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Wandering About: Analogy, Ambiguity and Humanistic Mathematics

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Synopsis

This article concerns the relationship between mathematics and language, emphasizing the role of analogy both as an expression of a mathematical property and as a source of productive ambiguity in mathematics. An historical discussion is given of the interplay between the notions of logos, litotes, and limit that has implications for our understanding and teaching of Dedekind cuts and, more generally, for a humanistic notion of the role of mathematics within liberal education.

Mathematics and language have interacted in many ways over the ages. Examples range from the effect of prehistoric "counting tokens" upon written symbols to today's strong presence of mathematics in linguistics and cryptography. We tend to overlook, however, the large role that ordinary rhetoric plays in this connection.

Consider, for example, Mark Twain's insight into the writer's craft:

A good word is to the best word as a lightning bug is to the lightning. (T)

The statement (T) is, of course, an *analogy*, that is, an assertion of the form "A is to B as C is to D." By succinctly indicating enlightening but unsuspected connections between pairs of things, the analogy has become a major rhetorical tool in the humanities. One side of an analogy is often more familiar to us, making the other side easier to comprehend. Is there something "mathematical" going on here?

James Gregory [17, page 215] used an analogy to give us insight into the vitality of 17$^{\text{th}}$-century mathematics:

$$\textit{The power of all previous methods has the same ratio to that of series as the glimmer of dawn to the splendor of the noonday sun.} \tag{G}$$

The appearance of the word *ratio* in Gregory's analogy reminds us of an almost forgotten connection between mathematics and the humanities: an analogy is an assertion of the sameness of ratios—ratios being understood perhaps in a very general sense.

Indeed, *logos*, the Greek word for the Latin *ratio*, is the main stem of *analogia*, from which our word *analogy* comes. Here the prefix *ana-* means "up" in a generalized sense, as used also in *analysis* ("loosening up"). Like *analysis*, a word whose meaning was extended from geometry to philosophy in ancient times, *analogia* originally applied to mathematical assertions and later to rhetorical analogies.

Today the form of an analogy is often indicated in symbols by writing

$$A : B :: C : D,$$

where the four dots in the middle may remind us of the assertion of a "square" balance between the ratio $A : B$ and the corresponding ratio $C : D$. Completion of an analogy (giving three of its members and asking for the fourth) is sometimes featured in questions on intelligence tests. Putting the question in a multiple-choice setting lessens worry about existence and uniqueness of the answer.

An open-ended variant of this is to take two "extremes" and ask whether there exists a "mean" between them. For example, does there exist an entity X serving as intermediary between the sciences and the humanities in the sense that

$$\textit{SCIENCES} : X :: X : \textit{HUMANITIES}? \tag{P}$$

Beginning in the fourth century BCE the ancient Greeks exploited relations among the ideas behind logos, litotes, and limit that allowed them to initiate mathematical analysis without developing our numerical epsilon-delta approach. Recalling these efforts brings mathematics closer to the humanities, suggests possibilities for new approaches to teaching, and—as we move toward modern times—sheds new light on the old topic of Dedekind cuts and on the place of mathematics within the liberal arts today. Perhaps a solution to proposition (P) will suggest itself along the way.

Figure 1: Plato and Aristotle "wandering about" in Raphael's *School of Athens*. See [15]. Bronze sculpture, 18" tall, by Jeanie Stephenson, installed October, 2012, in Courtyard of Woods Laboratories, the University of the South, Sewanee TN. Photograph by Mary Priestley, reproduced by permission.

1. Logos

In Greek mathematics, *logos* (plural, *logoi*) means ratio—"the size of one thing relative to another" according to *Euclid V*—but the word is more familiar outside mathematics. "In the beginning was the logos," the phrase from the New Testament that opens the fourth gospel, contains perhaps its most famous usage, written at a time when the word already signified a wealth of related notions.

Alternative English translations of *logos* include *word*, *language*, *speech*, and *reason*. In the fourth century BCE it was seen as a unifying principle—along with notions of harmony, rhythm, etc.—permeating the *enkuklios paideia*, as the Greeks referred to the distinctive education that they developed [21]. Logos still plays a unifying role in liberal education, as suggested by such English derivatives as logic and analogy, and by the host of modern academic words, such as ecology, possessing the suffix *-logy*.

