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Population Growth Models: 
Relationship between sustainable fishing and making a profit

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Abstract:  
In this paper, we develop differential equations that model the sustainable harvesting of species having different characteristics. Specifically, we assume the species satisfies one of two different types of density dependence. From these equations, we consider maximizing sustainable harvests. We then introduce a cost function for fishing and study how maximizing profit affects the harvesting strategy. We finally introduce the concept of open access which helps explain the collapse of many fish stocks.

The equations studied involve relatively simple rational and exponential functions. We analyze the differential equations using phase-line analysis as well as graphing approximate solutions using Euler’s method, which is really a discrete form of the differential equation.

1 Introduction

Students rarely see how mathematical modeling relates to issues of public policy, particularly sustainability. In this paper, we discuss how this can be introduced into a course on differential equations or mathematical modeling. In particular, our discussion shows how modeling can help us develop general principles related to sustainability and profitability in the fishing industry. These ideas illuminate the difficulty the world has had related to overfishing and why these issues are difficult to overcome.

As a side issue, when students are given a differential equation

\[ f'(t) = g(f(t)). \]

we believe that they often do not understand where \( f'(t) \) comes from. We recommend introducing a discrete model and then taking the limit as the time unit goes to zero to help them better understand differential equation models.

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As an example, let $p(t)$ represent the size of a population of some species at time $t$ and let $r$ represent the per capita growth rate of the species as a function of the population size, that is, $r(p(t))$. Consider a fixed unit of time, say $h$. A model for population growth could then be

$$p(t + h) - p(t) = hr(p(t))p(t). \tag{1.1}$$

This means that the change in population size is proportional to the size of the population, the unit of time, and the per capita growth rate.

To develop a continuous model from equation (1.1), we divide by $h$ and then take the limit as $h$ goes to zero, giving

$$\lim_{h \to 0} \frac{p(t + h) - p(t)}{h} = r(p(t))p(t)$$

which simplifies to the density dependent population growth model,

$$p'(t) = dp/dt = r(p(t))p(t). \tag{1.2}$$

This helps students understand where $p'$ comes from in a differential equation and reinforces their understanding of the definition of the derivative. This step is often skipped when developing continuous models.

**Remark 1.1.** The discrete version of model (1.1) is often written as

$$p_{n+1} = p_n + r(p_n)p_n$$

where $n$ represents the number of time periods of length $h$, and the $h$ from (1.1) is incorporated into the per capita growth rate, which would then be the per capita growth rate per unit of time.

Section (2) discusses the development of two reasonable population growth models, compensation growth and depensation growth. In compensation growth, the per capita growth rate is decreasing while for depensation growth, the per capita growth rate increases, then decreases.

For each of these two models, we introduce four natural aspects a good model of population growth should satisfy and use these to develop differential equations that incorporate these aspects. In each case, there are two types of differential equations, one that uses rational functions and the other using exponential functions. The differential equations developed parallel four of the historical discrete population models, [2, 3, 5, 7]. Much of this section is adapted from [8, 9, 10].

Section (3) briefly discusses three approaches to studying first order autonomous nonlinear differential equations of the form (1.2):

- finding solutions using separation of variables,
- phase-line analysis, and
- graphing approximate solutions.
We will find that in most cases, finding solutions is difficult and not very illuminating, while phase-line analysis is usually quite easy, giving useful information. Finding approximate solutions reinforces results from phase-line analysis and sometimes gives additional insight.

In Section (4), we introduce harvesting or fishing into our model and use phase-line analysis to find stable and unstable equilibrium population sizes for different harvesting rates. We note depensation models give quite different results from compensation models which points to serious possible problems. From this, we determine sustainable equilibrium harvests in terms of the harvest rates and note the implications of these results.

Section (5) introduces maximizing sustainable revenue and profit and develops general principles, particularly that maximum profit occurs at a lower harvest rate than maximum revenue, which is beneficial to both the fish being harvested and the fishing industry, itself.

Section (6) introduces the concept of open access and its relation to the tragedy of the commons. This points out issues that not only can, but have happened in the fishing industry.

Section (7) summarizes some attempts that have been made to solve the problems identified in this paper, and points out some of the issues with implementing these attempts.

2 Population Growth Models

In this section, we make simple assumptions about the population growth of a single species and use these to develop a model for population growth. In particular, we consider the density dependent population growth model

\[ p'(t) = \frac{dp}{dt} = r(p(t))p(t) \]

where \( r \) is the per capita growth rate as a function of population size \( p \).

