{P}^1$ are scaled by (potentially independent) values in $\mathbb{C}^*$.

Thus, this mapping is acceptable. Now it must be shown that it is an isomorphism. First of all, because f is a rational map, it will necessarily be a homeomorphism, and with that, all that remains is to show that it is bijective. Let $((z_0 : z_1 : z_2), (x_1 : x_2)) \in \mathbb{CP}^2 \times \mathbb{P}^1$ such that $z_1x_2 = z_2x_1$. At least one of x_1 and x_2 must be nonzero. If they are both nonzero, then $z_1x_1^{-1}$ and $z_2x_2^{-1}$ can both be evaluated and are in fact equal. If one of them is 0, say x_2, then because $z_1x_2 = z_2x_1$ and because x_1 must be nonzero, it follows that $z_2 = 0$. Thus, let x_1 be nonzero. $(z_0, x_1, x_2, z_1x_1^{-1})$ must be a valid point in $(\mathbb{C}^4 - C)/G$. x_1 is nonzero, so it is impossible for all 4 coordinates to be 0. Furthermore, the only way for 3 coordinates to be 0 is if z_0, z_1, and x_2 are all 0. However, as demonstrated, if x_2 is zero, then $z_2 = 0$, and this would mean that all z_i are 0 which is not allowed. x_1 and x_2 cannot both be 0. The only other way that this point could not be in $(\mathbb{C}^4 - C)/G$ is if z_0 and $z_1x_1^{-1}$ are both 0 while x_1 and x_2 are nonzero. $z_1x_1^{-1} = 0$ means that $z_1 = 0$ because x_1 is nonzero. However, if x_1 and x_2 are nonzero, then $z_1 = 0$ implies $z_2 = 0$, which would cause all z_i to be 0, which is again not allowed. Thus, in x_1 is nonzero, $(z_0, x_1, x_2, z_1x_1^{-1})$ is indeed a valid point in $(\mathbb{C}^4 - C)/G$. When x_2 is nonzero, the argument follows in the same way but using $z_2x_2^{-1}$ instead of $z_1x_1^{-1}$.

Observe that

$$f(z_0, x_1, x_2, z_1x_1^{-1}) = ((z_0 : x_1z_1x_1^{-1} : x_2z_1x_1^{-1}), (x_1 : x_2)).$$

Note again that if x_1 and x_2 are both nonzero, then $z_1x_1^{-1} = z_2x_2^{-1}$, so

$$\begin{aligned} f(z_0, x_1, x_2, z_1x_1^{-1}) &= ((z_0 : x_1z_1x_1^{-1} : x_2z_2x_2^{-1}), (x_1 : x_2)) \\ &= ((z_0 : z_1 : z_2), (x_1 : x_2)) \end{aligned}$$

as desired. Note that if either x_1 or x_2 is 0, then that implies the corresponding z_i must also be 0, so rather than trying to take the inverse, the 0 yields the proper values. Thus, for any $((z_0 : z_1 : z_2), (x_1 : x_2)) \in \mathbb{CP}^2 \times \mathbb{P}^1$ such that $z_1x_2 = z_2x_1$, there is a point in $(\mathbb{C}^4 - C)/G$ which is mapped to it by f, so f is surjective.

Let $t = (t_0, t_1, t_2, t_{1,2})$ and $s = (s_0, s_1, s_2, s_{1,2})$ be points in $(\mathbb{C}^4 - C)/G$ which both map to the same point via f. Then

$$\begin{aligned} f(t) &= f(s) \\ ((t_0 : t_1t_{1,2} : t_2t_{1,2}), (t_1 : t_2)) &= ((s_0 : s_1s_{1,2} : s_2s_{1,2}), (s_1 : s_2)). \end{aligned}$$

From this, it is immediately true that $(t_1 : t_2) = k(s_1 : s_2)$ for some $k \in \mathbb{C}^*$ because $(t_1 : t_2)$ and $(s_1 : s_2)$ must be the same point in $\mathbb{P}^1$. Also, $(t_0 : t_1t_{1,2} : t_2t_{1,2}) = c(s_0 : s_1s_{1,2} : s_2s_{1,2})$ for some $c \in \mathbb{C}^*$ through the same type of logic. From here, by pushing the equations around, it can quickly be seen that t and s must actually be equivalent in $(\mathbb{C}^4 - C)/G$, so f is injective. Hence, f is bijective because it is both injective and surjective, so $\{((z_0 : z_1 : z_2), (x_1 : x_2)) \in \mathbb{CP}^2 \times \mathbb{P}^1 | z_1 x_2 = z_2 x_1\}$ and $(\mathbb{C}^4 - C)/G$ as defined in Example 3.4.2 are isomorphic.