Although a numerical ratio such as $154 : 49$ would be identified today with the fraction $22/7$, we should remember that the Greeks possessed no algebra of common fractions [20]. They would have had little inclination to add one ratio to another, although they saw clearly—see inequality (E4) below, for example—when one ratio might be said to be less than another. Indeed, as documented in [20] and described for the general reader in [9], our modern quantitative sense took a great many centuries to develop. The strength of this sense, however, may be seen in our quick identification of the rational number $22/7$ with its large equivalence class, including such fractions as $154/49$.

Archytas, a late Pythagorean and a friend of Plato (427–347 BCE), is traditionally credited with a striking analogy asserting the sameness of two logoi whose four members constitute a large part of the liberal education developed by the Greeks:

$$\mathit{ARITHMETIC} : \mathit{MUSIC} :: \mathit{GEOMETRY} : \mathit{ASTRONOMY}. \tag{A}$$

The *quadrivium*, as this fourfold collection of disciplines was much later named by Boethius (480–524), would eventually become an integral part of the classical liberal arts of medieval European universities. Archytas's analogy is usually understood by reflecting that arithmetic is about numbers "at rest" and geometry is about magnitudes "at rest"; whereas music and astronomy study the same things, respectively, but "in motion." That is, the science of music concerns itself with the numbers underlying pitches produced by the vibrations of the strings of a lyre, while astronomy studies magnitudes revolving in the great celestial sphere.

The use of the compact form (A) to represent Archytas's analogy would not have appeared in ancient times when the Greeks wrote everything out "rhetorically," rarely using abbreviations beyond allowing letters to stand for numbers or for the vertices of a geometric figure. In contrast, a college instructor today might breezily remark that if a plane cuts a cone parallel to one of the cone's generators, a Cartesian coordinate system can be easily imposed upon the plane so that the curve of intersection has an equation $cy = x^2$ for a certain constant c. Apollonius (260?–200? BCE), however, in making essentially this observation long ago, required a great many words to express virtually the same idea, calling attention to the equality of areas of a certain square (of size x) and a certain rectangle (with sides c and y). Apollonius went on to note that the analogue of this equality of square and rectangle becomes an inequality in the case of the other types of conic sections. Accordingly, he gave them the names that became our familiar *parabola*, *hyperbola*, and *ellipse*, depending upon (as we would say today) whether the symbol $=$, $>$, or $<$ applies. The connections between the three types of conic sections, these three modern symbols, and the familiar rhetorical terms *parable*, *hyperbole*, and *ellipsis* are noteworthy, both for seeing ties between mathematics and rhetoric and for apprehending how symbols can sometimes reveal analogies with surprising succinctness. See [18] for a more complete discussion of Apollonius's ideas.

While their mathematics became more involved, the Greeks remained reticent to use symbolism to break the ties joining mathematics and rhetoric. This is often seen as a hindrance. It is worth emphasizing that it took a very long time for abbreviative notations and their consequent large-scale algebraic manipulations to manifest themselves. They have proved to be so efficient, however, that we have all but forgotten about past struggles in mathematics between symbolists and rhetoricians; see [7, pages 426–431]. The square of four dots in (A), for example, although it may remind us of ancient times when the Pythagoreans identified the figurate number Four with Justice, was accepted as an abbreviation for equality in proportions only in the 17^{th} century.

The analogy (A) would be taken by the ancient Greeks to exemplify much the same kind of information as is conveyed by the ancient law of the lever

$$W : w :: d : D, \tag{E1}$$

where W and w are weights whose centers of gravity are located distances D and d from the lever's fulcrum, respectively. Condition (E1) of course, reflects the assertion that a lever balances when the ratio of the weights upon it are in inverse proportion to the ratio of their distances from the fulcrum.

The famous "three-halves power" planetary law of Johannes Kepler (1571–1630) might be expressed by writing

$$T^2 : t^2 :: R^3 : r^3, \tag{E2}$$

where T and t are the periods of time taken for a full revolution about the sun of planets whose mean distances from the sun are R and r, respectively.

