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Steady State Solutions for a System of Partial Differential Equations Arising from Crime Modeling

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Abstract

I consider a model for the control of criminality in cities. The model was developed during my REU at UCLA. The model is a system of partial differential equations that simulates the behavior of criminals and where they may accumulate, hot spots. I have proved a prior bounds for the partial differential equations in both one-dimensional and higher dimensional case, which proves the attractiveness and density of criminals in the given area will not be unlimitedly high. In addition, I have found some local bifurcation points in the model.

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Chapter 1

Introduction

Crime modeling is a growing field of mathematical modeling. Notable contributions can be attributed to the UCLA models (Short et al., 2008; Bertozzi et al., 2014), which has helped thirty cities worldwide curb crime. This model, hereafter referred to as the “Random Walk Model (RWM)”, assumes that the criminals follow a random walk that is biased toward regions with high "attractiveness". The concept of "attractiveness" is rooted in criminology research. When a crime happens at a given place, that place as well as places nearby become more attractive to similar crimes (Short et al., 2008; Budd, 1999; Weisel et al., 1999). This phenomenon is known as the repeat or near-repeat victimization, which depends on whether a criminal revisits the previous place or a neighboring place respectively. Some have likened this phenomenon to the "broken windows effect" whereby disorder and minor crimes such as jaywalking and littering lead to an increase in major crimes like burglary and murder. In previous crime models, the broken windows effect is usually treated together with repeat and near-repeat victimization as a critical factor determining the change in attractiveness (Harcourt, 1998; Gau and Pratt, 2010; Budd, 1999; Weisel et al., 1999). The Random Walk Model successfully picks up the crime hotspots: i.e., disjoint areas with high crime rates (Short et al., 2010b; H. Berestycki, 2010); and compares favorably with the real data (Short et al., 2010a).

However, the assumption that criminals take a random walk with constant velocity is a very restricted one. In real life, criminals can take a train or other vehicle and move much farther than a similar criminal on foot. As a result, they may move a long distance in a single step. Previous researches have shown that human motions are better modeled with Lévy flights, instead of random walks (James et al., 2011; Brockmann et al., 2006; González et al., 2008). Also, data of distances between homes of criminals and their targets suggests that they are willing to make long trips for valuable targets (Leary, 2011; Snook, 2004). To this end, Chaturapruek et al. developed a new model by assuming that the criminals undergo a Lévy flight (Chaturapruek et al., 2013), i.e., criminals can go anywhere in one time step with a probability proportional to attractiveness and inversely proportional to some power μ of distance (Chechkin A V, Metzler R and Yu, 2008). We refer this model as the "Lévy Flight Model (LFM)" from now on.

Nevertheless, a real criminal can only move as fast as traffic or public transit, i.e., the property of LFM that criminals can take arbitrarily long jumps in a time step does not accurately reflect the reality. Furthermore, movement patterns of different types of criminals can vary greatly. As was shown in (Snook, 2004; Koppen et al., 1998), professionals and older criminals can travel faster than amateur and younger criminals. With these facts in mind, Chaohao Pan and I et.al proposed a Truncated Lévy Flight Model (TLFM) for the movement of criminals during the REU at UCLA in 2015. In this model, we set a speed limit for the Lévy flight to get rid of arbitrarily long jumps by setting a speed limit on the Lévy flight. People have proven that the sum of independent truncated Lévy flights converges to a Gaussian process (Mantegna and Stanley, 1994). In the continuum limit, our model involves only a regular Laplace operator. And actually it differs from the Random Walk model just by a constant. Also, when we make the speed limit large, it is closer to the Lévy flight model. This is because our model takes into account both the "nonlocal" feature of criminals by allowing them long range jumps (which makes the model looks like the Lévy flight model) and the "local" feature of them that is restricted by the speed limit (which makes the model looks like the Random Walk model). Additionally, by setting different speed limits, TLFM can simulate different categories of criminals. For instance, to simulate dynamics of amateur or younger criminals, we take the speed limit to be small, which leads to a pattern that is similar to that generated by RMW. Likewise, if there is a group of professional criminals with better mobility, we can increase the speed limit accordingly. In this case, the generated pattern will look like the pattern generated by LFM, except that the unrealistic long jumps will no longer occur. We also show that, quantitatively, both RWM and LFM are special cases of TLFM. Another remarkable feature of TLFM is that the Laplace operator in the continuum system, other than the fractional Laplace operator in the Lévy Flight Model (Chaturapruek et al., 2013; Zoia et al., 2007), is numerically more amenable to various kinds of boundary conditions.

The rest of paper is organized as follows. In Chapter 2, we review the basic assumption of the repeat and near-repeat victimization, and redo the derivation of the TLFM model, in both discrete and continuum settings, the biased truncated Lévy flight for the criminals. For the continuum system, we perform a linear stability analysis on the homogeneous steady state solution to study the formation of hotspots. We also explore the relations between TLFM with previous RWM and LFM. Then in Chapter 4, we review literatures on the similar models (Berestycki et al., 2014; Cantrell et al., 2012). In Chapter 5, we show our result of the a priori bound in both one-dimensional and higher-dimensional cases. Finally in Chapter 6, we show the existence of bifurcation points of the model.

Chapter 2

Review of the Model

The work in this and next chapter was done in during the REU at UCLA, and I reproduce it here for the completeness of the paper. In this chapter, we describe the model constructed in the REU at UCLA. First, we describe the model in the discrete case. In particular, the discrete model is valid on every connected graph, so we can use it on any city. However, in this paper, we only define the models on a one-dimensional grid graph with grid length l. We apply periodic boundary conditions, by which we see the city as a torus, unless otherwise specified. First, we define some basic terms, including local attractiveness, and the broken window effect. Then, we discuss the evolution of the distribution of the criminals.

2.1 Local attractiveness

As in (Short et al., 2008; Chaturapruek et al., 2013; Jones et al., 2010), we define the attractiveness of each spot at certain time to criminals as following. For each burglary site k, at each time step t, we denote its attractiveness as $\mathbf{A}_k(t)$. The higher the number, the criminals are more likely to travel to that position at the given time, and also more likely to commit crimes at there.

Also, the attractiveness contains two components, the static attractiveness and the dynamical attractiveness, which we separately denote as A_k^0 and $B_k(t)$. A_k^0 is only determined by the position while $B_k(t)$ is related to the effects of repeat and near-repeat victimization. Hence, we will write

$$\mathbf{A}_k(t) = A_k^0 + \mathbf{B}_k(t), \tag{2.1}$$

At each time step, a criminal at a certain position can either commit a crime or move to some other places. The decision he or she makes depends on the attractiveness of each position. We use $\mathbf{E}_k(t)$ to denote the number of crimes committed in the time interval $(t, t+\delta t)$ at site k.

Intuitively, the more crimes committed at a site, the larger attractiveness. Therefore, temporarily neglecting the near-repeat victimization effect, we can then express the evolution of $\mathbf{B}_k(t)$ as

$$\mathbf{B}_k(t+\delta t) = \mathbf{B}_k(t)(1-\omega\delta t) + \theta\mathbf{E}_k(t) \tag{2.2}$$

where ω is the decay rate of the attractiveness field. In addition, θ describes the increase of $\mathbf{B}_k$ for each crime that occurs at k.

Now, if we also consider the near-repeat victimization, the evolution equation of $\mathbf{B}_k(t)$ can be expressed as

$$\mathbf{B}_k(t+\delta t) = \left[(1-\eta)\mathbf{B}_k(t) + \frac{\eta}{2}(\mathbf{B}_{k-1}(t) + \mathbf{B}_{k+1}(t))\right](1-\omega\delta t) + \theta\mathbf{E}_k(t), \tag{2.3}$$

where $\eta \in (0,1)$ is a constant measuring the significance of the near-repeat victimization effect.

In addition to the attractiveness, we use ρ to describe the density of the criminals. Initially, a given number of criminals are distributed over the graph. In the discrete model, we assume that criminals can only take one action in the time interval $[n\delta t, (n+1)\delta t)$, where δt is the length of each time step and $n \in \mathbb{N}$. To make the model more easily to be analyzed, we work on the average number of criminals rather than seeing each criminal as independent person. We use $\rho_k(t)$ to denote the average number of criminals at site k during the time interval $[t, t+\delta t)$.