3.5 Cohomology

Definition 3.5.1. *Let X be a toric variety with a 1-skeleton whose primitive generators are $\{v_1, \ldots, v_d\}$. Note that there are d elements of the 1-skeleton. The* cohomology *of X is defined as follows*

$$H^*(X) := \mathbb{Z}[D_1, \ldots, D_d]/I$$

where $D_1, \ldots, D_d$ are variables (and thus $\mathbb{Z}[D_1, \ldots, D_d]$ is the polynomial ring in d variables with coefficients in $\mathbb{Z}$.), and I is the ideal of $\mathbb{Z}[D_1, \ldots, D_d]$ generated by all elements of the following two forms

- *$D_{i_1} \ldots D_{i_k}$ where the set of 1-dimensional cones with primitive generators $\{v_{i_1}, \ldots, v_{i_k}\}$ do not span a cone in the fan of X.*
- *$\sum_{i=1}^{d} \langle u, v_i \rangle D_i$ where u is a vector (of the same length as v_i) with exactly one coordinate equal to 1 and all other coordinates equal to 0, and $\langle u, v_i \rangle$ denotes a standard dot product.*

This definition is not overwhelmingly complicated, but even so, an example makes it more clear how it actually works.

Example 3.5.2. *We've seen that $\mathbb{CP}^2$ is a toric variety. The cohomology of $\mathbb{CP}^2$ then is*

$$H^*(\mathbb{CP}^2) = \mathbb{Z}[D_1, D_2, D_3]/I$$

where I is the ideal as described above. The first type of generating element is all $D_{i_1} \ldots D_{i_k}$ where the set of 1-dimensional cones with primitive generators $\{v_{i_1}, \ldots, v_{i_k}\}$ do not span a cone in the fan of $\mathbb{CP}^2$. As we've seen, the only subset of the 1-skeleton of $\mathbb{CP}^2$ which does not span a cone in the fan is the entire 1-skeleton. Thus, the only generating element of I of this form is $D_1 D_2 D_3$.

The second type of generating element is all $\sum_{i=1}^{3} \langle u, v_i \rangle D_i$ where u is a length 2 vector with exactly one coordinate equal to 1 and all other coordinates equal to

0. There are only 2 valid u, $u_1 = (1, 0)$ and $u_2 = (0, 1)$. Thus, the following are generating elements of I.

$$\begin{aligned}\sum_{i=1}^{3} \langle u_1, v_i \rangle D_i &= \langle (1,0), (-1,-1) \rangle D_1 + \langle (1,0), (1,0) \rangle D_2 + \langle (1,0), (0,1) \rangle D_3 \\ &= -D_1 + D_2\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{3} \langle u_2, v_i \rangle D_i &= \langle (0,1), (-1,-1) \rangle D_1 + \langle (0,1), (1,0) \rangle D_2 + \langle (0,1), (0,1) \rangle D_3 \\ &= -D_1 + D_3\end{aligned}$$

Thus, I has three generating elements. Because $H^*(\mathbb{CP}^2)$ is the polynomial ring quotiented by I, it can be thought of as all polynomials in $\mathbb{Z}[D_1, D_2, D_3]$ with each of the generating elements of I being set equal to 0. This gives a set of equations. From $-D_1 + D_2 = 0$, it follows that $D_1 = D_2$ and by the same logic, $D_1 = D_3$. So all coordinates must be the same; let them all be referred to by a new coordinate D. Then, because $D_1 D_2 D_3 = 0$, this is really all polynomials in $\mathbb{Z}[D]$ with $D = 0$. Therefore,

$$H^*(\mathbb{CP}^2) = \mathbb{Z}[D]/(D^3)$$

Note that this is polynomials in one variable with integer coefficients and degree at most 2. Furthermore, notice that it will be the case that $H^*(\mathbb{CP}^n) = \mathbb{Z}[D]/(D^{n+1})$.