In *The Measurement of a Circle* Archimedes (287–212 BCE) proved that

$$A : r^2 :: C : D, \tag{E3}$$

where A stands for the area of a circle with radius r, r^2 stands for the square whose side is r, and C stands for the circumference of a circle with diameter D. Here on the left-hand side is the ratio of the area of a circle to the area of a square, while on the right-hand side is the ratio of the length of the circumference of a circle to the length of its diameter. The analogy (E3) thus asserts that the ratio of two "two-dimensional" magnitudes is the same as the ratio of two "one-dimensional" magnitudes.

Note that in fact all six of the ratios involved in (E1), (E2), and (E3) are ratios of "like" magnitudes in the same way. This observation assumes some importance in discussing similarities and differences between rhetoric and mathematics.

2. Rhetorical analogy and mathematical proportion[1]

The long history of liberal education has seen several ups and downs in the relationship between mathematics and rhetoric. Even in earliest times the holistic nature of the ancient Greek *enkuklios paideia* began to fracture under allied, but competing, interests. Plato emphasized the philosophical side of education, dominated by the quadrivium, while Isocrates was touting rhetoric, especially its power of oral persuasion. "The right word is a sure sign of good thinking," said Isocrates, anticipating somewhat the sentiment of our analogy (T).

[1] While the connection between rhetorical analogies and mathematical proportions seems little noticed today, it was clearly pointed out in 1929 in the fourth chapter, "Proportions," of Scott Buchanan's *Poetry and Mathematics* [5], where analogies (E1) and (E2) are also discussed.

The victor, generally speaking, was Isocrates [21, page 194], and this victory was solidified during the long period of Roman domination that followed. Between *ratio* and *oratio* there seemed to be little contest:

> With the Greeks geometry was regarded with the utmost respect, and consequently none were held in greater honor than mathematicians, but we Romans have restricted this art to the practical purposes of measuring and reckoning.
>
> But on the other hand we speedily welcomed the orator...
>
> —Cicero, *Tusculan Disputations*

It was the Romans who gave us the beautiful Latin phrase *artes liberales*, but under their domination mathematics became increasingly valued for its practical utility rather than its liberating qualities. Cicero (106–43 BCE) used the Latin *humanitas* to translate the Greek *paideia*, which signified both education and culture [21, page 99].

The ages since have seen more fluctuations in the relative importance attached to the varying disciplines constituting the liberal arts. Suffice it to say that today the humanities and the modern sciences have grown far apart, with mathematics stretched ever more tenuously in between. Of course, there are good reasons for distinguishing between the two. Whereas (E1), (E2), and (E3) can each be reinterpreted in modern terms as asserting the equality of two real numbers, the statements (T), (G), and (A) cannot. If we try to reduce analogy (A) in this way, for example, we should ask exactly what sort of entity is the ratio

$$ARITHMETIC : MUSIC \tag{R1}$$

and in precisely what sense is it the same as

$$GEOMETRY : ASTRONOMY? \tag{R2}$$

Applying such a reductionist approach to (A), however, misses its rhetorical point. We understand the analogy as a whole without feeling any need to give a fixed and exact meaning to its constituent entities (R1) and (R2). In fact, we might playfully rephrase, and re-interpret, the substance of (A) by saying that, just as music is a trick that arithmetic plays upon the ear, so astronomy is a trick that geometry plays upon the eye. Plato takes this point of view in his *Republic* [530d]. The best rhetorical analogies invite multiple harmonic interpretations, resisting reduction to a single level.

On the other hand, when discussing statement (E3), we might follow Archimedes in attempting to give a common numerical meaning to the geometric ratios $A : r^2$ and $C : D$. He approximated the numerical value of C : D to high accuracy by inscribing and circumscribing a circle with regular polygons of 6, 12, 24, 48, and 96 sides, finally concluding that

$$223 : 71 < C : D < 22 : 7. \tag{E4}$$

Thus, Archimedes proved that $22 : 7$ is roughly equal to the ratio of the circle to its diameter. Or, as we put it today, π is approximately 3.14. As is well-known, the use of the symbol π as an abbreviation for the ratio of the circle to its diameter is only about 300 years old.[2] Archimedes's result (E3) is now generally interpreted to mean that π can be defined as the common numerical value of the two geometric ratios in (E3), from which our familiar formulas $A = \pi r^2$ and $C = \pi D$ follow.

