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Line-of-Sight Pursuit and Evasion Games on Polytopes in R^n

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Abstract

We study single-pursuer, line-of-sight Pursuit and Evasion games in polytopes in $\mathbb{R}^n$. We develop winning Pursuer strategies for simple classes of polytopes (monotone prisms) in $\mathbb{R}^n$, using proven algorithms for polygons as inspiration and as subroutines. More generally, we show that any Pursuer-win polytope can be extended to a new Pursuer-win polytope in more dimensions. Though we provide bounds on which polytopes are Pursuer-win, these bounds are not tight. Closing the gap between those polytopes known to be Pursuer-win and those known not to be remains an problem for future researchers.

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Chapter 1

Introduction

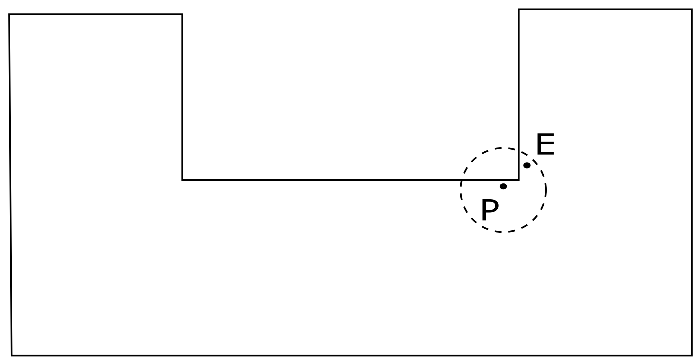
1.1 Pursuit and Evasion Games

Mathematicians and Computer Scientists have long studied Pursuit and Evasion Games (PEGs). In these games, one or more Pursuers attempt to capture an Evader in a given play environment (metric space).

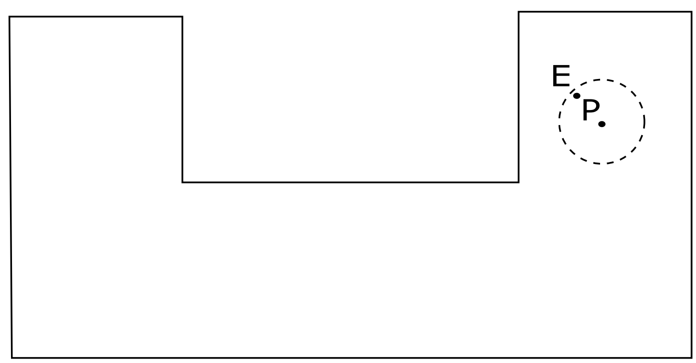
Play environments range from general graphs (where the game is called "cops and robbers") to arbitrary subsets of $\mathbb{R}^n$. (Berry et al., 2015) Typically, the Pursuers and Evader move at the same speed and in discrete time; they take turns moving in straight lines a distance up to their maximum speed. If we drop the requirement for discrete turns, then an Evader can avoid Pursuers in almost any polytopal environment – see (Bollobas et al., 2009) for an overview of the continuous Pursuit and Evasion (Lion and Man) problem.

The game ends when the Pursuer team ends their turn satisfying the capture condition. The capture condition is either co-location or a maximum distance $\epsilon > 0$ between a Pursuer and the Evader (with the Evader in the line-of-sight of that Pursuer). In $\mathbb{R}^n$, the choice of ϵ only changes the number of steps necessary for capture, not which environments permit capture. (Berry et al., 2015) The Pursuer team wins if they capture the Evader. For any environment, the most basic question is whether a single Pursuer can capture the Evader. This is a special case of the question: for any given environment, what is the minimum number of Pursuers needed to guarantee that the Pursuer team can capture the evader? (Berry et al., 2015)

At a higher level, we say that a particular game or environment is



a. Capture conditions are not met because the Pursuer cannot see the Evader (even though they are close).



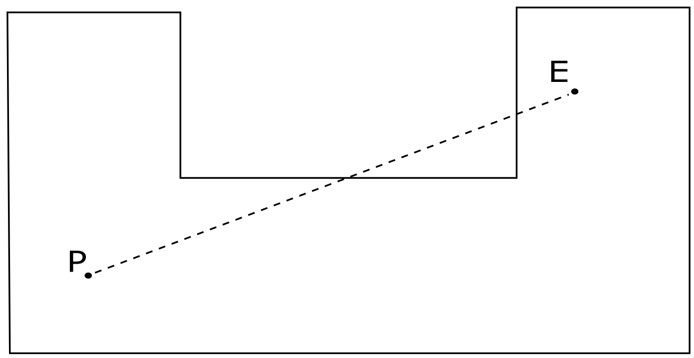
b. Capture conditions are met because the Pursuer is close to the Evader and can see them.