At each time step, a criminal either moves to another burglary site or commits a crime. We consider a criminal committing a crime in the time interval $(t, t+\delta t)$ at site k as a standard Poisson process. And the parameter of the Poisson process, $p_k(t)$, is defined to be

$$p_k(t) = 1 - e^{-A_k(t)\delta t}, \tag{2.4}$$

where $A_k(t)$ denotes the expectation of $\mathbf{A}_k(t)$. We also denote the expectation of $\mathbf{B}_k(t)$ as $B_k(t)$. Then it follows immediately from (2.1) that

$$A_k(t) = A_k^0 + B_k(t). \tag{2.5}$$

Moreover, by the property of standard Poisson process, the expectation of $\mathbf{E}_k(t)$ is $\delta t A_k(t)\rho_k(t)$.

Thus, by taking the expectation of both sides in (2.3), we obtain

$$B_k(t+\delta t) = \left[(1-\eta)B_k(t) + \frac{\eta}{2}(B_{k-1}(t) + B_{k+1}(t))\right](1-\omega\delta t) + \theta\delta t A_k(t)\rho_k(t). \tag{2.6}$$

2.2 The Discrete Truncated Lévy Flight

In the Random Walk Model, the criminals can only move to a neighboring site in each time step (i.e., they have speed limit 1). By contrast, in the Lévy Flight Model, we allow the criminals to move to any site on the graph (i.e., they have no speed limits).

As in (Chaturapruek et al., 2013), we define the relative transition likelihood of a criminal moving from cite i to cite k, $w_{i\to k}$ subject to the following Lévy power law

$$w_{i\to k} = \frac{A_k}{(l|i-k|)^{\mu}}, \tag{2.7}$$

where $\mu \in (1,3)$. The intuition at here is that although arbitrarily long jumps are theoretically allowed in a Lévy flight, the probability of traveling to a distant site in one time step is low.

However, in contrast with the Lévy Flight Model, for the truncated Lévy flight, we make the rule that no criminals can move more than L gridspaces within one time step, where $L \in \mathbb{Z}, L \geq 1$. Therefore, the relative transition likelihood, still denoted as $w_{i\to k}$, with abuse of notation, can defined as follows

$$w_{i\to k} = \begin{cases} \frac{A_k}{l^{\mu}|i-k|^{\mu}}, & 1 \leq |i-k| \leq L, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

The (normalized) transition probability is then defined as

$$q_{i\to k} = \frac{w_{i\to k}}{\sum_{j\neq i} w_{i\to j}}. \tag{2.9}$$

Following the settings in the Random Walk Model as in (Short et al., 2008), the criminals obey the following rules: in the time interval $(t, t+\delta t)$, a criminal will either

- Commit a crime which obeys the standard Poisson process with parameter $p_k(t) = 1 - e^{-A_k(t)\delta t}$.
- Or else moves on according to a biased truncated Lévy flight.

Also, some new criminals appear with a constant rate Γ. Given the criminal density at time t, the criminal density after one time step can be calculated as

$$\rho_k(t+\delta t) = \sum_{\substack{i\in\mathbb{Z} \\ 1\le|i-k|\le L}} [1 - A_i(t)\delta t]\rho_i(t) q_{i\to k}(t) + \Gamma\delta t. \tag{2.10}$$

2.2.1 The Continuous Limit of TLFM

After that, we take the continuous limit for the above discrete model as δt and l both converge to 0. Firstly by following the same procedure as in (Chaturapruek et al., 2013), we can derive the continuum limit for (2.6) as follows

$$A_t = \frac{l^2\eta}{2\delta t} A_{xx} - \omega(A - A_0) + A\rho\theta, \tag{2.11}$$

where we write $\frac{\partial A}{\partial t}$ as A_t for consistency of notation.

The derivation of the continuous limit for ρ, however, is more difficult, and much different from the process in (Chaturapruek et al., 2013) due to the truncation.

First, we define

$$z_L := 2\sum_{k=1}^{L} \frac{1}{k^\mu}, \tag{2.12}$$

and

$$\mathcal{L}(f_i) := \sum_{\substack{j\in\mathbb{Z} \\ 1\le|i-j|\le L}} \frac{f_j - f_i}{(|j-i|l)^\mu}. \tag{2.13}$$

Then it follows immediately from (2.8) that

$$\sum_{\substack{i\in\mathbb{Z} \\ 1\le|i-k|\le L}} w_{i\to k} = l^{-\mu} z_L A_i + \mathcal{L}(A_i). \tag{2.14}$$

With (2.14) and (2.9), we obtain

$$
\begin{aligned}
q_{i\to k} &= \frac{w_{i\to k}}{\sum_{\substack{j\in\mathbb{Z}\\ 1\le|j-k|\le L}} w_{i\to j}} = \frac{w_{i\to k}}{l^{-\mu} z_L A_i (\frac{\mathcal{L}(A_i)}{l^{-\mu} z_L A_i} + 1)} \\
&\sim w_{i\to k}\left[\frac{1}{l^{-\mu} z_L A_i} - \frac{\mathcal{L}(A_i)}{(l^{-\mu} z_L A_i)^2}\right] \qquad (2.15)\\
&= \frac{A_k}{|i-k|^{\mu}}\left(\frac{1}{z_L A_i} - \frac{\mathcal{L}(A_i) l^{\mu}}{A_i^2 z_L^2}\right), \quad 1 \le |i-k| \le L,
\end{aligned}
$$

where, in the second step, we have applied the approximation $\frac{1}{1+x} \sim 1 - x$ for small x.
We then obtain by (2.10)

$$
\frac{\rho_k(t+\delta t) - \rho_k(t)}{\delta t} = \frac{1}{\delta t}\left[\sum_{1\le|i-k|\le L} \rho_i (1 - A_i \delta t) q_{i\to k} - \rho_k\right] + \Gamma. \qquad (2.16)
$$

Now applying (2.15) to the RHS of (2.16), we obtain

$$
\begin{aligned}
\frac{\rho_k(t+\delta t) - \rho_k(t)}{\delta t} &= \frac{1}{\delta t}\sum_{1\le|i-k|\le L} \rho_i (1 - A_i\delta t)\frac{A_k}{|i-k|^{\mu}}\left(\frac{1}{z_L A_i} - \frac{\mathcal{L}(A_i) l^{\mu}}{A_i^2 z_L^2}\right) - \frac{\rho_k}{\delta t} + \Gamma \\
&= \frac{A_k}{\delta t}\left[\sum_{1\le|i-k|\le L} (1 - A_i\delta t)\frac{\rho_i}{A_i}\frac{1}{z_L|i-k|^{\mu}} - \frac{\rho_k}{A_k}\right] \\
&\quad - \frac{A_k}{\delta t}\sum_{1\le|i-k|\le L}\left[(1 - A_i\delta t)\frac{\rho_i}{|i-k|^{\mu}}\frac{\mathcal{L}(A_i) l^{\mu}}{A_i^2 z_L^2}\right] + \Gamma.
\end{aligned}
\qquad (2.17)
$$

We also find that (2.12) and (2.13) implies

$$
\sum_{\substack{i\in\mathbb{Z}\\ 1\le|i-k|\le L}} \frac{\rho_i}{|i-k|^{\mu}} = \sum_{\substack{i\in\mathbb{Z}\\ 1\le|i-k|\le L}} \frac{\rho_i - \rho_k}{|i-k|^{\mu}} + \sum_{\substack{i\in\mathbb{Z}\\ 1\le|i-k|\le L}} \frac{\rho_k}{|i-k|^{\mu}} = l^{\mu}\mathcal{L}(\rho_k) + z_L\rho_k \sim z_L\rho_k, \qquad (2.18)
$$

where we ignore the $O(l^{\mu})$ term in the final step. Then by (2.18) and (2.17), we obtain

$$
\begin{aligned}
\frac{\rho_k(t+\delta t)-\rho_k(t)}{\delta t} &= \frac{A_k}{\delta t}\sum_{1\le|i-k|\le L}\left[\frac{\rho_i}{A_i}\frac{1}{z_l|i-k|^\mu}-\delta t\frac{\rho_i}{|i-k|^\mu z_L}-\frac{\rho_k}{A_k}\frac{1}{z_L|i-k|^\mu}\right]\\
&\quad-\frac{A_k}{\delta t}\sum_{1\le|i-k|\le L}\left[\frac{\rho_i}{|i-k|^\mu}\frac{\mathcal{L}(A_i)l^\mu}{A_i^2z_L^2}-\frac{\rho_i\mathcal{L}A_i}{A_iz_L^2|i-k|^\mu}l^\mu\delta t\right]+\Gamma\\
&\sim\frac{A_k}{\delta t}\sum_{1\le|i-k|\le L}\left[\frac{\frac{\rho_i}{A_i}-\frac{\rho_k}{A_k}}{|i-k|^\mu z_L}-\frac{\rho_i}{|i-k|^\mu}\frac{\mathcal{L}(A_i)l^\mu}{A_i^2z_L^2}-\delta t\frac{\rho_i}{|i-k|^\mu z_L}\right]+\Gamma\\
&\sim\frac{l^\mu}{z_L\delta t}\left[A_k\mathcal{L}\left(\frac{\rho_k}{A_k}\right)-\frac{\rho_k\mathcal{L}(A_k)}{A_k}\right]-A_k\rho_k+\Gamma,
\end{aligned}
\tag{2.19}
$$