In bracketing the size of π by inequality (E4), Archimedes hints at an idea that would require some two millennia to come to fruition with Richard Dedekind (1831–1916), as we shall see in Section 4. Archimedes's efficient algorithm that produces inequality (E4) can in principle be continued forever, bracketing π ever more closely. Wherever we stop in this algorithm, however, we do not, of course, find the exact numerical value of π. Instead we find out more surely what π is not. This leads us to litotes.

3. Litotes

Rhetoricians borrowed the term *analogy* from mathematics, and mathematicians might now consider returning the compliment. *Litotes*, an old Greek word meaning "plainness," refers in modern rhetoric to something like double negation.[3] Explicitly, litotes refers to the expression of an affirmative by the negation of its opposite. The term was not unfamiliar to high-school English classes years ago, where it may have been pronounced with a strong accent upon the second of its three syllables. Modern dictionaries still give it three syllables, but seem to prefer to put the accent upon the first.

Whether we are familiar with the term or not, each of us uses litotes from time to time. Occasionally litotes occurs almost unnoticed in a word where the prefix *a-* means "not," as in *atom*, which literally refers to something that

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[2] A good reference for the history of the symbol π can be found in [4, page 442].

[3] As in many cases, the relevant Wikipedia article is a good first reference: `http://en.wikipedia.org/wiki/Litotes`, accessed January 30, 2013.

is "not divisible." Or when the prefix *in-* means "not," as in *infinite*. More commonly, however, its use in speech leads to a more modest expression of what is much the same thing, for example, by saying "not unfamiliar" rather than "familiar."

The reason that we introduce the word here is that, as we shall see, the Greeks were often able to use the notion of litotes to substitute for our notion of a limit. It is true that the Greek *peras* can be translated as *limit*, but in a geometrical, not numerical, sense. If we take its opposite, *apeiron*, to mean *indeterminacy*, we see that the ancient Greeks had more or less our idea.

How did the Greeks do this? Litotes arises naturally in the discussion of terms whose opposites are more quickly apprehended. Maturity and good manners, for example, are barely noticeable, but their absence is conspicuous. It is easier to recognize, and hence define, their absence rather than their presence. Thus we may find ourselves listing a series of *don't*s for immature readers and/or writers of mathematics, rather than attempting the harder and perhaps unrewarding task of explaining positively what mathematical maturity means. The same considerations apply to codes of conduct. It is no wonder that most of the Ten Commandments are statements about what should not be done. And a college honor code today might explain little about honor, but simply assert that one should not lie, cheat, or steal.

Such considerations apply to mathematics itself perhaps more than we notice. Litotes reveals itself, of course, in indirect proofs first seen in ancient times where, to affirm a proposition p, we show that "not-p" leads to absurdity. What we are really doing is proving "not-not p." Constructivists today would vigorously challenge the main premise of litotes when used in this context—that "not-not p" is the same as p.

A more subtle use in Greek mathematics of litotes has to do with the notion of equality of geometric magnitudes, where "equality" means "not being unequal," and properties following from inequality are stressed. Archimedes postulated in effect, what may be implicit in Euclid, that if two magnitudes are unequal, then some integral multiple of their difference exceeds either. In discussing areas, volumes, and tangents, the ancient Greeks were able to avoid speaking of limits by using the idea behind litotes. Archimedes proved (E3) by showing first that the area A of a circle is equal to the area B of a triangle whose base is the circumference of the circle and whose height is the radius. To prove $A = B$ he gave two brief arguments, showing that a contradiction follows from each of the two possibilities $A > B$ and $A < B$.[4]

Archimedes's name was indelibly stamped [31] upon the condition he postulated during the late 19$^{\text{th}}$-century *arithmetization of analysis*, about which we shall say more below. Suitably rephrased in modern terms, this condition is now well known as the *Archimedean property*; see [1, page 19], for example. A more sophisticated example of the use of litotes during this modern period is seen in the statement of the important Baire category theorem, where *second category* means *not first category.* Stripped of Baire's "category" terminology, a common version of the theorem states that a (non-empty) complete metric space is not the countable union of nowhere-dense sets.