While we ignore aspects of age-related growth, predator-prey interactions, and competition between species, our models can still help us generate principles related to the fishing industry. In particular, we make assumptions about the growth of a species and interpret these assumptions in the development of a reasonable per capita growth rate, \( r \), in terms of the population size, \( p \).

2.1 Compensation Models

We first consider a species which exhibits compensation growth. For such a species, as the size of the population increases, there is less food and space available, so we assume its per capita growth rate, \( r \), is a decreasing function of the population size. This means that the smaller the population size, the larger the per capita growth. This leads to the assumption that there is a positive *intrinsic* per capita growth rate \( b \), which is the maximum per capita growth rate, and it occurs when the population size is (near) zero. We acknowledge that assuming a growth rate for zero population makes no sense, but in reality we are assuming that the limit as \( p \) decreases to zero equals \( b \).
The next condition is that for a given species in a given location, there is an environmental carrying capacity $L$ for that species, which is determined by, among other things, food and space availability. Thus, if $p \in (0, L)$ then $r(p) > 0$, and if $p > L$ then $r(p) < 0$. For simplicity, we re-scale so that $p(t)$ gives the proportion that the population size is above or below the carrying capacity. This means, $r(1) = 0$.

**Remark 2.1.** If we are actually modeling a specific species, we would generally not know $L$, so would be unable to rescale our equation. For developing general principles, rescaling is not a problem.

For a discrete model, $r(p_0) < -1$ would mean that for the population size, $p_0$, more than all of the population would die off. Since most populations have aspects of discrete behavior, we therefore believe it is reasonable to assume that $r(p) \geq -1$ for all $p > 0$. Thus, our last condition is

$$\lim_{p \to \infty} r(p) = -1.$$  

**Remark 2.2.** For a continuous model, there is nothing that implies $r(p) > -1$. We could just as easily have assumed

$$\lim_{p \to \infty} r(p) = a$$

for some constant $a < -1$. In fact, for the frequently considered logistic model for population growth, the per capita growth rate is

$$r(p) = b(1 - p)$$

and therefore

$$\lim_{p \to \infty} r(p) = -\infty.$$  

While the assumption that $r(p)$ goes to negative infinity is also reasonable for a continuous model, there is nothing about population growth that implies a linear per capita growth rate, such as for the logistic model. We leave it to the interested reader to investigate how either of these alternate assumptions would affect the following discussion.

**Definition 2.3.** The differential equation (1.2)

$$p' = dp/dt = r(p)p$$

is a **compensation population model** if the function $r$ satisfies the conditions:

1. $r(0) = b > 0$,
2. $r$ is a decreasing function of population size $p$,
3. $r(1) = 0$, and
4. $\lim_{p \to \infty} r(p) = -1$.  


Figure 1: Two possible per capita growth rates, \( r \). Graph (a) is concave up while graph (b) has a point of inflection.

Students should be able to conclude that \( r \) should have one of the shapes in Figure (i). We might ask if \( r \) should be concave up as in graph (a) or should have a point of inflection, as in graph (b). This will be dealt with briefly, later. We do note that \( r \) could have several points of inflection, but our models do not consider this.

The most common types of functions having shapes similar to Figure (i) are rational functions, exponential functions, and conditionally defined functions. While we could easily develop conditionally defined functions that work, we will focus on rational and exponential functions.

2.1.1 Rational per capita growth rate

The development of a rational per capita growth rate for differential equation (1.2) was developed for discrete population models in [8]. We begin with

\[
r(p) = \frac{a_1 + a_2 p^j}{a_3 + a_4 p^k}.
\]  

(2.1)

Condition 1), \( r(0) = b \), gives that

\[
r(0) = b = \frac{a_1}{a_3}
\]

which is equivalent to \( a_1 = b \) and \( a_3 = 1 \). Substitution and using condition 3), \( r(1) = 0 \), gives

\[
r(p) = \frac{b - bp^j}{1 + a_4 p^k}.
\]

Next, condition 4),

\[
\lim_{p \to \infty} r(p) = -1,
\]

means \( j = k > 0 \) and

\[
\lim_{p \to \infty} r(p) = \frac{-b}{a_4} = -1
\]

so \( a_4 = b \). Thus, \( r \) is of the form

\[
r(p) = \frac{b - bp^j}{1 + bp^j}.
\]  

(2.2)
Careful calculation and simplification gives that
\[
\frac{dr}{dp} = r'(p) = \frac{-b(1 + b)jp^{j-1}}{(1 + bp^j)^2} < 0 \tag{2.3}
\]
for \( r > 0 \), so \( r \) is decreasing on \((0, \infty)\) and \( r \) satisfies condition 2), so \( r \) satisfies all four conditions of a compensation model.