As another example, consider the complete fan with a 1-skeleton whose primitive generators are $\{(-1,-1), (1,0), (0,1), (1,1)\}$. This corresponds to the toric variety that is $\mathbb{CP}^2$ with a blowup at the point $(1 : 0 : 0)$.

Example 3.5.3. *Let Σ be the stated fan. Let these be $v_0, v_1, v_2, v_{1,2}$ respectively. There are 4 elements of the 1-skeleton. The cohomology of X is*

$$H^*(X) = \mathbb{Z}[D_0, D_1, D_2, D_{1,2}]/I$$

where I is the ideal generated by elements of two forms. The first type is all elements $D_{i_1} \dots D_{i_k}$ where the set of 1-dimensional cones with primitive generators $\{v_{i_1}, \dots, v_{i_k}\}$ do not span a cone in the fan of X. The sets of 1-dimensional cones not spanning a cone in the fan are $\{v_0, v_{1,2}\}$ and $\{v_1, v_2\}$ as well as any set of size 3 and the set of size 4 (which is the entire 1-skeleton). Hence, $D_0 D_{1,2}$ and $D_1 D_2$ are generating elements. Note that $D_0 D_1 D_2 D_{1,2}$, $D_0 D_1 D_2$, $D_0 D_1 D_{1,2}$, etc. are also potential generators, but they are all generated by $D_0 D_{1,2}$ and $D_1 D_2$, so they do not need to be included as generating elements.

The other type of generating element is $\sum_{\alpha}^{d} \langle u, v_i \rangle D_\alpha$ *where* u *is a vector (of the same length as* v_i*) with exactly one coordinate equal to* 1 *and all other coordinates equal to* 0*, and* $\langle u, v_i \rangle$ *denotes a standard dot product. This yields the following*

$$\sum_{\alpha}^{d} \langle u_1, v_i \rangle D_\alpha = -D_0 + D_1 + D_{1,2}$$

$$\sum_{\alpha}^{d} \langle u_2, v_i \rangle D_\alpha = -D_0 + D_2 + D_{1,2}$$

Thus, with all of this, we have

$$I = \langle D_0 D_{1,2}, D_1 D_2, -D_0 + D_1 + D_{1,2}, -D_0 + D_2 + D_{1,2} \rangle$$

The cohomology is $H^*(X) = \mathbb{Z}[D_0, D_1, D_2, D_{1,2}]/I$*, so each of the generators of* I *is set to* 0 *in the cohomology. From the fact that* $-D_0 + D_1 + D_{1,2}$ *and* $-D_0 + D_2 + D_{1,2}$ *are both equal to the same thing,* 0*, it follows that* $D_1 = D_2$*, and notably, they are both equal to* $D_0 - D_{1,2}$*. Substituting for* $D_1 D_2 = 0$ *(and noting that* $D_0 D_{1,2} = 0$*) yields that*

$$\begin{aligned} 0 &= (D_0 - D_{1,2})(D_0 - D_{1,2}) \\ &= D_0^2 - 2D_0 D_{1,2} + D_{1,2}^2 \\ &= D_0^2 - 0 + D_{1,2}^2 \\ D_0^2 &= -D_{1,2}^2 \end{aligned}$$

This means that $\mathbb{Z}[D_0, D_1, D_2, D_{1,2}]/I$ *can be understood to be* $\mathbb{Z}[D_0, D_{1,2}]$ *with the conditions that* $D_0 D_{1,2} = 0$ *and* $D_0^2 = -D_{1,2}^2$*.*

Even without the fan, it is possible to determine the cohomology of a toric variety, as long as we have the geometric information, particularly with regard to blowups.

Definition 3.5.4. *Let* X *be a toric variety that is some projective space with* k *subspaces blown up. The cohomology of* X *is* $H^*(X, \mathbb{Z}) = \langle H, E_1, \ldots, E_k \rangle$ *where* H *is the proper transform of a hyperplane of the original projective space, and* E_i *corresponds to the exceptional divisor of one of the* k *blowups. In addition, there are conditions for the presentation of the group generated by these elements. Many of these properties can be understood fairly naturally from the characterization of the generators. In this presentation of the group, multiplication corresponds to what would be the result of a general intersection of the two spaces. So for example, if* *a hyperplane in the space would in general not intersect with a given blowup E_i, then $H \cdot E_i = 0$. Note that $X = 1$, as intersecting X with any general subspace will yield that same subspace.*

As an example, consider the toric variety that is $\mathbb{CP}^2$ with a blowup at the point $(1:0:0)$.