For us, a particularly important use of litotes may be seen in the following definition of the Riemann integral. We first define the straightforward notions of lower and upper sums of a bounded, real-valued function f on $[a, b]$ and then declare that $\int_a^b f(x)dx$ is the number—if there is only one such real number—that lies between every lower and upper sum. Rather than defining the integral directly, as we might do by first formulating a technical, numerical definition of the limit of a sequence of approximating sums, we simply call attention to all real numbers that are too large and all that are too small. The integral is the transition point between these two sets of "wrong answers."

Here the use of litotes obviates the need for a definition of limit. When it can be used in this manner, litotes allows us to avoid limits—at least "one-dimensional" limits—by specifying the desired limit as the complement of all

[4] Since this article concerns analogies, we should ask whether there is a three-dimensional analogue of the planar equality $A = B$. In *The Method* [16, pages 20–21 of Supplement], Archimedes states

> ...judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius.

Archimedes emphasizes the importance in mathematics of both heuristics (here illustrated in a discovery by analogy) and rigorous proof. A rigorous proof, of course, is essential, lest a heuristic analogy lead us astray. Archimedes's careful proof that the surface of a sphere is four times its greatest circle is given in his paper *On the Sphere and Cylinder.* For accounts of the re-discovery and modern analysis of the Archimedes palimpsest containing *The Method*, see [23] and [24].

numbers that are too large and all those that are too small.[5] The idea is close to Sherlock Holmes's principle in detection: *When you have eliminated the impossible, whatever remains, however improbable, must be the truth.*

4. Litotes again: Eudoxus' condition and Dedekind cuts

Since Dedekind and Riemann were great friends, one cannot help wondering whether Dedekind's 1872 monograph on cuts might have been spurred in part by the definition of the integral just mentioned. Defining the Riemann integral to be a transition point, or "cut," implicitly relies upon the existence of such cuts and calls attention to the importance of the completeness, or connectedness, of the real numbers.

What do we mean by connectedness? Here litotes arises again in a natural way, for the definition of a *disconnected* topological space, like the set of rationals with the usual metric, is straightforward: it is the union of two non-empty, disjoint open sets. To say that a topological space is *connected* means that the space is *not disconnected*.

These modern issues, remarkably, are related to ancient mathematics. *Euclid V*, which deals with the theory of proportion, states first that two geometric magnitudes possess a ratio when some (positive, integral) number of copies of each exceeds the other. This statement banishes the notion of a fixed infinitesimal from the theory and formalizes what is meant by saying that the two magnitudes must be of "the same kind" in order to speak of their ratio. But what should be meant by "sameness" of ratios in light of the fact, discovered about 430 BCE, that not all ratios of geometric magnitudes are ratios of integers? The brilliant response to this question is the condition given in *Euclid V* that is traditionally ascribed to Eudoxus (395?–342? BCE):

> Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order [12, page 99].

[5] Surprisingly, one can use this trick of elimination (litotes) to give a "limit-less" definition of the derivative in elementary calculus as well. See [22].

With the help of litotes this definition can be interpreted more briefly in modern terms. To say that A : B : : C : D means that there is no ratio of natural numbers lying between the ratio A : B and the ratio C : D.

In the setting of 19$^{\text{th}}$-century set theory, and with emphasis upon modern analysis rather than the venerable geometry of old, Dedekind turned this condition on its head and defined a real number to be a cut (*Schnitt*) in the rationals. By a cut is meant a partition of the rational numbers into two (non-empty) segments. If we stipulate that the lower segment does not contain a greatest rational, we have a one-to-one correspondence between the Dedekind cuts and the system $\mathbb{R}$ of real numbers that we wish to define. A final refinement, identifying the real number with its lower segment of rationals, is often made so that the ordering of $\mathbb{R}$ corresponds simply to set inclusion.[6]

Thus the foundations of mathematics have been moved from a geometric framework of points, lines, etc. to an arithmetical construct where numerically defined real numbers have taken the place of geometric points. This is one of the accomplishments of the late 19$^{\text{th}}$-century movement that Felix Klein later called the *arithmetization of analysis*. In practice, of course, we still find ourselves speaking of the "points" in $\mathbb{R}$.