where, at the second step, we ignore the $O(l^\mu\delta t)$ terms in the summation. We also observe that

$$
\mathcal{L}(A_k)=\sum_{\substack{j\in\mathbb{Z}\\1\le|j-k|\le L}}\frac{A_j-A_k}{(|j-k|l)^\mu}=\frac{1}{l}\sum_{\substack{j\in\mathbb{Z}\\1\le|j-k|\le L}}\frac{A_j-A_k}{(|j-k|l)^\mu}l. \tag{2.20}
$$

We make the following changes of variable

$$
x=kl,\, y_j=jl,\, A_j=A(y_j),\, A_k=A(x).
$$

Then the right hand side of (2.20) can be regarded as a midpoint Riemann sum on the interval defined as follows:

$$
I:=\left[x-(L+\frac{1}{2})l,\, x-\frac{1}{2}l\right]\cup\left[x+\frac{1}{2}l,\, x+(L+\frac{1}{2})l\right]. \tag{2.21}
$$

Hence, (2.20) implies

$$
\mathcal{L}(A_k)\sim\frac{1}{l}\int_M\frac{A(y)-A(x)}{|y-x|^\mu}dy, \tag{2.22}
$$

$$
\frac{l^\mu}{z_L\delta t}\mathcal{L}(A_k)=\frac{l^{\mu-1}}{z_L\delta t}\int_M\frac{A(y)-A(x)}{|y-x|^\mu}dy.
$$

As l converges to 0, the integration is local at x, and we can apply Taylor expansion at x on the integrand to obtain

$$
\begin{aligned}
\frac{l^\mu}{z_L \delta t}\mathcal{L}(A_k) &= \frac{l^{\mu-1}}{z_L \delta t}\int_M |y-x|^{-\mu}\left[A_x(x)(y-x) + A_{xx}(x)\frac{(y-x)^2}{2} + O((y-x)^3)\right]dy \\
&\sim \frac{l^{\mu-1}}{z_L \delta t}\left[\int_M \frac{A_x(x)(y-x)}{|y-x|^\mu}dy + \int_M \frac{A_{xx}(x)(y-x)^2}{2|y-x|^\mu}dy\right] \\
&= \frac{l^{\mu-1}}{z_L \delta t}\int_{x+\frac{1}{2}l}^{x+(L+\frac{1}{2})l} |y-x|^{2-\mu}A_{xx}(x)dy \\
&= \frac{l^2}{z_L \delta t(3-\mu)}\left[(L+\frac{1}{2})^{3-\mu} - (\frac{1}{2})^{3-\mu}\right]A_{xx}(x),
\end{aligned}
\tag{2.23}
$$

where, at the second step, we ignore the $O((y-x)^{3-\mu})$ terms, since $|y-x| \ll 1$ and $\mu < 3$. Applying (2.23) to (2.19), we obtain

$$
\rho_t = \frac{l^2}{\delta t z_L(3-\mu)}\left[(L+\frac{1}{2})^{3-\mu} - (\frac{1}{2})^{3-\mu}\right]\left[A(\frac{\rho}{A})_{xx} - \frac{\rho}{A}A_{xx}\right] - A\rho + \Gamma. \tag{2.24}
$$

To simplify the expressions, we reparametrize (2.24) as follows:

$$
A = \bar{A}\omega, \rho = \frac{\bar{\rho}\omega}{\theta}, t = \frac{\bar{t}}{\omega}, \bar{\eta} = \frac{l^2\eta}{2\delta t\omega}.
$$

This together with (2.11) and (2.24) implies (we drop the bars for now)

$$
A_t = \eta A_{xx} - A + \alpha + A\rho, \tag{2.25}
$$

$$
\rho_t = D\left[A(\frac{\rho}{A})_{xx} - \frac{\rho}{A}A_{xx}\right] - A\rho + \beta, \tag{2.26}
$$

where

$$
D = \frac{l^2}{\omega\delta t z_L(3-\mu)}\left[(L+\frac{1}{2})^{3-\mu} - (\frac{1}{2})^{3-\mu}\right], \quad \alpha = \frac{A^0}{\omega}, \quad \beta = \frac{\Gamma\theta}{\omega^2}. \tag{2.27}
$$

Chapter 3

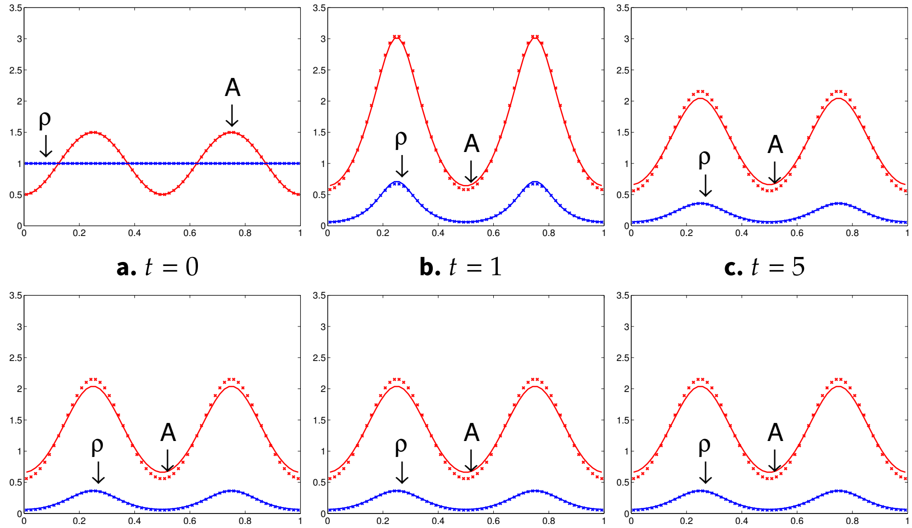
Numerical Work

To verify the derivation of the continuous limit, we compare the solutions of the discrete model (2.6) and (2.10), and the continuum limit (2.25) and (2.26) numerically. For the discrete model, we assume the grid points x_i, $i = 1, 2, \cdots, 60$ with $x_i - x_{i-1} = 1/60$. For the continuum system we consider the computational domain $x \in [0, 1]$ with $\Delta x = 1/60$, $\Delta t = 1/3600$. We use forward Euler method for time discretizations and spectral method for space derivatives. Periodic boundary conditions are implemented in both cases.

For the continuous m periodic boundary conditions for the solution. Therefore, we also apply the periodic boundary condition to the discrete model.

Figure 3.1 shows a comparison between our numerical simulations for the discrete and continuum models. We can observe a great agreement even at the boundary. Eventually, we observe a steady state with two hotspots for both models when time t gets large.