Example 3.5.5. *Let $X = \mathrm{Bl}_{(1:0:0)}\,\mathbb{CP}^2$. Then $H^*(X,\mathbb{Z}) = \langle H, E \,|\, H^3 = 0,\ H^2 = 1_{pt},\ H \cdot E = 0,\ E^2 = -1_{pt}\rangle$. A few of these conditions can be understood geometrically. The reason for $H^3 = 0$ and $H^2 = 1_{pt}$ is that in X, the hyperplane of $\mathbb{CP}^2$ is 1-dimensional subspace, a line. In 2-dimensional space, a pair of general lines will intersect at a single point. So $H \cdot H = H^2$ is a point, represented here by 1_{pt}. Then, a line and a point will not in general intersect in 2-dimensional space. Thus, $H \cdot H \cdot H = H^3 = 0$. For $H \cdot E$, consider a general line in $\mathbb{CP}^2$ and the blowup of the point. The blowup is a copy of $\mathbb{P}^1$, but it is perpendicular to $\mathbb{CP}^2$, and it only intersects $\mathbb{CP}^2$ at a single point. As such, the intersection of a hyperplane in $\mathbb{CP}^2$ and the blowup of a point will generally be nothing. Therefore, $H \cdot E = 0$.*

As it turns out, Examples 3.5.3 and 3.5.5 were using the same toric variety, so the found cohomologies ought to be the same. The presentations of the two are different on the surface, but it can be shown that they are in fact isomorphic.

Example 3.5.6. *There is some geometric reasoning which can quickly lead to the connection between the two. Note that in the presentation from Example 3.5.3, $D_{1,2}$ is the variable corresponding to the 1-dimensional cone that was added in the blowup. Similarly, in Example 3.5.5, E is the generator corresponding to the blowup. So it is natural to suppose that $D_{1,2}$ and E are the same. Then, because Example 3.5.3 has $D_0 D_{1,2} = 0$ and Example 3.5.5 has $H \cdot E = 0$, it is also nautral to suppose that D_0 and H might be the same. Then, the remaining properties fall out readily. $D_0^2 = -D_{1,2}^2$ indicates that $H^2 = -E^2$, and indeed this is true as $H^2 = 1_{pt}$ and $E^2 = -1_{pt}$. Keep in mind that what $1_p t$ is exactly doesn't matter all that much for the presentation itelf. Finally, $H^3 = 0$ indicates that $D_0^3 = 0$ should be true. Observe that*

$$\begin{aligned} D_0^3 &= D_0(-D_{1,2}^2) \\ &= (D_0 D_{1,2})(-D_{1,2}) \\ &= 0(-D_{1,2}) = 0 \end{aligned}$$

Thus, the fact that the cohomologies are actually the same has been loosely demonstrated.

For the most part, calculating the cohomology of a toric variety from its fan will produce a presentation that is easier to work with, so that will be the more commonly used method, but both methods should nevertheless be kept in mind.

Chapter 4

Connections

4.1 Toric Varieties from Graph Associahedra

As mentioned, a complete fan in $\mathbb{R}^n$ can be converted to a convex n-dimensional polytope, and vice versa. This then creates a natural connection to graphs because every graph has a graph associahedra which can be converted to a complete fan. This is why only complete fans need to be investigated. In addition, subdividing cones of a complete fan directly corresponds to truncating faces of the polytope. Considering that every graph associahedron can be constructed by truncating faces of a simplex, and noting that the simplex corresponds to the fan of some $\mathbb{CP}^n$, it then follows that every fan corresponding to a graph associahedron can be made by subdividing cones in some $\mathbb{CP}^n$.