Figure 1.1 In line-of-sight Pursuit, a Pursuer only captures the Evader if they have line-of-sight.

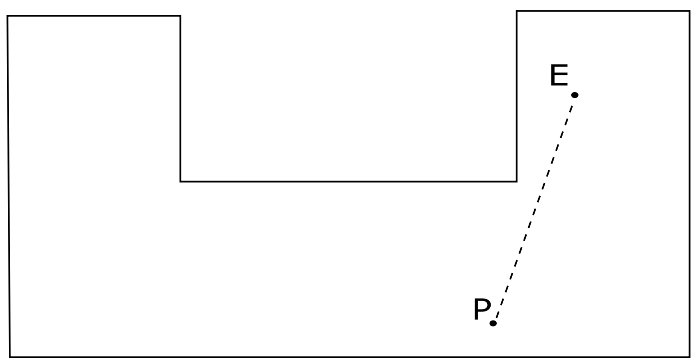
Pursuer-win if there exists a deterministic strategy Pursuer that guarantees that the Evader will be captured. Likewise, a game is Evader win if there is an Evader strategy that prevents capture indefinitely. Note that some games may be neither Pursuer-win nor Evader-win; some environments permit probabilistic Pursuer capture. (Bollobas et al., 2009)

1.2 Line-of-Sight Pursuit and "Lucky" Evaders

The variant we study introduces asymmetric information through the Pursuers's line-of-sight constraint. The Pursuers only know the location of the evader when a line connecting a Pursuer and Evader is contained in the environment. In these models, the Evader retains full information of the Pursuers’ movements and strategy. This simulates the worst case of a “lucky” Evader who always guesses correctly where the Pursuer is and what they will do next.



a. The Pursuer cannot see the Evader



b. The Pursuer can see the Evader

Figure 1.2 In line-of-sight Pursuit, the Pursuer only knows the location of the Evader when the line connecting the Pursuer and Evader is contained in the environment.

In the language of deterministic strategies, we can think of a “lucky” Evader as one that has a strategy designed precisely to counter the Pursuers’. Since the Pursuers must use a deterministic strategy, there exists an Evader strategy that acts as though it has perfect information of the Pursuer’s strategy. Thus, an Evader employing this strategy would appear to be “lucky” and always thwart the Pursuers to the extent possible. However, if we permitted non-deterministic Pursuer strategies this would not be the case; there is no Evader strategy that can predict the outcome of a non-deterministic Pursuer strategy that relies on private random bits. In this model (with non-deterministic Pursuer strategies) (Berry et al., 2015)

1.3 Our Research Interest

We study single-Pursuer, line-of-sight, discrete time, unit-capture Pursuit and Evasion games in $\mathbb{R}^n$. We are interested to know which polytopes admit a deterministic winning strategy for a single pursuer.

Chapter 2

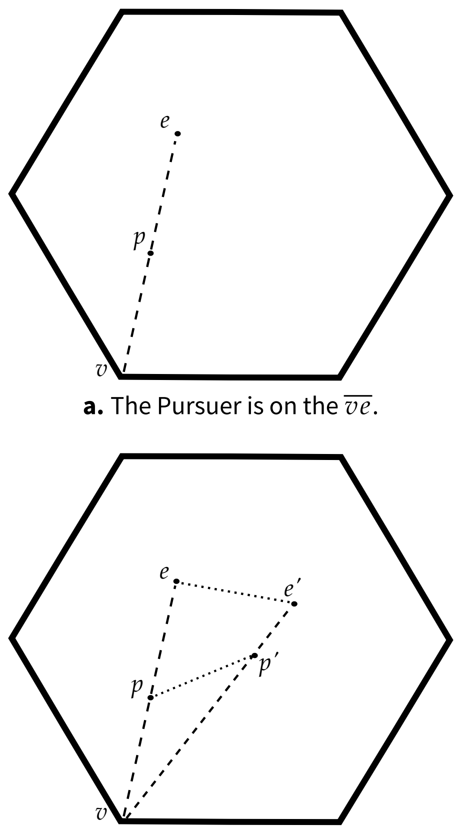
Preliminaries and Prior Results

2.1 The Game

Recall that the Pursuer and Evader are points that each move at the same fixed speed within a polytope. They move in straight lines up to unit distance (i.e. they must take the shortest path from their start point to end point and cannot take a path if it is not contained in the environment). The Pursuer captures the evader if at the end of the pursuer's move, the distance between the evader and pursuer is no more than the *capture distance* $\epsilon > 0$ and the shortest path between them is contained in the environment. While the Evader has full knowledge of the Pursuer's moves, the Pursuer only knows the location of the evader when the line segment connecting the Pursuer to the Evader is contained in the environment.