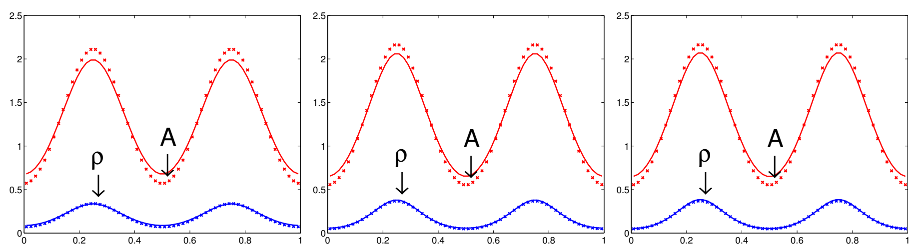
Figure 3.2 compares the steady states of our models when the speed limit L is set to be distinct values, and Figure 3.3 compares the steady states for different values of μ. We see that the models fit fairly well for a large range of L and μ.



a. $t = 0$ **b.** $t = 1$ **c.** $t = 5$

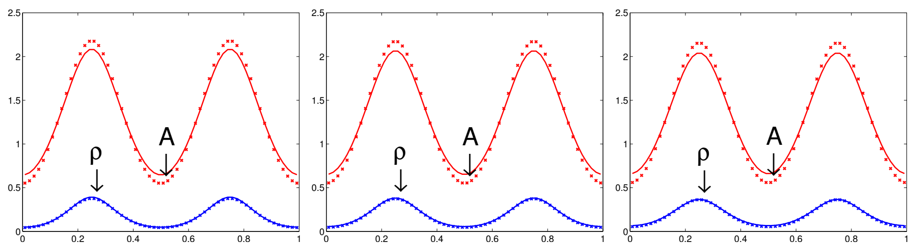
d. $t = 10$ **e.** $t = 50$ **f.** $t = 200$

Figure 3.1 Simulations of biased truncated Lévy flight when $\mu = 2.5$, $l = 1/60$, $\delta t = 0.01$, and the speed limit L is 9. The solid curve represents the continuum limit, and the dots represent the discrete model. The initial conditions are taken to be $A^0 = 1 - 0.5\cos(4\pi x)$ and $\rho = 1$. Other relevant parameters are: $\eta = 0.1, \Gamma = 0.3, \omega = 1$



a. $L = 3l$ **b.** $L = 18l$ **c.** $L = 27l$

Figure 3.2 Comparisons of the steady state obtained using different values of L. All parameters are the same as in Figure 3.1 except for L. The shots are taken at $t = 200$.



a. $\mu = 1.5$ **b.** $\mu = 2$ **c.** $\mu = 2.5$

Figure 3.3 Comparisons of the steady state obtained using different values of μ. All parameters are the same as in Figure 3.1 except for μ. The shots are taken at $t = 200$.

Chapter 4

Literature review

In this chapter, we review two papers that work on the equations very similar to our model.

4.1 Contribution by Berestcyki et. al.

In "Existence of Symmetric and Asymmetric Spikes for a Crime Hotspot Model" by Henri Berestycki et.al, the paper, the authors describe a model very similar to ours, which is

$$\begin{cases} A_t = \epsilon^2 A_{xx} - A + \rho A + A_0(x) \text{ in } (-L, L) \\ \rho_t = D(\rho_x - 2\frac{\rho}{A} A_x)_x - \rho A + \gamma(x) \text{ in } (-L, L) \end{cases} \tag{4.1}$$

As in our equation, A is the attractiveness and ρ is the density of criminals. In addition, ϵ represents nearest neighbor interactions and $\gamma(x)$ is the source term representing in the birth rate of criminals. Then, they do the change of variables by letting $v = \rho/A^2$ so that the equation becomes

$$\begin{cases} A_t = \epsilon^2 A_{xx} - A + vA^3 + A_0(x) \\ (A^2 v)_t = D(A^2 v_x)_x - vA^3 + \gamma(x) \end{cases} \tag{4.2}$$

Unlike us, they apply the Neumann boundary condition for the equations. They established the existence of steady states with multiple spikes in symmetric spikes and asymmetric spikes, using the Lyapunov-Schmidt reduction method.

In particular, by defining ω to be the unique even solution of the ODE

$$\omega_{yy} - \omega + \omega^3 = 0,$$

the authors have proved the following theorems

Theorem 4.1. *If $D = \frac{\hat{D}}{\epsilon^2}$ for some fixed $\hat{D} > 0$ and*

$$A_0(x) \equiv A_0, \gamma(x) \equiv \overline{A} - A_0, where \overline{A} > A_0 \tag{4.3}$$

and ϵ is small enough, then (4.2) *has a K-spike steady state (A_ϵ, v_ϵ) which satisfies*

$$A_\epsilon(x) = A_0 + \frac{1}{\epsilon}\sum_{j=1}^{K}\frac{1}{\sqrt{v_j^\epsilon}}\omega(\frac{x - t_j^\epsilon}{\epsilon}) + O(\epsilon \log\frac{1}{\epsilon}), \tag{4.4}$$

$$v_\epsilon(t_i^\epsilon) = \epsilon^2 v_i^\epsilon, i = 1, \cdots, K, \tag{4.5}$$

where

$$t_i^\epsilon \to t_i^0, i = 1, \cdots, K, \tag{4.6}$$

with

$$t_i^0 = \frac{2i - 1 - K}{K}L, i = 1, \cdots, K, \tag{4.7}$$

and

$$v_i^\epsilon = v_i^0(1 + O(\epsilon \log\frac{1}{\epsilon})), i = 1, \cdots, K, \tag{4.8}$$

with

$$v_i^0 = \frac{\pi^2 K^2}{2(\overline{A} - A_0)^2 L^2}, i = 1, \cdots, K \tag{4.9}$$

Theorem 4.2. *Under the same assumption as in* 4.1 *and*

$$\frac{2\sqrt{\pi}(\hat{D}A_0^2)^{1/4}}{(\overline{A} - A_0)^{3/4}L} \leq 1 \tag{4.10}$$

and

$$\frac{2\sqrt{\pi}(\hat{D}A_0^2)^{1/4}}{(\overline{A} - A_0)^{3/4}L} \neq \frac{2}{\sqrt{5}}, \tag{4.11}$$

problem (4.2) *has an asymmetric* 2*-spike steady state (A_ϵ, v_ϵ), which satisfies*

$$A_\epsilon(x) = A_0 + \frac{1}{\epsilon}(\sum_{j=1}^{2}\frac{1}{\sqrt{v_i^\epsilon}}\omega(\frac{x - t_i^\epsilon}{\epsilon}) + O(\epsilon \log\frac{1}{\epsilon})), \tag{4.12}$$

$$v_\epsilon(t_i^\epsilon) = \epsilon^2 v_i^\epsilon, i = 1, \cdots, K, \tag{4.13}$$

where t_i^ϵ and v_i^ϵ satisfy (4.6) *and* (4.8)*, respectively.*

Theorem 4.3. *Assume that $\epsilon > 0$ is small enough and $D = \frac{\hat{D}}{\epsilon^2}$ for some fixed $\hat{D} > 0$, then* (4.2) *has a single-spike steady state (A_ϵ, v_ϵ) which satisfies*

$$A_\epsilon(x) = A_0(x) + \frac{1}{\epsilon}\frac{1}{\sqrt{v_0^\epsilon}}\omega(\frac{x - t_0^\epsilon}{\epsilon}) + O(\epsilon \log \frac{1}{\epsilon}), \tag{4.14}$$

$$v_\epsilon(t_0^\epsilon) = \epsilon^2 v_0^\epsilon, \tag{4.15}$$

where

$$t_0^\epsilon \to t_0, \int_{-L}^{t_0} \gamma(x)dx = \int_{t_0}^{L} \gamma(x)dx \tag{4.16}$$

and

$$v_0^\epsilon = \frac{2\pi^2}{(\int_{-L}^{L} \gamma(x)dx)^2}(1 + O(\epsilon \log \frac{1}{\epsilon})) \tag{4.17}$$

In addition, by rescaling the solution and the second diffusion coefficient, their model becomes

$$\begin{cases} 0 = \epsilon \hat{A}_{xx} - \hat{A} + \hat{v}(\epsilon A_0 + \hat{A})^3 + \epsilon^3 A_0'', x \in \Omega, \\ 0 = \hat{D}\big((A_0(x) + \frac{1}{\epsilon}\hat{A})^2 \hat{v}_x\big)_x - \frac{1}{\epsilon}\hat{v}(\epsilon A_0(x) + \hat{A})^3 + \gamma(x), x \in \Omega. \end{cases} \tag{4.18}$$

They observe that by doing such scaling, the equations become very similar to the Schnakenberg model,

$$\begin{cases} 0 = \epsilon \hat{A}_{xx} - \hat{A} + \hat{v}(\epsilon A_0 + \hat{A})^3 + \epsilon^3 A_0'', x \in \Omega, \\ 0 = \hat{D}(A_0^2 \hat{v}_x)_x - \frac{1}{\epsilon}\hat{v}(\epsilon A_0(x) + \hat{A})^3 + \gamma(x), x \in \Omega. \end{cases} \tag{4.19}$$

Here, they assume $\epsilon << 1$ and Neumann boundary conditions.

In the later chapter of the paper, the authors also compute the positions and amplitudes of both symmetric and asymmetric spikes in the equation. In addition, they express the existence and nondegeneracy conditions of the equations (4.18). And they compute the approximate solutions for the equations.