From the way that they are constructed, all cones in the fan of $\mathbb{CP}^n$ correspond to linear subspaces of $\mathbb{CP}^n$, and as stated earlier, blowups correspond to subdivisions of cones. Thus, because every fan being investigated is formed by subdividing cones in the fan of $\mathbb{CP}^n$, the corresponding toric varieties can all be constructed by blowing up torus-invariant linear subspaces of $\mathbb{CP}^n$. Hence, by adapting the algorithm for constructing graph associahedra, it is possible to create an algorithm that directly constructs the toric variety corresponding to a graph through a sequence of blowups of projective space without ever needing to construct the graph associahedron itself.

There are a couple of ways that this can be done. First, because it is often very useful to work with the toric variety through its fan, the algorithm for constructing graph associahedra can be converted to one which constructs

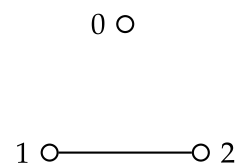


Figure 4.1 Example graph on 3 vertices with 1 edge.

fans for toric varieties as follows: For a graph Γ on n vertices, the fan of the toric variety corresponding to the graph associahedron can be constructed as follows:

1. Let Σ_Γ be the fan of $\mathbb{CP}^{n-1}$. This will be the complete fan in $\mathbb{R}^{n-1}$ with a 1-skeleton which has primitive generator set $S = \{(-1, \ldots, -1), (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$. These are all length $n-1$ vectors, and as such, this set of vectors has size n.

2. Assign each vertex in Γ to one of the 1-dimensional cones (or equivalently to one of the vectors in S) of Σ_Γ.

3. Starting with highest dimensional cones (cones of codimension 1) of Σ_Γ, for each such cone u, examine the set of vertices assigned to the 1-dimensional cones contained in u. If this set is a valid tube of Γ, subdivide u.

4. Repeat the previous step for cones of Σ_Γ of codimension 2 and so on up through cones of codimension $n-2$ (2-dimensional cones).

5. The resulting fan is the toric variety as determined by the graph associahedron for Γ.

Thus, we can find the fan of the toric variety corresponding to the graph on 3 vertices with a single edge as drawn in Figure 4.1,

Example 4.1.1. *There are* 3 *vertices in the graph, so begin with the fan for* $\mathbb{CP}^2$*. This is the complete fan with* 1*-skeleton whose primitive generators are* $(-1, -1)$, $(1, 0)$, *and* $(0, 1)$*. Assign vertices* 0, 1, *and* 2 *to those* 1*-dimensional cones respectively. There are three codimension* 1 *cones in the fan of* $\mathbb{CP}^2$*, one containing each pair of* 1*-dimensional cones. There is only one pair of vertices that form a valid tube, vertices* 1 *and* 2*. As such, the only cone in the fan of* $\mathbb{CP}^2$ *to subdivide is the one containing* *the corresponding* 1*-dimensional cones,* $(1, 0)$ *and* $(0, 1)$*. In order to subdivide that cone, it is removed and replaced with the* 1*-dimensional cone with primitive generator* $(1, 1)$ *as well as a* 2*-dimensional cone spanned by* $\{(1, 0), (1, 1)\}$ *and a* 2*-dimensional cone spanned by* $\{(0, 1), (1, 1)\}$*. This fan,* Σ *is the fan of the toric variety corresponding to the graph.*

Alternatively, if a geometric description of the toric variety is preferred, it is also possible to directly construct the toric variety based on blowups in projective space. For a graph Γ on n vertices, the corresponding toric variety can be found as follows:

1. Let V_Γ be $\mathbb{CP}^{n-1}$.
2. Order the vertices in Γ, $v_0, \ldots, v_{n-1}$, and assign each vertex v_i to a coordinate x_{v_i}, such that points in $\mathbb{CP}^{n-1}$ are of the form $(x_{v_0} : \ldots : x_{v_{n-1}})$. This is possible because a point in $\mathbb{CP}^{n-1}$ will have n coordinates.
3. Consider the subspace of $\mathbb{CP}^n$ where x_{v_i} is free to vary (subject to still being a point in $\mathbb{CP}^n$), and all other coordinates are equal to 0. If the set of vertices $\{v_0, \ldots, v_{i-1}, v_{i+i}, \ldots, v_{n-1}\}$, that is the set of all vertices except for v_i, is a valid tube of Γ, then blow up the described subspace in $\mathbb{CP}^{n-1}$. Perform this for all subspaces where exactly 1 coordinate is allowed to vary.
4. Repeat the previous step for subspaces of $\mathbb{CP}^n$ in which exactly 2 coordinates are allowed to vary and all others must be 0, and so on up through subspaces with $n - 2$ coordinates allowed to vary. There is no need to check subspaces with $n - 1$ coordinates allowed to vary because blowups of such subspaces would behave as blowups of an $n - 2$-dimensional subspace of various copies of $\mathbb{C}^{n-1}$, and as previously mentioned, such blowups do not change the structure of the space. Furthermore, there is no need to worry about when all n coordinates are allowed to vary because the corresponding set of vertices in Γ would be empty, and a tube must be nonempty.
5. The resulting toric variety is that which corresponds to Γ.