Substitution of (2.2) into (1.2) gives
\[
p'(t) = \frac{bp(t) - bp^j(t)}{1 + bp^j(t)}. \tag{2.4}
\]
and differential equation (2.4) is a compensation model for population growth.

A little work with derivatives shows that if \( 0 < j \leq 1 \), then \( r \) is concave up, similar to Figure (1a). Such per capita growth rates have been used to model populations which exhibit contest competition among themselves. In contest competition, there is a hierarchy in the species ability to forage or hunt. The discrete form of model (2.4) was first developed by P. Beverton and S. Holt [1957], with \( j = 1 \), to model exploited fish populations.

If \( j > 1 \), then \( r \) has a point of inflection. These per capita growth rates have been used to model species exhibiting scramble competition in which most of the individuals in the species are comparable in their ability to forage. The discrete form of model (2.4) with \( j > 1 \) was developed by J. Maynard-Smith and M. Slatkin [1973] as a simple model of a prey species in absence of a predator.

See [8, 9, 10] for more details on both contest and scramble competition.

2.1.2 Exponential per capita growth rate

The development of an exponential \( r \) is analogous to its development for the discrete model. Consider the exponential function
\[
r(p) = a_1(a_2)^p + a_3.
\]
Since \( \lim_{p \to \infty} r(p) = -1 \), then \( a_3 = -1 \). Then \( r(0) = b = a_1 - 1 \), so \( a_1 = 1 + b \) and
\[
r(p) = (1 + b)(a_2)^p - 1.
\]
Finally, \( r(1) = 0 = (1 + b)(a_2)^1 - 1 \), so
\[
a_2 = \frac{1}{1 + b}.
\]
This gives that
\[
r(p) = (1 + b)^{1-p} - 1, \tag{2.5}
\]
with \( j > 0 \). Careful calculation and simplification gives that
\[
r'(p) = \ln(1 + b)(1 + b)^{1-p}(-j)p^{j-1} < 0 \tag{2.6}
\]
so \( r \) is decreasing and therefore satisfies the condition of definition (2.3).
Substitution of equation (2.5) into (1.2) gives

\[ p'(t) = (1 + b)^{1-p(t)} p(t) - p(t). \]  

(2.7)
as an exponential population model for compensation growth.

Again, if \(0 < j \leq 1\), then \(r\) is concave up and models a species exhibiting contest competition. W. Ricker [1954] studied the discrete form of this equation with \(j = 1\) to study stock and recruitment of fish populations. When \(j > 1\), \(r\) has a point of inflection and models a scramble competition. The discrete form of this model was first considered by Bellows [1981].

2.2 Depensation Models

In Section (2.1), we assumed the per capita growth rate was decreasing. In fact, for some species, if the size of the population decreases by too much, the per capita growth rate may also decrease. This may happen, and for several reasons. If the population size is too small and spread over a large area, it may be difficult to find a mate. For prey species that travel in herds, there may not be enough individuals to protect themselves. For predator species that hunt in packs, there may not be enough to hunt effectively.

**Definition 2.4.** The differential equation (1.2)

\[ p' = r(p)p \]

is a depensation population model if the function \(r\) satisfies the conditions:

1. the maximum of the per capita growth rate occurs at \(0 < p = m < 1\), at which it equals the intrinsic per capita growth rate, \(b\).
2. \(r\) is increasing on \((0, m)\) and decreasing on \((m, \infty)\).
3. \(r(1) = 0\), and
4. \(\lim_{p \to \infty} r(p) = -1\).

Figure (2) shows the graphs of two possible depensation per capita growth rates. Note that in graph (a), \(r\) is positive on the interval \(0 \leq p < 1\), but in graph (b), \(r\) is negative for \(0 \leq p < v < 1\). This means that if the population size is less than \(v\), the population will continue to decrease to extinction. The value, \(v\) is called the minimum viable population size. Such a case is called critical depensation.

The process of developing a per capita growth rate to model depensation is a variation on what was done for compensation, but pulling in other geometric concepts such as translation and dilation of functions.

We begin similarly to [8] in which depensation models were developed for discrete models. Let’s consider a rational per capita growth rate similar to (2.1)

\[ r(p) = \frac{a_1 + a_2 p^j}{1 + a_3 p^j}. \]

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Figure 2: Possible depensation per capita growth rates. In graph (a), the population size will increase when $0 \leq p < 1$ but in graph (b), there is a point below which the population will die out.

For such a function to have a maximum, the exponent must be even and positive, so we will instead use

$$r(p) = \frac{a_1 + a_2 p^{2j}}{1 + a_3 p^{2j}}$$

(2.8)

with $j > 0$. The idea is to translate, then dilate this function.