2.2 Pursuit in Convex Polytopes

In our model of the game (single-pursuer, line-of-sight, discrete time, ϵ-capture) all convex polytopes are Pursuer-win. One winning Pursuer strategy is known as the **Lion's Strategy**. First, the Pursuer moves to one of the vertices, v of the the polytope. At this point, the Pursuer (p) is on the line from v to the Evader (e). After each Evader move, the Pursuer simply maintains this invariant ($\overline{ve}$ contains p) and advances on p. Since the Pursuer can match any move made by the Evader by moving less than a unit distance, the Pursuer can advance along $\overline{ve}$ with each move. Eventually the Pursuer gets arbitrarily close to e and the Pursuer wins. (Bollobas et al., 2009)



a. The Pursuer is on the $\overline{ve}$.

b. The Pursuer can advance away from v and stay on $\overline{ve}$ without moving further than the Evader does.

Figure 2.1 The Lion's strategy guarantees capture in any convex polytope.

2.3 Evader-Win Polytopes

An environment is evader-win if there is an Evader strategy that prevents capture indefinitely, even by a non-deterministic Pursuer.

One can fairly easily construct polytopes that are Evader-win. One simple example is a torus in $\mathbb{R}^3$ or an annulus in $\mathbb{R}^2$. Since the Evader has perfect knowledge of all Pursuer moves, they can always move in the same direction as the Pursuer – if the Pursuer moves clockwise around the torus, so does the Evader. This strategy allows the Evader to flee from the Pursuer indefinitely along the circular track.

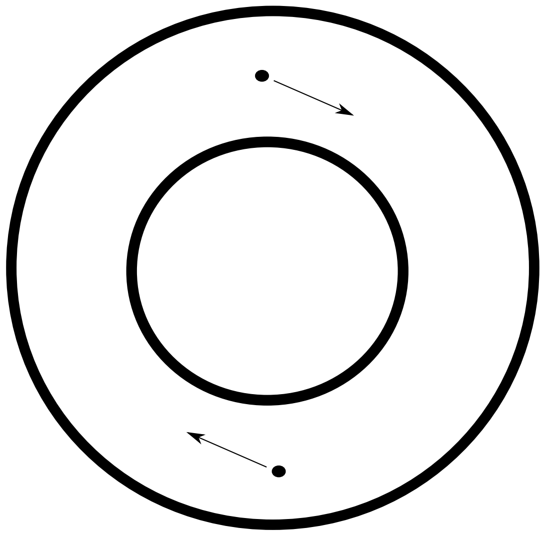


Figure 2.2 Futile pursuit in an Evader-win polytope.

2.4 Polytopes that Require Non-Determinism

Some polytopes (those that are neither Pursuer-win nor Evader-win) admit a non-deterministic Pursuer strategy that eventually captures the Evader. Consider a polygon A consisting of a central chamber and four long zig-zagging hallways branching from the main chamber (Figure 2.3). A crafty Evader can plant themselves in the first few turns of the hallway and move deeper into the hallway if the Pursuer begins to explore that hallway. By moving deeper into the hallway any time the Pursuer could make a move that would give sight of the Evader, the Evader forces the Pursuer to explore the hallway to its end to determine whether the Evader was originally in the hallway. If the Pursuer chooses to explore a hallway that does not contain the Evader, this gives the Evader enough time to switch which hallway it hides in. Thus, the Pursuer is forced to repeatedly guess which of the hallways contains the Evader, each time knowing only one hall in which the Evader cannot be lurking. It follows that for any deterministic Pursuer strategy, there exists an Evader strategy that anticipates the Pursuer strategy and hides in a hallway that the Pursuer will not check. This polygon is not Pursuer-win.