4.2 Contribution by Cantrell et. al

In (Cantrell et al., 2012), the authors consider about the system

$$\begin{cases} \frac{\partial A}{\partial t} = \eta \Delta A - A + A^0 + \rho A & \text{in } \Omega \times (0, T] \\ \frac{\partial \rho}{\partial t} = \nabla \cdot [\nabla \rho - \frac{2\rho}{A}\nabla A] - \rho A + \overline{A} - A^0 & \text{in } \Omega \times (0, T] \\ \frac{\partial A}{\partial n} = 0, \frac{\partial \rho}{\partial n} - \frac{2\rho}{A}\frac{\partial A}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T] \end{cases} \tag{4.20}$$

where $\overline{A} \equiv \lambda$ is the average attractiveness. They work on finding the spatially non constant solutions for the problem, and they call this system EQ

$$\begin{cases} \eta\Delta A - A + A^0 + \rho A = 0 & \text{in } \Omega \\ \nabla \cdot [\nabla\rho - \frac{2\rho}{A}\nabla A] - \rho A + \overline{A} - A^0 = 0 & \text{in } \Omega \\ \frac{\partial A}{\partial n} = 0, \frac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

Associated with the problem, they consider the problem LN

$$\begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\mu > 0$ is a simple eigenvalue. They assume that

$$\overline{A} > A^0, \eta < \frac{2}{\mu}, \text{and } A^0 < \frac{(\eta\mu^2 - 2\mu)^2}{12\mu(\eta\mu + 1)}.$$

Then, they use critical point theory to make a shift of the variables of E and get the following system MP

$$\begin{cases} \frac{\partial A}{\partial t} = \eta\Delta A - A + \rho A + \overline{\rho} A + \overline{A}\rho & \text{in } \Omega \\ \frac{\partial \rho}{\partial t} = \nabla \cdot [\nabla\rho - \frac{2(\rho+\overline{\rho})}{A+\overline{A}}\nabla A] - \rho A - \overline{A} - \overline{\rho} A & \text{in } \Omega \\ \frac{\partial A}{\partial n} = 0, \frac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

Also, they define

$$F(\lambda, A, \rho) = \begin{bmatrix} \eta\Delta A - A + \rho A + (1 - \frac{A^0}{\lambda})A + \lambda\rho \\ \nabla \cdot [\nabla\rho - \frac{2(\rho+(1-\frac{A^0}{\lambda}))}{A+\lambda}\nabla A] - \rho A - \lambda\rho - (1 - \frac{A^0}{\lambda})A \end{bmatrix}.$$

Their main result in the paper is:

Let $\lambda_0 > A^0$ be a solution of

$$3\frac{A^0}{\lambda^2} + \frac{(\eta\mu - 2)}{\lambda} + \eta + \frac{1}{\mu} = 0$$

such that $\frac{\lambda_0}{\eta\mu}$ is not an eigenvalue of LN. Then a branch of spatially nonconstant solutions of F bifurcates from the equilibrium $(\overline{A}, 1 - \frac{A^0}{\overline{A}})$ at $\overline{A} = \lambda_0$. In a neighborhood of the bifurcation point, the bifurcating branch can be parametrized as $(\overline{A}, A, \rho) = (\overline{A}(s), \overline{A}(s) + s\lambda_0\phi + s\xi_1(s), (1 - \frac{A^0}{\overline{A}(s)} + s(\eta\mu + \frac{A^0}{\lambda_0})\phi +$

$s\xi_2(s)))$), where ϕ is a normalized eigenfunction of LN and $(\xi_1, \xi_2) \in W$, and where $\overline{A}(0) = \lambda_0$ and $(\xi_1(0), \xi_2(0)) = (0, 0)$. Furthermore, the bifurcating branch is part of a connected component C_0 of the set $\overline{S}$, where $S = \{(\overline{A}, A, \rho) : (\overline{A}, A - \overline{A}, \rho - 1 + \frac{A^0}{\overline{A}}) \in V, F(\overline{A}, A - \overline{A}, \rho - 1 + \frac{A^0}{\overline{A}}) = 0$, $(A, \rho) \neq (\overline{A}, 1 - \frac{A^0}{\overline{A}})\}$, and C_0 either extends to infinity in $\overline{A}, A$, or ρ, or contains a point where $(\overline{A}, A - \overline{A}, \rho - 1 + \frac{A^0}{\overline{A}}) \in \partial V$, or contains a point $(\lambda^*, \overline{A}, 1 - \frac{A^0}{\overline{A}})$ with $\lambda^* \neq \lambda_0$.

Chapter 5

A Priori Bound for the Equations

In this chapter, we prove the a priori bounds for the stable solution of the equations, both in one dimensional and higher dimensional cases.

5.1 One Dimensional Case

We work on the system

$$\eta A_{xx} - A + \alpha + A\rho = 0 \tag{5.1}$$

$$D[A(\frac{\rho}{A})_{xx} - \frac{\rho}{A}A_{xx}] - A\rho + \beta = 0 \tag{5.2}$$

subject to 2π-periodic boundary conditions. Defining $B = \frac{\rho}{A}$, our system now becomes

$$\eta A_{xx} - A + \alpha + A^2 B = 0 \tag{5.3}$$

$$DAB_{xx} - DBA_{xx} - A^2 B + \beta = 0 \tag{5.4}$$

We assume

$$A(0) = A(2\pi), A_x(0) = A_x(2\pi), B(0) = B(2\pi), B_x(0) = B_x(2\pi) \tag{5.5}$$

First, we prove a priori bounds for the integral of A^2B over the whole space, and then we are going to prove the upper bound for A and B respectively.

Lemma 5.1. *If* (A, B) *is a solution of* (5.3), (5.4), *then*

$$\int_0^{2\pi} A^2 B dx = 2\pi\beta \tag{5.6}$$

Proof. Integrating (5.4) from 0 to 2π, we get

$$\int_0^{2\pi} DAB_{xx} dx = \int_0^{2\pi} DBA_{xx} dx + \int_0^{2\pi} -A^2B + \beta dx. \tag{5.7}$$

Integrating by parts, we find

$$\int_0^{2\pi} DAB_{xx} dx = DAB_x|_{x=0}^{2\pi} - \int_0^{2\pi} DA_x B_x dx \tag{5.8}$$

$$\int_0^{2\pi} DBA_{xx} dx = DBA_x|_{x=0}^{2\pi} - \int_0^{2\pi} DA_x B_x dx \tag{5.9}$$

Then, as we are applying periodic boundary condition at here, we know

$$DAB_x|_{x=0}^{2\pi} = DBA_x|_{x=0}^{2\pi} = 0 \tag{5.10}$$

Therefore,

$$\int_0^{2\pi} DAB_{xx} dx = \int_0^{2\pi} DBA_{xx} dx,$$

so

$$\int_0^{2\pi} A^2 B dx = \int_0^{2\pi} \beta dx = 2\pi\beta. \tag{5.11}$$

This proves the lemma □

Remark 5.1. *Noticing that this lemma can be generalized to the case where* β *is not a constant function, which is a more realistic assumption in most cities. In this case, the equality will become*

$$\int_0^{2\pi} A^2 B dx = \int_0^{2\pi} \beta dx$$

Now, before proving that A is bounded from above, we first need to prove a lemma about the minimum of A.

Lemma 5.2. *If* (A, B) *is a solution of* (5.3), (5.4), *then*

$$\min A \leq \Big(\int_0^{2\pi} \alpha dt \Big) / (2\pi) + \beta \tag{5.12}$$

Proof. By (5.3), we get

$$\eta A_{xx} = A - \alpha - A^2 B \tag{5.13}$$

Integrating both sides of (5.13) from 0 to 2π, we get

$$\int_0^{2\pi} A_{xx} dx + \int_0^{2\pi} \alpha dx + \int_0^{2\pi} A^2 B dx = \int_0^{2\pi} A dx \tag{5.14}$$

Then, as we apply periodic boundary condition here, it follows

$$\int_0^{2\pi} \eta A_{xx} dx = A_x(2\pi) - A_x(0) = 0. \tag{5.15}$$

By Lemma 5.1, $\int_0^{2\pi} A^2 B dx = 2\pi\beta$, so

$$\int_0^{2\pi} A dx = \int_0^{2\pi} \alpha dx + 2\pi\beta \tag{5.16}$$

Therefore, $\min A \leq (\int_0^{2\pi} \alpha dt)/(2\pi) + \beta$. □

Now, with Lemma 5.2, we are ready to prove the a prior bound for A.