Using the graph in Figure 4.1 again, the toric variety conrresponding to the graph associahedron can be found.

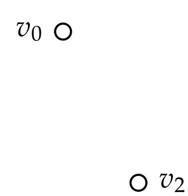
Example 4.1.2. *The graph has three vertices, so start with* $\mathbb{CP}^2$*. Let a point in* $\mathbb{CP}^2$ *be* (x_0, x_1, x_2)*. Then, vertices* 0*,* 1*, and* 2 *are assigned to* x_0*,* x_1*, and* x_2 *respectively. The only set of vertices forming a valid tube is* $\{1, 2\}$*, so the only subspace to be blown up is that in which* x_1 *and* x_2 *are set to* 0*. As it must still be a subspace of* $\mathbb{CP}^2$*, this is the point* $(1 : 0 : 0)$*. Thus, the toric variety corresponding to the described graph is* $\mathbb{CP}^2$ *with the point* $(1 : 0 : 0)$ *blown up.*

Notice that the toric variety/fan corresponding to the graph associahedra of the graph in Figure 4.1, as found in these examples, is the same toric variety/fan as was used in Examples 3.5.3 and 3.5.5.

4.2 Cohomologies from Graphs

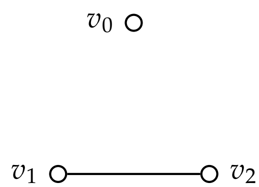
Now that a relationship between graphs and toric varieties (via graph associahedra) has been established, it is reasonable to consider computing the cohomology of the toric variety corresponding to a particular graph's graph associahedron, thereby determining a cohomology for the graph. It is possible to do so by leveraging the numerous characterizations described in previous sections, and doing so for small graphs is not unreasonable. The following examples demonstrate how this process is carried out for the possible graphs on 3 vertices.

Example 4.2.1. *For the following graph on* 3 *vertices,*



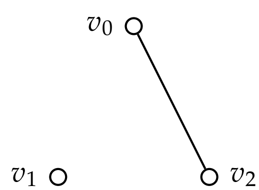
the fan is in $\mathbb{R}^2$*. The vertices are assigned to primitive generators* $\{(-1, -1), (1, 0), (0, 1)\}$ *in that order. There are no edges (and thus no other tubes), so there are no other primitive generators in the fan. Thus, the cohomology is formed from the polynomial ring in* 3 *variables, labeled* D_i *where* i *matches the labels for the vertices of the graph. Using the construction for the ideal relevant to the cohomology from Definition 3.5.1, the ideal is* $\langle D_0 D_1 D_2, -D_0 + D_1, -D_0 + D_2\rangle$*. Through some manipulation, this can be shown to mean that the overall cohomology is* $\mathbb{Z}[D]/\langle D^3\rangle$*. Note that this is the cohomology of* $\mathbb{CP}^2$ *as found in Example 3.5.2, and this is expected, as the graph has no edges, so the corresponding toric variety is* $\mathbb{CP}^2$ *without any blowups.*