However, A is also not Evader-win. A simple non-deterministic Pursuer strategy (guess a hallway, explore it to the end, repeat) defeats any potential winning Evader strategy. Thus A provides an example of a simply con-

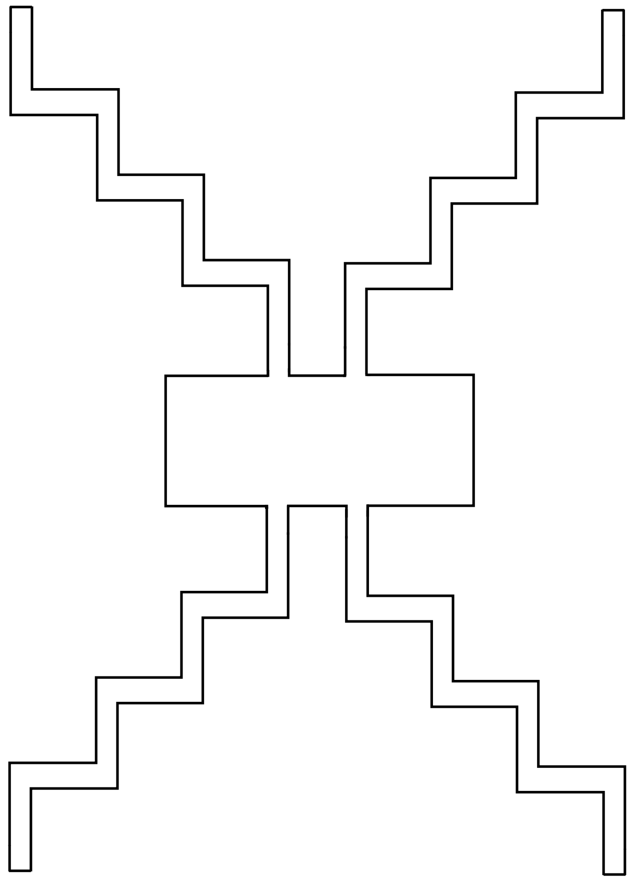


Figure 2.3 A polytope that requires a non-deterministic Pursuer strategy.

nected polygon (polytope) that is neither Pursuer-win nor Evader-win and which admits a non-deterministic winning Pursuer strategy.

This example is particularly relevant because we use analogous polytopes to show that certain classes of polytopes are not Pursuer-win.

2.5 Best Known Results on Polygons

Berry et al. (2015) provides the current most general result for single-Puruser, line-of-sight pursuit in two dimensions. The authors prove that if a Polygonal environment is monotone, scallop, or strictly sweepable, then it is Pursuer-win (under the same restrictions as above). These classes of polygons are defined below.

Definition 1 (Monotone). *A polygon Q is monotone if a straight line segment H can be continuously along moved some axis L such that*

1. $H \cap Q$ *is always convex*
2. *Every point in Q is swept out exactly once.*

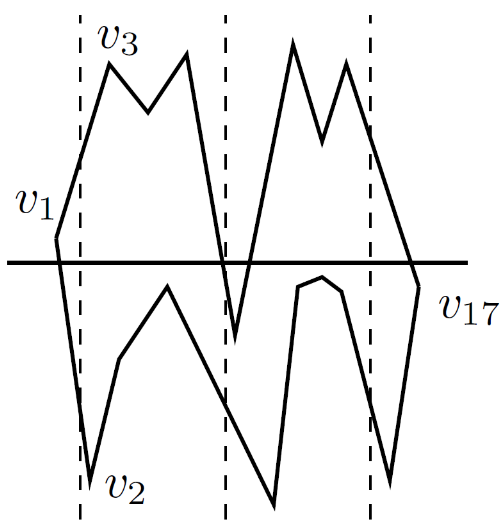


Figure 2.4 A Monotone Polygon can be swept out by translating a line segment along an axis. Figure from (Berry et al., 2015).

Definition 2 (Scallop)**.** *A polygon Q is scallop if a straight line segment H can be rotated continuously around one of its endpoints such that*

1. *$H \cap Q$ is always convex*
2. *Every point in Q is swept out exactly once.*

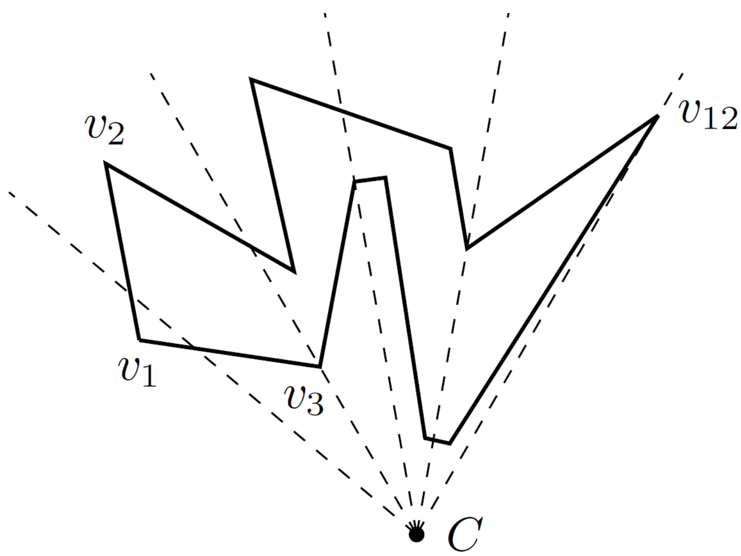


Figure 2.5 A Scallop polygon can be swept out by rotating a line segment about a point. Figure from (Berry et al., 2015).