Theorem 5.1. *If* (A, B) *is a solution of* (5.3), (5.4), *then*

$$A \leq \min A e^{2\pi^2}, \tag{5.17}$$

and hence by Lemma 5.2 *is bounded.*

Proof. We start this problem from the global minimum of A. Let c_1 be the global minimum for A, and d_1 be the next local maximum on the right of c_1. Next, we iterate through each local minimum and local maximum by the following rule, as shown in Figure

- Denote c_{i+1} as the next local minimum on the right of d_i.
- Denote d_i as the next local minimum on the right of c_i.

We do this until we reach the global maximum, d_n.

As we are applying the periodic boundary condition at here, we may assume $d_n - c_1 < 2\pi$. Then, for each $i = 1, 2, \cdots, n$, consider about the interval $[c_i, d_i]$. We integrate (5.3) from c_i to m where $m \in [c_i, d_i]$ and get

$$\int_{c_i}^{m} \eta A_{xx} dx = \int_{c_i}^{m} A dx - \int_{c_i}^{m} \alpha dx - \int_{c_i}^{m} A^2 B dx \tag{5.18}$$

However, as c_i is a local minimum, $A_x(c_i) = 0$. Also, for $x \in [c_i, m]$, we know $A(x) \leq A(m)$. Hence,

$$\eta A_x(m) \leq \int_{c_i}^{m} A dx \leq A(m)(m - c_i) \tag{5.19}$$

which means

$$\frac{A_x(m)}{A(m)} \leq (m - c_i)/\eta \tag{5.20}$$

for each $m \in [c_i, d_i]$. Then, integrate (5.20) from c_i to d_i, we find that

$$\ln A|_{x=c_i}^{d_i} \leq \frac{(m - c_i)^2}{2}|_{x=c_i}^{d_i}, \tag{5.21}$$

so

$$\ln \frac{A(d_i)}{A(c_i)} \leq \frac{(d_i - c_i)^2}{2}. \tag{5.22}$$

Therefore, we can conclude that $A(d_i) \leq A(c_i)e^{(d_i - c_i)^2/2}$.

Also, we know $A(c_{i+1}) < A(d_i)$ for all i. Hence,

$$\begin{aligned} A(d_n) &\leq A(c_n)e^{(d_n - c_n)^2/2} \\ &\leq A_{d_{n-1}} e^{(d_i - c_i)^2/2} \\ &\leq \cdots \\ &\leq e^{(d_1 - c_1)^2/2} e^{(d_2 - c_2)^2/2} \cdots e^{(d_n - c_n)^2/2} \\ &\leq A(c_1)e^{[(d_1 - c_1)^2 + (d_2 - c_2)^2 + \cdots + (d_n - c_n)^2]/2} \end{aligned}$$

Then we know

$$\begin{aligned} &(d_1 - c_1)^2 + (d_2 - c_2)^2 + \cdots + (d_n - c_n)^2 \\ =&(2\pi)^2[(d_1 - c_1)^2/(2\pi)^2 + (d_2 - c_2)^2/(2\pi)^2 + \cdots + (d_n - c_n)^2/(2\pi)^2] \end{aligned}$$

Since $\frac{d_i - c_i}{2\pi} < 1$ for each $i = 1, 2, \cdots, n$, it follows

$$\frac{(d_i - c_i)^2}{(2\pi)^2} < \frac{d_i - c_i}{2\pi}$$

for each $i = 1, 2, \cdots, n$. Therefore,

$$\begin{aligned} &(d_1 - c_1)^2 + (d_2 - c_2)^2 + \cdots + (d_n - c_n)^2 \\ \leq&(2\pi)^2[(d_1 - c_1)/(2\pi) + (d_2 - c_2)/(2\pi) + \cdots + (d_n - c_n)/(2\pi)] \\ \leq&(2\pi)^2 \end{aligned}$$

Hence, we can conclude that

$$A(d_n) \leq A(c_1)e^{(2\pi)^2/2} = A(c_1)e^{2\pi^2}, \tag{5.23}$$

which means $A(d_n)$ is bounded by $A(c_1)$, and hence by Lemma 5.2 is bounded from above. □

Theorem 5.2. *If* (A, B) *is a solution of* (5.3), (5.4), *then at the local minimum of* B, x_0, *we have*

$$B(x_0) \leq \frac{\eta}{D} + \frac{1}{4 \max \alpha} \tag{5.24}$$

Proof. By (5.3),

$$A_{xx} = \frac{A - \alpha - A^2 B}{\eta}$$

Plugging that into (5.4), we have

$$DAB_{xx} = \frac{DB}{\eta}(A - \alpha - A^2 B) + A^2 B - \beta. \tag{5.25}$$

Since $B_{xx} > 0$, we have

$$\frac{DB(x_0)}{\eta} A(x_0) - \frac{DB(x_0)}{\eta} \alpha(x_0) - \frac{DA(x_0)^2 B(x_0)^2}{\eta} + A^2(x_0)B(x_0) > 0$$

Therefore,

$$\begin{aligned} B(x_0) &\leq \frac{1}{A(x_0)} + \frac{\eta}{D} - \frac{\alpha(x_0)}{A(x_0)^2} \\ &\leq \frac{1}{A(x_0)} + \frac{\eta}{D} - \frac{\min \alpha}{A(x_0)^2} \end{aligned}$$

Since $z + \frac{\eta}{D} - \min \alpha(x_0) \leq \frac{1}{4 \min \alpha} z^2$ for any z, we can conclude that

$$B(x_0) \leq \frac{\eta}{D} + \frac{1}{4 \min \alpha} \tag{5.26}$$

□

5.2 Higher Dimensional Case

Now, we generalize our work to the two-dimensional case, which is a more realistic case in daily life. The equations now become

$$\eta(A_{xx}+A_{yy})-A+\alpha+A^2B=0, x, y\in(0,2\pi) \tag{5.27}$$

$$DA(B_{xx}+B_{yy})-DB(A_{xx}+A_{yy})-A^2B+\beta=0, x, y\in(0,2\pi) \tag{5.28}$$

Still, we apply periodic boundary condition, at here, which means we have

$$A(0,y)=A(2\pi,y), A(x,0)=A(x,2\pi)$$

$$B(0,y)=B(2\pi,y), B(x,0)=B(x,2\pi)$$

$$A_x(0,y)=A_x(2\pi,y), A_y(x,0)=A_y(x,2\pi)$$

$$B_x(0,y)=B_x(2\pi,y), B_y(x,0)=B_y(x,2\pi)$$

Some conclusions we have gotten in the one-dimensional case can be generalized to the two-dimensional case, while some others need alternative proofs.

Lemma 5.3. *If* (A,B) *is a solution of* (5.27), (5.28), *then*

$$\int_0^{2\pi}\int_0^{2\pi} A^2B\,dxdy=4\pi^2\beta \tag{5.29}$$

Proof. Similar to the proof of Lemma 5.1, Integrating 5.29 over the space domain, we can get

$$\int_0^{2\pi}\int_0^{2\pi} DAB_{xx}+DAB_{yy}dxdy=\int_0^{2\pi}\int_0^{2\pi} DBA_{xx}+DBA_{yy}dxdy+\int_0^{2\pi}\int_0^{2\pi}-A^2B+\beta dxdy. \tag{5.30}$$

Then, doing integration by parts, we find

$$\int_0^{2\pi}\int_0^{2\pi} DAB_{xx}dxdy=\int_0^{2\pi}(DAB_x|_{x=0}^{2\pi}-\int_0^{2\pi} DA_xB_xdx)dy \tag{5.31}$$

$$\int_0^{2\pi}\int_0^{2\pi} DBA_{xx}dxdy=\int_0^{2\pi}(DBA_x|_{x=0}^{2\pi}-\int_0^{2\pi} DA_xB_xdx)dy \tag{5.32}$$

Then, as we are applying periodic boundary condition at here, we know

$$DAB_x|_{x=0}^{2\pi}=DBA_x|_{x=0}^{2\pi}=0 \tag{5.33}$$

Therefore,

$$\int_0^{2\pi}\int_0^{2\pi} DAB_{xx}dxdy = \int_0^{2\pi}\int_0^{2\pi} DBA_{xx}dxdy.$$

Similarly, we can get

$$\int_0^{2\pi}\int_0^{2\pi} DAB_{yy}dxdy = \int_0^{2\pi}\int_0^{2\pi} DBA_{yy}dxdy.$$

Hence,

$$\int_0^{2\pi} A^2Bdx = \int_0^{2\pi} \beta dx = 2\pi\beta \tag{5.34}$$

which shows $\int_0^{2\pi}\int_0^{2\pi} A^2Bdxdy = 4\pi^2\beta$. □

Finally, we are going to show that both A and B are bounded from above in the two-dimensional case.