Example 4.2.2. *For the following graph on* 3 *vertices,*



the vertices are assigned to primitive generators $\{(-1,-1),(1,0),(0,1)\}$ *in that order. From the possible tubes, there is also a primitive generator in the fan corresponding to the subdivision of the fan for vertices* v_1 *and* v_2*, so there is a primitive generator* $(1,1)$*. Thus, the cohomology is formed from the polynomial ring in* 4 *variables, labeled* D_i *where* i *matches either the labels for the vertices of the graph, or the set of labels of vertices forming a tube. In this case, that means the variables are* $\{D_0, D_1, D_2, D_{1,2}\}$*, but I phrased it that way because that is the more general notion. Using the construction for the ideal, it is* $\langle D_0D_1D_2, D_0D_1D_{1,2}, D_0D_2D_{1,2}, D_1D_2D_{1,2}, D_0D_{1,2}, D_1D_2, D_0D_1D_2D_{1,2}, -D_0 + D_1 + D_{1,2}, -D_0 + D_2 + D_{1,2}\rangle$*. There is clearly a great deal of redundancy in these generators, and it can be simplified to* $\langle D_0D_{1,2}, D_1D_2, -D_0 + D_1 + D_{1,2}, -D_0 + D_2 + D_{1,2}\rangle$*. Observe that this is the same ideal as is found in Example* 3.5.3*, and in fact, the set of primitive generators used in both of these examples were the same, so this is as expected. Notably, this suggests that this graph (and its graph associahedron) corresponds to the toric variety described in Example* 3.5.5*, namely* $\mathbb{CP}^2$ *with a blowup at* $(1:0:0)$*, which is indeed the case.*

Example 4.2.3. *For the following graph on* 3 *vertices,*

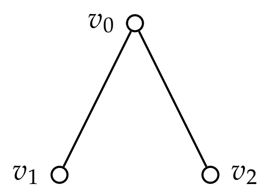


the vertices are again assigned to primitive generators $\{(-1,-1),(1,0),(0,1)\}$*. From the possible tubes, there is also a primitive generator in the fan corresponding to the subdivision of the fan for vertices* v_0 *and* v_2*, so there is a primitive generator* $(-1,0)$*. Thus, the cohomology is formed from the polynomial ring in* 4 *variables, labeled* D_i *where* i *matches either the labels for the vertices of the graph, or the set of labels of vertices forming a tube. In this case, that means the variables are* $\{D_0, D_1, D_2, D_{0,2}\}$*. Using the construction for the ideal, it is*

$\langle D_0D_1D_2, D_0D_1D_{0,2}, D_0D_2D_{0,2}, D_1D_2D_{0,2}, D_0D_2, D_1D_{0,2}, D_0D_1D_2D_{0,2}, -D_0+ D_1 - D_{0,2}, -D_0 + D_2\rangle$. This can be simplified to $\langle D_0D_2, D_1D_{0,2}, -D_0 + D_1 - D_{0,2}, -D_0 + D_2\rangle$. Though this appears different from the cohomology found in Example 4.2.2 *at a glance, it is in fact isomorphic, as the generators of the ideal end up amounting to the same conditions (up to relabeling of variables). This is what we would expect, as the graphs are isomorphic, so it is both unsurprising and fortunate that the cohomologies are isomorphic as well.*

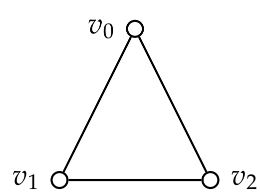
The cohomology for the other graph on 3 vertices with a single edge turns out to be a relabeling of this previous example, so I will not write it out here.

Example 4.2.4. *For the following graph on* 3 *vertices,*



the vertices are assigned to primitive generators $\{(-1, -1), (1, 0), (0, 1)\}$ as usual. From the possible tubes, there are also primitive generators in the fan corresponding to the subdivisions which are $(0, -1)$ and $(-1, 0)$. Thus, the cohomology is formed from the polynomial ring in 5 *variables, labeled D_i where i matches either the labels for the vertices of the graph, or the set of labels of vertices forming a tube. In this case, that means the variables are $\{D_0, D_1, D_2, D_{0,1}, D_{0,2}\}$. Using the construction for the ideal and eliminating redundant generators, it is $\langle D_0D_1, D_0D_2, D_{0,1}D_{0,2}, D_1D_{0,2}, D_2D_{0,1}, -D_0 + D_1 - D_{0,2}, -D_0 + D_2 - D_{0,1}\rangle$. At this point, it is becoming quite difficult to cleanly simplify the presentation of the cohomology, though characterizing it in terms of blowups may provide some insight.*