Definition 3 (Strictly Sweepable)**.** *A polygon Q is strictly sweepable if a straight line segment H can be moved continuously over Q such that*

1. *$H \cap Q$ is always convex*
2. *Every point in Q is swept out exactly once.*

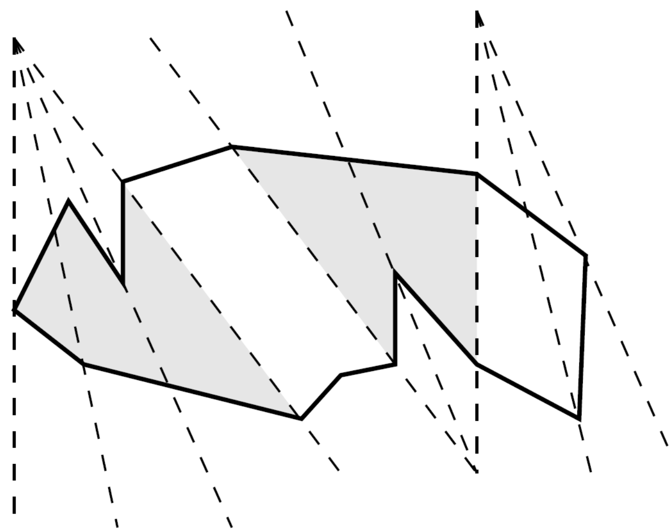


Figure 2.6 A strictly sweepable polygon can be swept out by translating and rotating a line segment over it. Figure from (Berry et al., 2015).

For all three types of polygons, (Berry et al., 2015) use a technique they call the Rook's Strategy. The strategy starts by constructing a search path that traverses the polygon left-to-right to locate the Evader. Following the search allows the Pursuer's line-of-sight to sweep out the entire polygon, meaning that it will eventually spot the Evader. Once the Evader is spotted, the Pursuer switches to "Rook Mode" in which it matches every horizontal move made by the Evader. During this process, the Pursuer maintains a horizontal offset from the Evader, though not one large enough that the Evader could cross the Pursuer' horizontal without inviting capture. By matching every horizontal motion of the Evader, the Pursuer maintains two important invariants:

- The Pursuer's horizontal distance from the Evader is no more than one unit.

- The Evader cannot cross the Pursuer's horizontal without inviting capture.

The offset is particularly important because it provides an opportunity for the Pursuer to advance its horizontal and corner the Evader. If the Evader moves in the direction of the offset (that is, the Evader is to the left of the Pursuer and moves farther left), the Pursuer can simply follow the Evader moving left. However, if the Evader moves opposite to the offset (the Evader is to the left of the Pursuer and the Evader moves right), the "doubles back" on itself and cannot end a full unit distance from the Pursuer. This gives the Pursuer a chance to advance their horizontal (move vertically towards the Evader). Finally, since the polygon is finite, the Evader must double back on the offset eventually – it can't run in one direction forever. (Berry et al., 2015)

Unfortunately, the Evader has ways to fight back. The Evader can delay pursuit by breaking line-of-sight with the Pursuer. In this case, the Pursuer must revert to search mode and continue with a search path. Thankfully, (Berry et al., 2015) show that reverting to a carefully constructed search path forces the Evader farther left in the polygon. Since the polygon is finite, eventually the Evader is forced to the left of the polygon and has no more corners to hide around. From here Rook mode guarantees capture.

Our work leverages the Rook's strategy from (Berry et al., 2015), but does not draw on the idea of a search path. Rather, we adapt the the Rook mode approach of maintaining an offset, matching moves, and waiting for a double-back to extend strategies for low-dimensional polytopes to high-dimensional polytopes.

Chapter 3

Pursuit in Generalizations of Monotone Polygons

In this chapter, we study polytopes that are natural generalizations of monotone polygons to higher dimensions. We'll show that some simple generalizations, such as *monotone prisms*, are Pursuer-win via a strategy that leverages the work of (Berry et al., 2015). However, we'll also show that other natural generalizations, such as *monotone products* are not Pursuer-win – they do not permit a deterministic Pursuer strategy that guarantees capture.