Theorem 5.3. *Given A, B 2π-periodic continuous functions, there exists $M(\alpha,\beta)$ such that if A, B is a solution of* (5.27), (5.28), *then $||A||_\infty \leq M(\alpha,\beta)$.*

Proof. First, multiplying (5.27) by A and do integration over the space domain $(0,2\pi)\times(0,2\pi)$, we get

$$\eta\int_0^{2\pi}\int_0^{2\pi} |\nabla A|^2dxdy + \int_0^{2\pi}\int_0^{2\pi} A^2dxdy = \int_0^{2\pi}\int_0^{2\pi} \alpha Adxdy + \int_0^{2\pi}\int_0^{2\pi} A^3Bdxdy. \tag{5.35}$$

Note that by Lemma 5.3, we have

$$\int_0^{2\pi}\int_0^{2\pi} A^3Bdxdy \leq ||A||_\infty \int_0^{2\pi}\int_0^{2\pi} A^2Bdxdy \leq ||A||_\infty 4\pi^2\beta$$

Therefore,

$$\eta\int_0^{2\pi}\int_0^{2\pi} |\nabla A|^2dxdy < (4\pi^2||\alpha||_\infty + 4\pi^2\beta)||A||_\infty.$$

Then suppose $A(x_0)$ is the global maximum. Then, we find for $||y|| = 1$, by Fundamental Theorem of Calculus,

$$A(x_0 + ry) = A(x_0) + \int_0^1 (\nabla A(x_0 + sy)\cdot y)ds \tag{5.36}$$

Then, define

$$K = \left(\frac{4\pi^2||\alpha|| + 4\pi^2\beta}{\eta}\right)^{1/2},$$

and integrating on the circle of radius r, we get

$$\begin{aligned}
\int_{S_r} A(x_0+y)dy &= \int_{S_r}\int_0^1 (\nabla A(x_0+sy)\cdot y)dsdr + ||A||_\infty 2\pi r \\
&\geq ||A||_\infty 2\pi r - r\left(\int\int |\nabla A(x_0+ys)|^2||y||dsdy\right)^{1/2} \\
&\geq ||A||_\infty 2\pi r - r\left(\frac{4\pi^2||\alpha|| + 4\pi^2\beta}{\eta}||A||_\infty\right)^{1/2} \\
&\geq ||A||_\infty 2\pi r - rK||A||_\infty^{1/2} \\
&= r||A||_\infty^{1/2}(2\pi||A||_\infty^{1/2} - K)
\end{aligned}$$

Then, there are two cases

(a) If

$$2\pi||A||_\infty^{1/2} - K \leq 1,$$

then

$$||A||_\infty \leq \left(\frac{1+K}{2\pi}\right)^2,$$

so A is bounded.

(b) If

$$||A||_\infty^{1/2}2\pi - K > 1,$$

then we know

$$\int_{S_r} A(x_0+y)dy \geq r||A||_\infty^{1/2}$$

Then, integrate over on the whole circle of $r = 1$, we get

$$\begin{aligned}
\int_{C_r} A &\geq \int_0^1 r||A||_\infty^{1/2}dr \\
&\geq \frac{1}{2}||A||_\infty^{1/2}
\end{aligned}$$

However, integrate (4.27) over the space domain $(0,2\pi)\times(0,2\pi)$, by the periodic boundary condition, we know

$$\int_0^{2\pi}\int_0^{2\pi} A_{xx} + A_{yy}dxdy = 0,$$

so

$$\int_0^{2\pi}\int_0^{2\pi} A dxdy = \int_0^{2\pi}\int_0^{2\pi} \alpha dxdy + 4\pi^2\beta \le 4\pi^2(||\alpha||_\infty + \beta) \quad (5.37)$$

Therefore, $\frac{1}{2}||A||_\infty^{1/2} \le 4\pi^2(||\alpha||_\infty + \beta)$, so $||A||_\infty$ is bounded.

Therefore, by (a), (b), we can conclude that A is bounded from above. □

In addition, we also prove that B is bounded in the two-dimensional case, and this proof can be generalized to the one-dimensional case.

Theorem 5.4. *Given A, B 2π-periodic continuous functions, there exists $M(\alpha, \beta)$ such that if A, B is a solution of* (5.27), (5.28)*, then $||B||_\infty \le M(\alpha, \beta)$.*

Proof. Integrate (5.28) over the whole space domain, we get

$$\int_\Omega [DA(B_{xx} + B_{yy}) - DB(A_{xx} + A_{yy}) - A^2B + \beta] d\mu = 0 \quad (5.38)$$

By (5.27), we know

$$A_{xx} + A_{yy} = \frac{A - \alpha - A^2B}{\eta}.$$

Plug this into (5.38), we get

$$\int_\Omega [DA(B_{xx} + B_{yy}) - DB\frac{A - \alpha - A^2B}{\eta} - A^2B + \beta] d\mu = 0 \quad (5.39)$$

Multiply (5.39) by $\frac{\eta}{AD}$, as we are applying periodic boundary conditions, it follows

$$-\int_\Omega B + \int_\Omega \frac{B\alpha}{A} + \int_\Omega AB^2 + \int_\Omega \frac{\eta\beta}{AD} = \int_\Omega \frac{\eta AB}{D} \quad (5.40)$$

Then, let

$$W = \{x; 0 \le A(x) \le \alpha(x)\}, W' = \Omega - W,$$

we have

$$-\int_W B - \int_{W'} B + \int_W \frac{B\alpha}{A} + \int_{W'} \frac{B\alpha}{A} + \int_W AB^2 + \int_{W'} AB^2 + \int_\Omega \frac{\eta\beta}{AD} = \int_\Omega \frac{\eta AB}{D}.$$

Hence,

$$\begin{aligned}\int_\Omega AB^2 + \int_\Omega \frac{\eta\beta}{AD} &\le \int_\Omega \frac{\eta AB}{D} + \int_{W'} \frac{BA}{A} \\ &\le \int_\Omega \frac{\eta AB}{D} + \int_{W'} \frac{AB}{\alpha} \\ &\le K \int_\Omega BA \\ &\le K(\int AB^2)^{1/2}(\int A)^{1/2}\end{aligned}$$

Therefore, $\int AB^2$ is bounded. Also, it immediately follows that $\int AB$, $\int \frac{1}{A}$ are bounded because they are all less than or equal to $K(\int AB^2)^{1/2}(\int A)^{1/2}$.Then, we know

$$|\int_\Omega B d\mu|^2 \le (\int_\Omega |AB^2| d\mu)^{1/2} \cdot (\int_\Omega |\frac{1}{A}| d\mu)^{1/2}$$

is bounded. Finally, by (5.40), we can tell $\int \frac{B}{A}$ is also bounded. Then, multiplying (5.28) by B and integrating, we get $\int_\Omega |\nabla B|^2$ is bounded, so B is bounded by the same reason of Theorem 5.27. □

Chapter 6

Bifurcation Points

For β constant, it is readily seen that, for any $t > 0$, taking $\alpha = t$, $A = t+\beta$, and $B = \frac{\beta}{(t+\beta)^2}$ satisfy (5.3)-(5.5). These solutions will be called trivial solutions. A trivial solution $(\hat{t}, \hat{t} + \beta, \frac{\beta}{(\hat{t}+\beta)^2})$ is called a bifurcation point of (5.3)-(5.5) if any neighborhood of $(\hat{t}, \hat{t} + \beta, \frac{\beta}{(\hat{t}+\beta)^2})$ contains nontrivial solutions.

In this chapter, we prove the existence of bifurcation points for the system (5.3)-(5.5). We do so by proving the existence of non-constant solution to (5.3)-(5.4) subject to the Neumann boundary condition where

$$A_x(0) = A_x(\pi) = B_x(0) = B_x(\pi) = 0$$

and symmetrizing A and B about π. We use the following theorem provided in (Crandall and Rabinowitz, 1971).