Example 4.2.5. *For the following graph on* 3 *vertices,*



the vertices are assigned to primitive generators $\{(-1,-1),(1,0),(0,1)\}$. *From the possible tubes, there are also primitive generators in the fan corresponding to the subdivisions which are* $(1,1)$, $(0,-1)$ *and* $(-1,0)$. *Thus, the cohomology is formed from the polynomial ring in* 6 *variables, labeled* D_i *where* i *matches either the labels for the vertices of the graph, or the set of labels of vertices forming a tube. In this case, that means the variables are* $\{D_0, D_1, D_2, D_{1,2}, D_{0,1}, D_{0,2}\}$. *Using the construction for the ideal and eliminating redundant generators, it is* $\langle D_0D_1, D_0D_2, D_1D_2, D_{1,2}D_{0,1}, D_{1,2}D_{0,2}D_{0,1}D_{0,2}, -D_0+D_1+D_{1,2}-D_{0,2}, -D_0+D_2+D_{1,2}-D_{0,1}\rangle$.

As can be seen, it's not always clear how to simplify the presentation of the ring. As it is, the construction simply gives the variables and generators for the ideal by which to quotient, but there seems to typically be a lot of redundancy in both. Even so, finding a way to eliminate this redundancy is not straightforward.

While this allows us to compute the cohomology of a graph, it requires some amount of knowledge about toric varieties and their fans. It would be preferable if we could characterize the cohomology solely from information about the graph itself.

Theorem 4.2.6. *For a graph* Γ, *the cohomology can be found as follows. Let*

$$D = \{D_t \mid t \text{ is a valid tube of } \Gamma\}$$

be a set of formal symbols. Note that this set is indexed by tubes of Γ, *including individual vertices* (*tubes with* 1 *vertex*) *and edges* (*tubes with* 2 *vertices*) *as well as larger tubes. Then the cohomology is* $\mathbb{Z}[D]/I$ *for an ideal* I *which is generated by all elements of the following forms*:

- $\prod_{t\in T} D_t$, *where* T *is a collection of tubes which is not a valid tubing of* Γ
- $\sum_{t_i\in T_i} D_{t_i} - \sum_{t_0\in T_0} D_{t_0}$ *for* $i \neq 0$, *where* T_i *is the collection of tubes containing vertex* v_i *and not containing* v_0, *and* T_0 *is similarly defined* (*as the collection of tubes containing vertex* v_0 *and not containing* v_i).

The multiplicative generators result from the fact that (just as there is a face in the graph associahedron,) there is a cone in the fan for every valid tubing of Γ. The multiplicative generators are formed from the product of variables corresponding to cones which collectively do not span a cone, and therefore do not correspond to a valid tubing. One key point to note is that any tubing which contains all vertices in the graph is invalid, which explains the connected component idea that was previously suspected.

The additive generators are because of the conventions I have been following in which v_0 corresponds to the cone with primitive generator $(-1, -1)$, and v_i corresponds to the cone with primitive generator that has a 1 in the i-th position and 0 elsewhere. Thus, any tube which contains v_i and not v_0 will have a 1 in the i-th position and the corresponding D_t should be added to the sum for that additive generator. Any tube which contains v_0 and not v_i will have a -1 in the i-th position and the corresponding D_t should therefore be subtracted from that additive generator. A tube containing both v_i and v_0 will have a 0 in the i-th position and does not positively or negatively contribute to that additive generator.

With this, we have constructively defined a cohomology for graph solely in terms of the graph itself.

Chapter 5

Future Work

While this construction of graph cohomology is nice to have, there is still work to be done. First, because the construction draws heavily from the construction of a cohomology given a fan, it has the same faults, namely that there is a significant amount of redundancy in the generators it provides. A useful next step would be to find means of simplifying the presentation of a cohomology so that it is more manageable to work with. Even if this is not feasible for arbitrary graphs, there may be particular classes of graphs for which the presentation can be more readily simplified, so looking for graphs whose properties line up well with the cohomology could yield interesting results.

In a similar vein, another direction to go with the results of this project is to use the graph cohomology as an invariant for graphs and investigate whether the properties of the cohomology might illuminate connections between graphs. If a certain class of graphs can be identified by the properties of the graph cohomology, it would be possible to take arbitrary graphs and determine whether or not they belong to that class of graphs. Such results could shed light on a number of questions in graph theory.

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