3.1 Monotone Prisms in 3 Dimensions

In this section, we prove that a restricted set of polytopes are Pursuer-win. These polytopes, which we call monotone prisms, are among the simplest generalizations of monotone polygons in $\mathbb{R}^2$. Intuitively, we'll think of monotone prisms in $\mathbb{R}^3$ as prisms formed by extruding a monotone polygon from $\mathbb{R}^2$ into the z direction. In $\mathbb{R}^n$, $n > 3$ we can think of monotone prisms as Cartesian products of monotone polygons with closed intervals in orthogonal directions.

Definition 4 (Monotone Prism)**.** *A polytope $T \subset \mathbb{R}^n$ is a Monotone Prism if there exists a monotone polygon $Q \subset \mathbb{R}^2$ such that $T = Q \times I_1 \times \cdots \times I_{n-1}$ for some intervals $I_1, \ldots, I_{n-2}$ in different dimensions of $\mathbb{R}^n$. In $\mathbb{R}^3$, T is a monotone prism if it is a monotone polygon in the xy-plane with uniform thickness in the z-direction.*

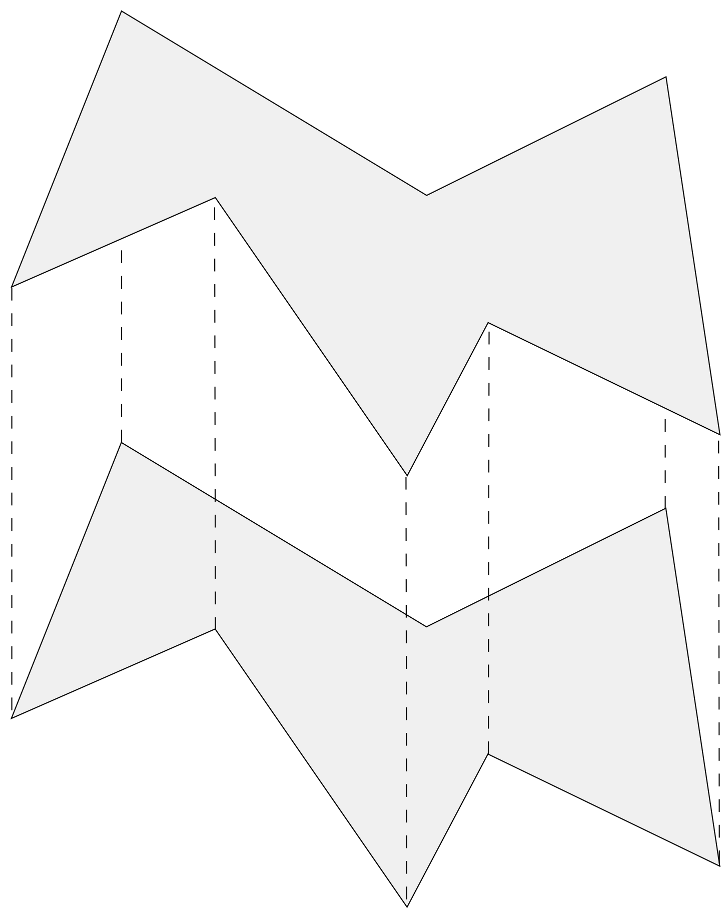


Figure 3.1 A example monotone prism in $\mathbb{R}^3$.

Our Pursuer strategy in monotone prisms will rely on chasing the Evader's projection (or "shadow") onto the original polygon Q. Here a small lemma proves helpful:

Lemma 1. *Let p be a point in a monotone prism T in $\mathbb{R}^3$. Then p can see the point $b \in T$ if and only if p can see the points corresponding to b in all copies of Q. (In other words - p can see b if and only if it can see the projection of b onto its copy of Q.)*

Proof: One direction is easy - if p can see all copies of b, then it can see b in particular. In the other direction, if p cannot see some copy b', then there must be a point q on $\overline{pb'}$ that is not in T. Since q must share x and y