The connection between Dedekind cuts and the condition of Eudoxus has long been known. See for instance [32, pages 38–40]. In his later years Eudoxus belonged to Plato's Academy, and Plato clearly appreciated aspects of this connection dependent upon the notion underlying litotes.[7]

The ancient Greek geometers presumably thought they were discovering relationships between formerly existing magnitudes. Certainly this is the view of Platonism. The arithmetization of analysis, however, tends to emphasize the creation[8] of numerically-defined entities, beginning with the myriad

[6]The beauty of Dedekind's scheme is that completeness of the set $\mathbb{R}$ thus defined is established with remarkable ease: Given a non-empty bounded set S in $\mathbb{R}$, take the union of all the lower segments corresponding to members of S. The supremum of S is in hand. On the other hand, the expected algebraic properties of $\mathbb{R}$ are tedious to verify, but follow easily from the similar well-known algebraic properties of the rationals. See Section 8.4 of [1], for example. In the end we have a complete ordered field $\mathbb{R}$, constructed by adding notions of set theory to the well-known arithmetic of integers.

[7]See [28] where this is explained in more detail.

[8]In light of the tendency to believe that in mathematics one is forced to choose between creation and discovery, it is worth noting that this is a false choice. We might take an intermediate position suggested by Kronecker's well-known quotation implying that the

of Dedekind's real numbers that had formerly been—numerically speaking—nothing at all, only "holes" in the rationals. Dedekind himself felt that he had created something new when he had the imagination to define an irrational number in terms of what it is not. One recalls the familiar lines from *Midsummer-Night's Dream*:

> ... as imagination bodies forth
> The forms of things unknown, the poet's pen
> Turns them to shapes and gives to airy nothing
> A local habitation and a name.

Dedekind's reification of the notion of a cut may thus be viewed as a poetic act and, in fact, Dedekind is called "the West's first modernist" in [11].

Discussions of mathematics and language often find themselves involved with discussions of poetry.[9] In this article we shall be content with indicating in our final two sections why a humanistic approach to mathematics sometimes leads to such an involvement.

5. Limit: A heuristic analogy

To have something concrete to deal with, let us first look at an example of the use of an analogy in teaching elementary calculus. The heuristic use of an analogy derives from the fact that one side of it is often more familiar than the other, giving students a new way to appreciate the less-familiar side.

Almost everyone who has taught elementary calculus has been frustrated by the fact that not every student finds the concept of a limit to be congenial, and that repeating a discussion of Cauchy's epsilons and deltas may only increase anxiety. The student's difficulty is that he simply does not "get the idea" of a limit, which is indeed a subtle notion.

First note, however, that misapprehension may be due to the fact that the notion of a function has not been fully absorbed. A function from the reals to the reals can be understood at least three ways: statically, kinematically, and geometrically. That is, as a pair of columns of numbers, as a rule of

natural numbers are discovered, while "alles anderes" is created by human beings. See [13] for many other views. The issue of creation versus discovery can be avoided by simply speaking of the discipline as being "developed."

[9] The literature on mathematics and poetry is large, but one might see [5], [25], and [34] as examples of different approaches over the years. Emily Grosholz has stated she has long been contemplating a book that will show how poetry stands in the same relation to the humanities as mathematics stands to the sciences.

correspondence suggesting motion, and as a curve in the plane—giving us three ways to illustrate remarks about functions. Nevertheless, some students will still misinterpret what is meant by the limit of a function at a point in or near its domain.

The fact that *limit*, with its attendant notion of *continuity*, is often difficult for students to comprehend is analogous to the fact that the notion of mathematical maturity is also often difficult. Litotes is involved again here: like maturity, the absence of continuity is much more noticeable than its presence.

Continuity in mathematics, however, is not unrelated to continuity in ordinary experience. Students may already have an intuitive feeling for continuity in everyday life—when what we actually do is what we intended to do. Or, in the external world, when what actually happens is what seemed to be on the verge of happening. Is there a simple heuristic device that will help the student naturally distinguish the two notions of tendency and actuality?

