CODEE Journal

Volume 17 *Engaging the World: Differential Equations Can Influence Public Policies*

Article 10

1-4-2024

ODE Models of Wealth Concentration and Taxation

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Keywords: wealth inequality, oligarchy, transcritical bifurcation
Manuscript received on April 25, 2023; published on January 4, 2024.

Abstract: We refer to an individual holding a non-negligible fraction of the country's total wealth as an *oligarch*. We explain how a model due to Boghosian *et al.* [2] can be used to explore the effects of taxation on the emergence of oligarchs. The model suggests that oligarchs will emerge when *wealth* taxation is below a certain threshold, not when it is above the threshold. The underlying mechanism is a transcritical bifurcation. The model also suggests that taxation of *income and capital gains* alone cannot prevent the emergence of oligarchs. We suggest several opportunities for students to explore modifications of the model.

1 Introduction

The concentration of wealth in the hands of a small number of individuals is considerable in many countries around the world, including the United States. In 2020, according to ref. [7], the 400 wealthiest Americans held 3.5% of the country's total wealth. (The same article also estimated all 10 million [5] Black households combined to hold a total of 3% of all U.S. wealth..) We present here a model extracted from Boghosian *et al.* [2], which sheds light on the effects of taxation on wealth concentration.

Individuals who hold a significant fraction of society's total wealth, and therefore considerable political power [3], will be called *oligarchs* here. The model suggests the existence of a minimal rate of wealth taxation below which oligarchs may emerge. The underlying mathematical mechanism is a *transcritical bifurcation* [6].

We present and analyze the model in Section 2, discuss some variations in Section 3, and suggest problems for students to think about in Section 4. We very briefly indicate in Section 5 what is captured by the full model and analysis in [2], but is not reproduced in our discussion here. The Appendix sketches solutions for the problems of Section 4.

2 Oligarchy and a flat wealth tax

We start by assuming that there are oligarchs holding a fraction $w \in (0, 1)$ of society's wealth. We assume that in a short time dt, the oligarchs, by virtue of the power that their wealth gives them, acquire a small fraction λdt of the wealth not yet in their hands:

$$\frac{dw}{dt} = \lambda(1 - w). \tag{2.1}$$

As w increases, acquiring wealth becomes easier for the oligarchs: They can hire more lobbyists and better laywers, they can afford riskier but more profitable investments, and so on. It is therefore reasonable to assume that λ is proportional to w, so

$$\lambda = gw \ \text{ for some constant } g > 0. \tag{2.2}$$

You might think of g as measuring the rate at which the oligarchs grab what's not theirs yet; hence g, the first letter in "grab". We thereby arrive at the logistic growth equation

$$\frac{dw}{dt} = gw(1 - w). \tag{2.3}$$

Suppose now that the government taxes the oligarchs, taking away a small fraction rdt of their wealth in time dt, where $r > 0$ is another parameter. We use r because it is the first letter in "rate" (of taxation). This is called a *flat* tax because r does not dependent on w. The equation now becomes

$$\frac{dw}{dt} = gw(1 - w) - rw. \tag{2.4}$$

Equation (2.4) has two fixed points, obtained by setting the right-hand side equal to zero and solving for w:

$$w = 0 \quad \text{and} \quad w = 1 - \frac{r}{g}. \tag{2.5}$$

This is illustrated in Figure 1.

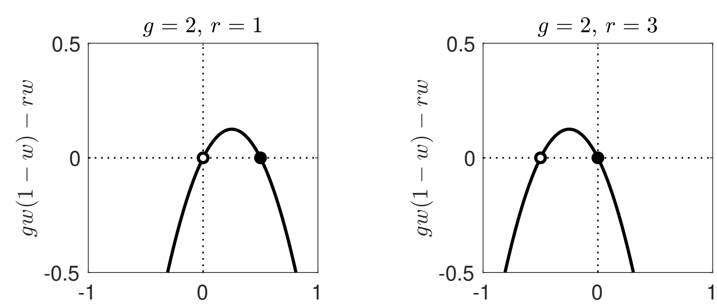

Figure 1: The right-hand side of equation (2.4) for $r < g$ (left) and for $r > g$ (right).

For $r < g$, the fixed point $w = 1 - \frac{r}{g}$ is positive and attracting, and 0 is repelling.[1] In keeping with common convention, this is indicated by open circles and closed circles in

[1]In general, if f is a differentiable function and w_* a number with $f(w_*) = 0$, then w_* is an attracting fixed point of $\frac{dw}{dt} = f(w)$ if $f'(w_*) < 0$, since solutions starting slightly left of w_* move up to w_*, and solutions starting slightly right of w_* move down to w_*. Similarly, w_* is a repelling fixed point if $f'(w_*) > 0$.

Figure 1. For $r > g$, the fixed point $w = 0$ is attracting, and $1 - \frac{r}{g}$ is negative and repelling. The interpretation is that for $r < g$, there will always be oligarchs (the attracting fixed point is positive), while for $r > g$, there will eventually be none. We have here an example of a *transcritical bifurcation* [6]. The two fixed points "collide" as r moves through g, and "exchange their stability properties"; see Figure 2.