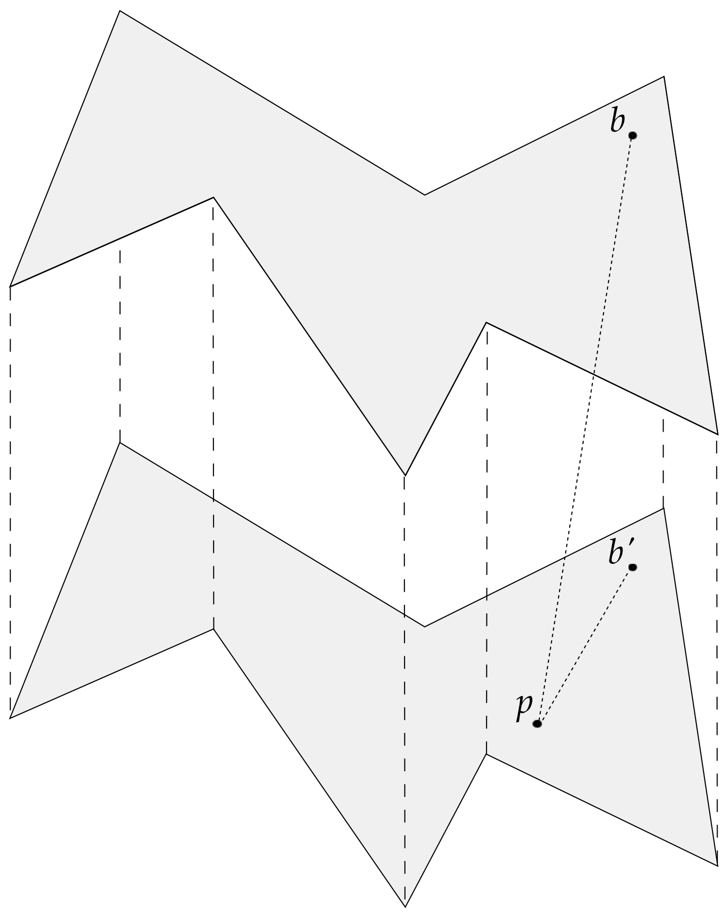


Figure 3.2 If anything in blocks the path from p to b', then it also blocks the path to b.

coordinates with some point q' on $\overline{pb}$ we know that $q' \notin T$ and p cannot see b.

■

This lemma means that the Pursuer can, when convenient, pretend that the Evader is confined to the same copy of Q as them and chase their shadow – no information is lost or gained by doing so.

Theorem 1 (Prisms of Small Thickness)**.** *Let $w(T)$ denote the thickness of a monotone prism T in $\mathbb{R}^3$. Then T is pursuer win if $w(T) < 1$.*

Proof: Our strategy is relatively simple: the Pursuer pretends that both it and the Evader are confined to the same copy of Q. Since the Pursuer can see the evader if and only if it can see the point corresponding to the evader on its own copy of Q, the fact that the Evader is in a different copy of Q does not make its shadow any more difficult to capture. Thus, the Pursuer can simply use the Rook's Strategy as described in (Berry et al., 2015) to converge on the point corresponding to the Evader its copy of Q. Since this strategy works regardless of the capture distance $\epsilon > 0$, we know that following this strategy the Pursuer will eventually get within $\sqrt{1 - w(T)^2}$ of the shadow of e. e can be no further than $w(T)$ away in the z direction, so this suffices to capture e.

■

Corollary 1. *If T is a monotone prism in $\mathbb{R}^3$ with $w(T) < 2$ then T is pursuer win.*

Proof: The Pursuer behaves as above except that it confines itself to the middle copy of Q. When p is in the middle copy of Q, the z-coordinate of e differs from p by no more than $\frac{w(T)}{2} < 1$, so approaching within $\sqrt{1 - \left(\frac{w(T)}{2}\right)^2} > 0$ of e's shadow guarantees capture.

■

These modest results demonstrate an important Pursuer technique: chasing the "shadow" of the Evader. In our next theorem, we use this idea and a slightly cleverer closing strategy to show that all all monotone prisms in $\mathbb{R}^3$ are pursuer-win.

Theorem 2. *If T is a monotone prism in $\mathbb{R}^3$ then T is Pursuer-win.*

Proof: As before, the Pursuer chases the shadow of the Evader in the Pursuer's copy of Q. However, instead of stopping when it captures e's shadow, the Pursuer (p) simply runs the 2-dimensional monotone pursuit algorithm (from Berry et al. (2015)) until it is within unit distance of e's shadow. From here on out, the Pursuer maintains an important invariant: after the each move, p is within unit distance of e's shadow. This invariant provides a few important properties:

1. No matter how e moves, this invariant can be maintained by moving to wherever e's shadow used to be. Since e can only move in straight lines up to a unit distance, its shadow on step i must always be able

to see its shadow at step $i + 1$. Thus, even if e makes a move that cuts off line of sight from p, the Pursuer can re-establish line-of-sight by moving to wherever e's shadow was before the line-of-sight-breaking move.

2. The Evader cannot cross the Pursuer's copy of Q without inviting capture – if e had just crossed p's copy of Q, then moving onto e's shadow guarantees capturing e – e's old shadow is at least as close to its new position as was its old position, and the move distance is equal to the capture distance.