Theorem 6.1. *Let W, Y be Banach spaces, Ω an open subset of and $G : \Omega \to Y$ be twice continuously differentiable. Let $\omega : [-1, 1] \to \Omega$ be a simple continuously differentiable arc in Ω such that $G(\omega(t)) = 0$ for $|t| \leq 1$. Suppose*

(*a*) $\omega'(0) \neq 0$,

(*b*) $\dim N(G'(\omega(0))) = 2$, *codim*$(R(G'(\omega(0)))) = 1$,

(*c*) $N(G'(\omega(0)))$ *is spanned by* $\omega'(0)$ *and* v, *and*

(*d*) $G''(\omega(0))(\omega'(0), v) \notin R(G'(\omega(0)))$.

Then $\omega(0)$ is a bifurcation point of $G(\omega) = 0$ with respect to $C = \{\omega(t) : t \in [-1, 1]$ and in some neighborhood of $\omega(0)$ the totality of solutions of $G(\omega) = 0$ from two continuous curves intersecting only at $\omega(0)$.

We apply this theorem as follows: define $W := (\mathbb{R}, M^2_{[0,\pi]}, M^2_{[0,\pi]})$, $Y := (M^0_{[0,\pi]}, M^0_{[0,\pi]})$, and $\omega(t) = (t, t+\beta, \frac{\beta}{(t+\beta)^2})$ where

$$M^2_{[0,\pi]} := \{u : [0,\pi] \to \mathbb{R}; u \text{ is twice continuously differentiable and } u'(0) = u'(\pi) = 0\}$$

and

$$M^0_{[0,\pi]} := \{u : [0,\pi] \to \mathbb{R}; u \text{ is contiunous and } u'(0) = u'(\pi) = 0\}$$

The curve ω is known as the curve of trivial solution. Define $G := W \to Y$ by

$$G(\alpha, A, B) = (\eta A_{xx} - A + \alpha + A^2 B, DAB_{xx} - DBA_{xx} - A^2 B + \beta).$$

Now, we try to find a point t where $\dim N(G'(\omega(t))) = 2$. Then, suppose

$$u = \sum_{k=1}^{\infty} a_k \cos(kx), v = \sum_{k=1}^{\infty} b_k \cos(kx),$$

we find that $G'(\alpha, \alpha+\beta, \frac{\beta}{(\alpha+\beta)^2})(0, u, v)$ equals

$$\sum_{k=1}^{\infty} \begin{bmatrix} -\eta k^2 + 1 + \frac{2\beta}{\alpha+\beta} & (\alpha+\beta)^2 \\ \frac{D\beta}{(\alpha+\beta)^2} k^2 - \frac{2\beta}{\alpha+\beta} & -D(\alpha+\beta)k^2 - (\alpha+\beta)^2 \end{bmatrix} \begin{bmatrix} a_k \cos kx \\ b_k \cos kx \end{bmatrix}. \tag{6.1}$$

Denote

$$A(k,\eta,\alpha) = \begin{bmatrix} -\eta k^2 + 1 + \frac{2\beta}{\alpha+\beta} & (\alpha+\beta)^2 \\ \frac{D\beta}{(\alpha+\beta)^2} k^2 - \frac{2\beta}{\alpha+\beta} & -D(\alpha+\beta)k^2 - (\alpha+\beta)^2 \end{bmatrix},$$

we can get the following lemma

Lemma 6.1. *Let $\beta < 2D$, given a positive integer N there exists $E(N)$ such that if $\eta < E(N)$, then there exists $\hat{t_1}, \cdots, \hat{t_n}$ such that if $\alpha = \hat{t_k}$, then $\det A_k = 0$ for $k = 1, 2, \cdots, n$.*

Proof. We first note that

$$\det A(k,\eta,\alpha) = D\eta(\alpha+\beta)k^4 + [D(\alpha - 2\beta) + \eta(\alpha+\beta)^2]k^2 + (\alpha+\beta)^2. \tag{6.2}$$

Denote it as $f(k,\eta,\alpha)$. Given any positive integer N, there exists $E(N)$ such that if $\eta \in (0, E(N))$ then

$$f(N,\eta,0) < 0$$

. Also for any $k \in \{1,2,\cdots,N\}$, $f(k,\eta,2\beta) > 0$. Hence by the intermediate value theorem, for each $k \in \{1,2,\cdots,N\}$, there exists $\hat{t_k} \in (0,2\beta)$ such that $f(k,\eta,\hat{t_k}) = 0$. Hence we have proved the lemma. □

Then with this lemma, we can prove the following theorem.

Theorem 6.2. *If $\beta, E(N), \hat{t_1},\cdots,\hat{t_n}$ are as given in Lemma* 6.1*, then there exists some η such that $(\hat{t_k}, t_k \hat{+} \beta, \frac{\beta}{(\hat{t_k}+\beta)^2})$ is a bifurcation point of system* (5.3)-(5.5).

Proof. Since $\det A(k,\eta,\alpha) = 0$, there exists a unique $a_k \in \mathbb{R}$ such that

$$A(k,\eta,\alpha)\begin{bmatrix}\cos(kx)\\ a_k\cos(kx)\end{bmatrix} = 0$$

which means $(a_k \cos kx, b_k \cos kx)$ is in the kernel of $G'(\omega(t))$. In addition, $(1,1,\frac{-2\beta}{(\alpha+\beta)^3})$ is in the kernel. Therefore, $\dim N(G'(\omega(0))) = 2$. and hence $\hat{t_k}, \hat{t_k}+\beta, \frac{\beta}{(\hat{t_k}+\beta)^2}$ is a bifurcation point in the space $M_{[0,\pi]}$.

Also, we find that for any $(u,v) \in M^0_{0,\pi} \times M^0_{0,\pi}$, we can represent it as

$$\begin{bmatrix}u\\ v\end{bmatrix} = \begin{bmatrix}\sum_{n=1}^{k-1} c_n\cos(nx)\\ \sum_{n=1}^{k-1} d_n\cos(nx)\end{bmatrix} + m\begin{bmatrix}-a_k\cos(kx)\\ \cos(kx)\end{bmatrix} + \begin{bmatrix}\sum_{n=k+1}^{\infty} c_n\cos(nx)\\ \sum_{n=k+1}^{\infty} b_n\cos(nx)\end{bmatrix} + p\begin{bmatrix}\cos(kx)\\ 0\end{bmatrix} \tag{6.3}$$

Now, we are going to show that for any $k \neq n$, $\begin{bmatrix}c_n\cos(nx)\\ d_n\cos(nx)\end{bmatrix}$ is in the range. To show it, it is enough for us to show that $\det A(j,\eta,\alpha) \neq 0$ for $k \neq j$. Notice that by (6.2), $\det A(j,\eta,\alpha) = \det A(j,\eta,\alpha) = 0$ if and only if $k^2 j^2 = \frac{\alpha+\beta}{D\eta}$. In addition, η is a $\hat{t_k}$ is a continuous function of η and hence $\frac{\alpha+\beta}{D\eta}$ is a continuous function on η. Therefore, we can choose η such that $\frac{\alpha+\beta}{D\eta}$ is not an integer. Thus, with that η, $\det A(j,\eta,\alpha) \neq 0$ for $k \neq j$. Therefore,

$$\begin{bmatrix}\sum_{n=1}^{k-1} c_n\cos(nx)\\ \sum_{n=1}^{k-1} d_n\cos(nx)\end{bmatrix} + m\begin{bmatrix}-a_k\cos(kx)\\ \cos(kx)\end{bmatrix} + \begin{bmatrix}\sum_{n=k+1}^{\infty} c_n\cos(nx)\\ \sum_{n=k+1}^{\infty} b_n\cos(nx)\end{bmatrix}$$

is in the range, which shows $\operatorname{codim}(R(G'(\omega(0)))) = 1$. Therefore, with some η, we can find $(\hat{t_k}, \hat{t_k}+\beta, \frac{\beta}{(\beta+\hat{t_k})^2}$ to be a bifurcation point with the Neumann Boundary condition.

Then, we extend the space $M_{[0,\pi]}$ to $[0, 2\pi]$ by taking the function symmetry about $x = \pi$. Notice that the function we get by the sum of $\cos kx$ and constant functions satisfy (5.5). Therefore, $\hat{t_k}, \hat{t_k} + \beta, \frac{\beta}{(\hat{t_k}+\beta)^2}$ is a bifurcation point of system (5.3)-(5.5). $\square$

Chapter 7

Conclusion and Future Work

In this paper, we have proved the existence of the a prior bounds for the system of equations (5.3) (5.4) in both one-dimensional and two-dimensional cases. In addition, we have found some local bifurcation points for those equations. In the future, we hope to find some non-constant stable solutions.

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