The idea of a limit of a function at a point is no harder to comprehend than the purpose of a human being at an instant in time. Consider this analogy, abbreviated as

$$\textit{LIMIT} : \textit{POINT} :: \textit{PURPOSE} : \textit{INSTANT}.$$

At this stage the student is presumed to be befuddled by the notion of a limit, but familiar with the ordinary distinction between one's purpose and one's action at an instant. Then the student may see that there are several familiar things that can happen on the right-hand side, and that these naturally correspond to possibilities on the left. Most of us can think of instants when our action did not reflect our purpose, or of times when we wandered aimlessly with no purpose whatever. Sometimes, with or without a purpose, we hesitate to act. Finally, there are the gratifying times when we have a purpose and act accordingly, giving us a sense of continuity.

Considering the limit of a function at or near a point in its domain gives rise to the same possibilities when we compare the action of the function (what it actually does at the point) with its limit or "purpose" (what it seemed on the threshold of doing at the point). When the limit exists and agrees with the action we have continuity. An easy modification of this simple heuristic device can help convey the idea behind a one-sided limit.[10]

[10] This analogy is discussed in more detail in [27, starting on page 15], an elementary textbook intended to implement the program described in [26].

6. Ambiguity, analogy, and the place of mathematics

It is time to consider whether $X = MATHEMATICS$ is a possible solution to our proposition (P), stated at the outset of this article. Not everyone will agree on any solution, of course, but consideration of such a possibility will surely involve our own conception of mathematics and of its place in liberal education.

In 1928 Max Dehn [10] stated that mathematics stands between the humanities and the natural sciences, "spheres that are unfortunately disjoint in our country." Dehn's remark is an early foreshadowing of the two cultures, a phrase made famous by C. P. Snow [30] to describe the large and growing gap between the sciences and the humanities. We may be surprised when reminded that during the Enlightenment the redoubtable Emilie du Châtelet not only translated Newton's *Principia* into French (published posthumously in 1759), but also encouraged her consort, Voltaire, to join her in becoming at home in both cultures [2]. Voltaire, the consummate man of letters, became an admirer of Newton and was to remark that there was more imagination in the head of Archimedes than in that of Homer. But of course, even in the Enlightenment there were not many who aspired to be so well rounded as this famous couple.

In fact, one might argue that the two cultures have been with us since the inception of liberal education in ancient Greece, as indicated, for example, by the dispute between Plato and Isocrates mentioned above. Yet it is worth noting that in this period so long ago the sciences and the humanities each respected and reinforced the other much more so than today. The critic Edmund Wilson has pointed out a common goal in the endeavors of the dramatist Sophocles and the geometer Euclid: Both are concerned with the consequences of given initial conditions. In the case of Euclid we see how the axioms force upon us the propositions that follow, while in a drama by Sophocles we see how the interaction of initially given human impulses must inevitably play out in the end. "The kinship, from this point of view, of the purposes of science and art," says Wilson, "appears very clearly in the case of the Greeks" [33, page 269].

What has happened to this kinship that owed much to the service of mathematics as an intermediary? Its decline is surely connected with the fact that many people today tend to identify mathematics with the sciences, or with quantitative literacy in education. They could only be puzzled by the suggestion that $X = MATHEMATICS$ might be a solution to proposition (P). Such a suggestion might be considered, it seems, only by taking account of the humanistic side of mathematics.

For most of us, forgetting the long evolution of our modern views, the ancient Greeks sometimes seem to have gone out of their way to couch mathematical ideas in geometric form. The condition of Eudoxus in *Euclid V* becomes more perspicuous when we reinterpret it, as Dedekind did, in terms of real numbers. A simpler example is given by the Pythagorean theorem, which we associate with the algebraic equation $a^2 + b^2 = c^2$ or with the distance formula in the Cartesian plane. Euclid's geometric proof of the theorem, however, says nothing about numbers. It shows how to decompose a square on the hypotenuse into two rectangles, each respectively equal in area to the squares built upon the legs of the given right triangle.