However, simply following e's shadow around is not enough – the Pursuer needs a way to advance its copy of Q and corner th Evader. Borrowing from the two-dimensional Rook's strategy, the Pursuer can closes with e whenever e doubles back on its offset from P. In other words, if e is to the left of p and makes a move to the right, p only needs to use a fraction of its move distance to match e's rightward move, since it was already to the right of e. With the left over move distance, p can advance its copy of Q by moving in the z direction towards e.

Moving on top of e's old shadow guarantees that this process will never be interrupted (since e cannot break line-of-sight) but does not guarantee progress (moving onto e's shadow does not advance the copy of Q). However, since the polytope is finite, we are guaranteed progress eventually – e must eventually double back its offset from p, providing an opportunity to advance p's copy of Q. This strategy allows a single line-of-sight Pursuer to capture an Evader in a monotone prism in $\mathbb{R}^3$.

■

3.2 Pursuit in General Monotone Prisms

To show that general monotone prisms (in $\mathbb{R}^n$) are Pursuer-win, we just need to show that we can compose shadow chasing techniques in successive dimensions.

Theorem 3 (Pursuit in Monotone Prisms)**.** *If T is a monotone prism in $\mathbb{R}^n$ then T is Pursuer-win.*

Proof: Let T be a monotone prism in $\mathbb{R}^n$. We proceed by induction on n. In the base $n = 3$ and our previous result shows that $T \in \mathbb{R}^3$ is Pursuer-win.

Now suppose that all monotone prisms in $\mathbb{R}^k$ are Pursuer-win. Given a monotone prism in $T \in \mathbb{R}^{k+1}$, let Q be the base monotone polygon. Then we can express T as $T' \times I$ where T' is a monotone prism of Q in $\mathbb{R}^k$ and I is an interval in the $k+1$-st dimension of $\mathbb{R}^{k+1}$. By an analogous argument to lemma 1, P can see E's shadow in its copy of T' if and only if it can see E. Thus P can capture E's shadow in its T' (because $T' \in \mathbb{R}^k$ is Pursuer-win). Once P has captured E's shadow in T', it proceeds as in the 3-dimensional case, advancing its copy of T' and moving to the previous location of E's shadow when necessary. This strategy captures E in T.

■

Chapter 4

Pursuit in General Prisms

In the previous chapter, we proved that any finite dimensional monotone prism is Pursuer-win. However, our proof only required that the base polygon Q from which the prism is formed be Pursuer-win, not necessarily monotone. This means that our results about monotone prisms readily generalize to any prism formed from a Pursuer-win base polygon.

Theorem 4 (Pursuit in General Prisms)**.** *If $T = Q \times I_1 \times \cdots \times I_{n-2}$ is a prism in $\mathbb{R}^n$ with Q a Pursuer-win polygon, then T is Pursuer win.*

Proof: As in monotone prisms, our sight lemma holds: a point p can see e if and only if it can see the shadow of e in every copy of Q. Equipped with this lemma, we can use the same algorithm as for monotone prisms. First, run the algorithm that allows p to get within one unit of $e's$ in p's copy of Q (one must exist since Q is Pursuer-win). Then employ our adapted Rook's strategy in successive dimensions (as in the inductive proof for monotone prisms) to capture e in each of these dimensions. Since the details of this proof are precisely the same as those in proof that monotone prisms are Pursuer-win (no part of our shadow-chasing strategy relied on monotonicity), we omit them here.

■

Chapter 5

Conclusion

We study line-of-sight pursuit in arbitrarily high dimensional polytopes. Though we have shown that monotone prisms are Pursuer-win, we doubt that this is the most general class of Pursuer-win polytopes. In particular, we would like to know if those polytopes which are weakly monotone in the convex sense admit Pursuer victory. Note that all monotone prisms are weakly monotone in the convex sense. These polytopes are a natural analog to the two dimensional monotone polygons on which Berry et al. (2015) enjoyed so much success.

5.1 Related Questions

Our work focuses on line-of-sight pursuit in 3 or more dimensions. However, there is still work to be done in 2 dimensional environments. Likewise, there are other models (omniscient pursuer, multi-pursuer, different underlying spaces) that engender interesting questions. Future researchers may consider the following open questions:

1. What is the most general class of polygons (in $\mathbb{R}^2$) in which a single line-of-sight pursuer can capture an evader? The authors of Berry et al. (2015) suggest that their strategy might generalize to (non-strictly) sweepable and straight walkable polygons.

2. What do Evader-win polytopes looks like? Are there any simply connected polytopes that are Evader-win?

3. Are all polytopes that are weakly monotone in the convex sense Pursuer-win? Or only monotone prisms?

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