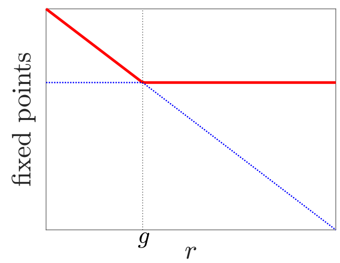

Figure 2: The fixed points as a function of r for a fixed g. The attracting fixed point is indicated in red and bold.

The transcritical bifurcation that occurs as r rises above g can also be achieved by reducing g below r, that is, by reducing the wealth-acquired advantage — perhaps through public policy measures such as regulations on lobbying [1]; see Figure 3.[2]

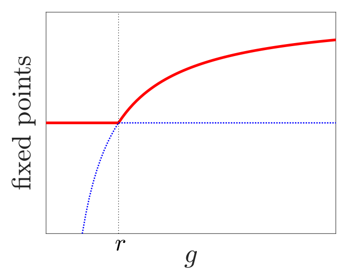

Figure 3: The fixed points as a function of g for a fixed r. The attracting fixed point is indicated in red and bold.

In summary, the model suggests that taxing wealth sufficiently heavily can prevent oligarchy.

3 Variations

3.1 Flat income tax

To model the taxation of *income* (or capital gains), assume the government takes away a fixed fraction $\theta \in (0, 1)$ of the wealth gained by the oligarchs in time dt:

$$\frac{dw}{dt} = (1 - \theta)gw(1 - w). \tag{3.1}$$

[2] Compare also with [2, Figure 1].

If $0 < \theta < 1$ and $g > 0$, equation (3.1) has the attracting fixed point $w = 1$ and the repelling fixed point $w = 0$. The model therefore suggests that income taxation alone does not prevent (total) oligarchy in the long run.

3.2 Progressive wealth tax

Progressive wealth taxation would mean that r itself grows with w. Assume $r = r_0 P(w)$, where r_0 is a fixed positive constant, and P is a continuous,[3] monotonically increasing function of $w \in [0, 1]$ with $P(0) = 1$. The equation now becomes

$$\frac{dw}{dt} = gw(1-w) - r_0 P(w) w. \tag{3.2}$$

Fixed points are obtained by setting the right-hand side equal to 0. One fixed point is always $w = 0$, corresponding to the absence of oligarchs. Other fixed points, if there are any, must solve

$$g(1-w) = r_0 P(w). \tag{3.3}$$

To understand the solutions of equation (3.3), we plot the left-hand side as a function of $w \in [0, 1]$, and, in the same plot, the right-hand side as well. Solutions of equation (3.3) are the abscissae of points where the two graphs meet. Examples are shown in Figure 4. If $r_0 < g$, there exists a positive fixed point and it is attracting, while the fixed point 0 is repelling then. If $r_0 > g$, there exists a negative fixed point, but no positive one, and it is repelling, while the fixed point 0 is attracting.

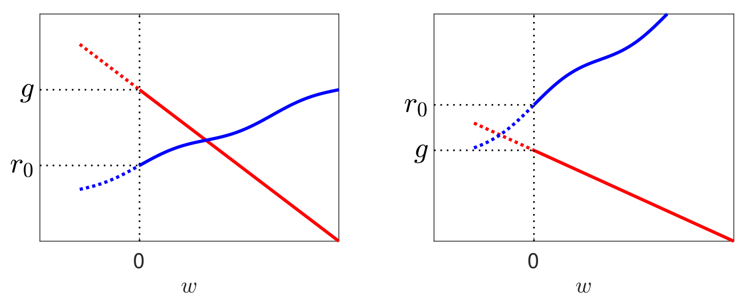

Figure 4: The left (red) and right (blue) sides of equation (3.3), for $r_0 < g$ and $r_0 > g$.

The conclusion is that a progressive wealth tax at rate $r_0 P(w)$ yields the same threshold as a flat wealth tax at rate r_0: Oligarchs emerge for $r_0 < g$, and not for $r_0 > g$. However, for $r_0 < g$, the positive fixed point will be smaller with a progressive wealth tax than it would be with a flat wealth tax at rate r_0, so the oligarchs will hold less of society's wealth; see Problem 4.3.

[3] We should assume that P is *Lipschitz* continuous to remove any doubts regarding the existence and uniqueness of solutions of equation (3.2), but such technical points are not our focus here.

4 Suggested problems

Problem 4.1. Modified model of the wealth-acquired advantage. Equation (2.2) seems a bit arbitrary. While it seems plausible that λ should be an increasing function of w, why should it be proportional to w? What happens if $\lambda = gw^2$ for instance? Then equation (2.4) is replaced by

$$\frac{dw}{dt} = gw^2(1-w) - rw.$$

Determine the fixed points of this equation, study the bifurcation that occurs as r is raised or g is lowered, and interpret the results in terms of the emergence or absence of oligarchs. Try other modifications that make sense to you as well.