Ambiguity is at play here. A similar sort of ambiguity can be seen when considering perhaps the most important notion in modern mathematics. As Hermann Weyl writes [32, page 8], "Nobody can explain what a function is, but ..." Weyl goes on to explain what really counts, but who can be said to comprehend fully the idea of a function without being able to hold in mind simultaneously a static, kinematic, and geometric concept? "By x," William Feller used to say, perhaps to irritate precisionists, "I mean the function that takes precisely the value x at the point x." Feller also made the provocative remark that ambiguity is what makes mathematics possible, but it seems unclear how seriously he meant this to be taken. Nevertheless, the way that we get and value ideas in mathematics[11], the way we learn and teach the subject, and the way we hold it in mind, relies upon an (inborn?) appreciation for a certain kind of productive ambiguity that is not the enemy of clarity (cf. [14]).

For example, a nontrivial "if-and-only-if" statement can be thought of as announcing an ambiguity in the sense that ostensibly different things can somehow be seen to be the same. Whenever we speak of an equivalence class by calling attention to a particular representative of it —"consider the rational number 22/7"—we are of course speaking ambiguously. And a similar remark applies to isomorphisms, which can often be regarded as elaborate analogies. A proof of the assertion that $\mathbb{R}$ is *categorical* in the sense that any

[11] See, for example, the chapter "Romance in Reason" of [27], whose title is intended to paraphrase the chapter's thesis, that elements of Romanticism are clearly evident in the development of 17$^{\text{th}}$-century analysis (calculus) during the so-called Age of Reason.

two complete ordered fields are isomorphic shows us a functional correspondence making it clear that pairs of elements of the two fields are analogous on several levels: with regard to their additive, multiplicative, and order properties.

How is it possible that mathematics, the most precise of subjects, carries along with it aspects of ambiguity? The relation of the word to *ambigere*, a Latin verb meaning "to wander about," hints at the answer. Is ambiguity not due—in part, at least—to the role of analogy, which is now commonly seen as crucial to all thought? "Any history of thought," writes Buchanan [5, page 174], "might begin and end with the statement that man is an analogical animal."

Analogies give us a priceless source of new ideas, but they must often be used with caution. We have noted how Archimedes says in his *Method* that we should be bold to seek out ideas from analogies—or from mathematical models, as we might call the fruits of his method today. At the same time Archimedes reminds us that in mathematics a proof must usually follow, for otherwise the analogy may mislead. Using the language of modern popular culture, we might say that Archimedes's remark has to do with our right brain and our left brain. Most of us are naturally stronger on one side than on the other, and the two-culture gap is exacerbated when our education tends to promote this disparity.

Should not a good liberal education seek to shore up our weaker side so we can better see these cooperating opposites? Perhaps the sculpture pictured at the outset of this article (Figure 1) has a deeper truth to tell. If either the seer or the sage depicted in that sculpture were eliminated, the unity of Raphael's composition would be destroyed. See Figure 2.

L. F. Richardson, writing in 1927, reminds us of this issue.

> There are, so to speak, in the mathematical country, precipices and pit-shafts down which it would be possible to fall; but that need not deter us from walking about. Yet if we wish to explore these steep descents, the pedestrian must be supplemented by the acrobatics of the pure mathematician [29, page 42].

More recently, André Weil is quoted by Armand Borel [B2009] as follows:

> Nothing is more fecund, all the mathematicians know it, than those obscure analogies, the blurred reflections from one theory to

Figure 2: "Cooperating Opposites" by Jeanie Stephenson. Reproduced by permission.

> another... nothing gives more pleasure to the researcher. One day the illusion drifts away, the premonition changes to a certitude: the twin theories reveal their common source before disappearing; as the Gita teaches it, knowledge and indifference are reached at the same time. The metaphysics has become mathematics, ready to form the subject matter of a treatise, the cold beauty of which cannot move us anymore.

> Fortunately for researchers, as the fogs clear away on some point, they reappear on another...

In ancient times, as we have noted, the notion of an analogy was the common property both of mathematics and of the humanities. On the one hand it was capable of rendering precise propositions, while on the other it could allude ambiguously to connections on more than one level. The search for hidden analogies motivates us all.

7. Concluding notes and dedication

An anonymous referee suggested references [5] and [20] in response to an earlier version of this article. The connection between litotes and simple notions of a limit was noted in [28]. For discussions of analogy and ambiguity relating to mathematics, see Chapter 4 of [8] and Section I of [6]. And see [19] for the role of analogies in understanding nature.

This paper is dedicated to the memory of Edward McCrady (1906–1981), whose variety of interests and love of language inspired many.

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