Problem 4.2. Progressive income tax. In Section 3.1, the assumption was that of the oligarchs' gains in time dt, namely $gw(1-w)dt$, a *fixed* fraction θ was taken away by the government. We now assume instead that the fraction taken away is not fixed, but is a function of $x = gw(1-w)$. We call this function $Q = Q(x)$. We assume Q to be a monotonically increasing function of $x \geq 0$ with $Q(0) \geq 0$. The greater the oligarchs' gain in a unit of time, the greater a percentage of their gains will be taken away from them; the income tax is *progressive* now. You may want to assume that the government will never take away as much as 100% of the oligarchs' gains.[4], so $Q(x) < 1$ for all x. Now the equation is

$$\frac{dw}{dt} = (1 - Q(gw(1-w)))gw(1-w).$$

Will oligarchs be able to persist?

Problem 4.3. Progressive wealth tax re-visited. Assume a progressive wealth tax with $r_0 < g$. Explain why the positive fixed point of equation (2.4) is closer to zero than it would be with a flat wealth tax at rate r_0.

Problem 4.4. Regressive wealth tax. Suppose that as w increases, the oligarchs, by virtue of their increasing political power, manage to reduce the wealth tax rate. So the equation is (3.2), but now P is a *decreasing* function of w with $P(0) = 1$ and $P(w) \geq 0$. Analyze what happens. (It will depend on what you assume about the function P.)

5 What the full model is about

We mentioned that the model suggested here is extracted from a model due to Boghosian *et al.* [2]. What would you learn by reading [2] that you cannot learn here? The essential answer is that you would learn about the *distribution* of wealth, not merely about the fraction of wealth held by the oligarchs. As a result, the model of [2] predicts things like the Lorenz curve and the Gini coefficient. Our equation (2.4), supported here by

[4]The Swedish children's book author Astrid Lindgren complained in 1976 that her marginal tax rate had risen to 102%. She wrote a satirical story about it, published in the Swedish tabloid *Expressen* [4]. It should be said that there is a distinction between the *marginal* tax rate — the rate at which the last dollar of income is taxed — and the *effective* one. Lindgren, by the way, supported the Swedish Social Democrats, who had introduced those steeply progressive income tax rates, throughout her life, even after 1976.

plausibility arguments only, is *derived* in [2] (see equation 17 in [2]) from a model of the time evolution of the wealth distribution.

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Appendix: Sketches of solutions for the problems

The problems are intended to inspire the reader to investigate on their own, and the solution sketches given here are not meant to undermine that, but to assist a reader who would like help getting started.

4.1. **Modified model of the wealth-acquired advantage**. For the modified equation, $w = 0$ is still a fixed point. Since the linearization of $gw^2(1-w) - rw$ around $w = 0$ is $-rw$, the fixed point $w = 0$ is always attracting now. Other fixed points must satisfy

$$w(1-w) = \frac{r}{g}.$$

Plot $w(1-w)$ as a function of w to see that there are no solutions of this equation when $r > \frac{g}{4}$, and there are two solutions, both strictly between 0 and 1, when $r < \frac{g}{4}$. The larger of the two positive fixed points is then attracting, and the smaller is repelling. If w is perturbed from 0 slightly, it returns to the attracting fixed point 0, but if w is raised beyond the smaller positive fixed point, which acts as a threshold, it converges to the larger one. This is an example of an *excitable* system [6]. The bifurcation that annihilates the two positive fixed points as r rises above $\frac{g}{4}$ is a *saddle-node bifurcation* [6].

Now think about other possible relations between λ and w.

4.2. **Progressive income tax**. If $Q(x) < 1$ for all x, the only two fixed points are still $w = 0$ and $w = 1$, and $w = 0$ is repelling, $w = 1$ is attracting. But what if we allow $Q(x) > 1$?

4.3. **Progressive wealth tax re-visited**. Ask yourself what the left panel of Figure 4 would look like if the wealth tax were flat.

4.4 **Regressive wealth tax**. The simplest example would be $P(w) = 1 - w$. Equation (3.2) then becomes

$$\frac{dw}{dt} = (g - r_0)w(1-w).$$

This equation has fixed points at $w = 0$ and $w = 1$. For $r_0 < g$, the fixed point at 0 is repelling, and that at 1 is attracting. For $r_0 > g$, the fixed point at 0 is attracting, and that at 1 is repelling. So even the regressive wealth tax with $P(w) = 1 - w$ still leads to the conclusion that oligarchy emerges if $r_0 < g$, but not if $r_0 > g$.

If you were to make plots like those in Figures 2 and 3 (with r_0 replacing r), what would they look like now? Is it still a transcritical bifurcation?

Next think about other examples, for instance $P(w) = 1 - w^2$; that would mean that the oligarchs become really effective at fighting the wealth tax only when they have acquired enough wealth.