Since $r$ has a maximum at $p = m$, we translate function (2.8), giving

$$r(p) = \frac{a_1 + a_2 (p - m)^{2j}}{1 + a_3 (p - m)^{2j}}.$$

As before,

$$r(m) = \frac{a_1 + a_2 (m - m)^{2j}}{1 + a_3 (m - m)^{2j}} = a_1 = b.$$

Since

$$\lim_{p \to \infty} r(p) = \frac{a_2}{a_3} = -1,$$

then $a_2 = -a_3$ and we have

$$r(p) = \frac{b - a_3 (p - m)^{2j}}{1 + a_3 (p - m)^{2j}}.$$

Finally,

$$r(1) = \frac{b - a_3 (1 - m)^{2j}}{1 + a_3 (1 - m)^{2j}} = 0$$

which gives

$$b - a_3 (1 - m)^{2j} = 0$$

and

$$a_3 = \frac{b}{(1 - m)^{2j}}.$$

Thus, this modified rational depensation per capita growth rate is

$$r(p) = b \left( \frac{1 - \frac{p - m}{1 - m}}{1 + b \left( \frac{p - m}{1 - m} \right)^{2j}} \right),$$

(2.9)
and the differential equation modeling this is

\[ p' = bp \left( \frac{1 - \left( \frac{p-m}{1-m} \right)^{2j}}{1 + b \left( \frac{p-m}{1-m} \right)^{2j}} \right). \]

Similar analysis gives the exponential per capita growth rate as

\[ r(p) = (1 + b)^{1 - \left( \frac{p-m}{1-m} \right)^{2j}} - 1, \]

and the corresponding differential equation

\[ p' = p(1 + b)^{1 - \left( \frac{p-m}{1-m} \right)^{2j}} - p. \]

### 3 Analyzing first order differential equations

The most obvious approach to studying differential equations is to try to find a solution. Let’s consider the simplest of our models, the compensation differential equation (2.4) with \( j = 1 \) and \( b = 1 \),

\[ \frac{dp}{dt} = \frac{p - p^2}{1 + p}. \] \hspace{1cm} (3.1)

To solve this equation, we use separation of variables, giving

\[ \frac{1 + p}{p - p^2} dp = 1 dt. \]

Rewriting using partial fractions gives

\[ \left( \frac{1}{p} + \frac{2}{1 - p} \right) dp = 1 dt. \]

Integrating both sides gives

\[ \ln |p| - 2 \ln |1 - p| = t + c_1 \]

which can be rewritten as

\[ \frac{p}{(1 - p)^2} = c_2 e^t. \]

We can solve for \( c_2 \) in terms of \( p(0) \) and use the quadratic formula to find a solution for \( p(t) \). This is a lot of work and the solution gives little information about the behavior of \( p \) other than

\[ \lim_{t \to \infty} p(t) = 1. \]

If \( b \neq 1 \), then we would not be able to use the quadratic formula to solve for \( p \). Even more, if \( j \neq 1 \), then the use of partial fractions would result in a relation for \( p \) that might not be solvable. In fact, for many of our differential equations, partial fractions would not even work. Thus, we need a better method to analyze our differential equations.
3.1 Phase-lines

In this approach, we describe the qualitative behavior of solutions without actually finding a solution. Let’s again consider the simple population model (3.1).

\[
\frac{dp}{dt} = \frac{p - p^2}{1 + p}.
\]

Note that since \( p \) is a function of \( t \), then \( p' \) can be considered as a function of \( t \). On the other hand, through model (3.1), \( p' \) can also be considered as a function of \( p \), as seen in Figure 3).

Figure 3: Graph of \( \frac{dp}{dt} \) as function of \( p \), indicating intervals on which \( p' \) is positive and negative, that is, \( p \) is increasing and decreasing.

Consider the functions \( p(t) = 0 \) and \( p(t) = 1 \). In both cases, \( p'(t) = 0 \). On the other hand, if we consider \( p' \) as a function of \( p \), then

\[
\frac{dp}{dt} = 0 = \frac{p - p^2}{1 + p},
\]

so \( p(t) = 0 \) and \( p(t) = 1 \) are constant solutions to model (3.1). In summary, if we have the differential equation \( dp/dt = g(p) \), then values, \( p_E \), for which \( g(p) = 0 \) are constant solutions, and are considered equilibrium values.

Note that when \( dp/dt = g(p) > 0 \) is positive, \( p \) is increasing and when it is negative, \( p \) is decreasing. Figure (3) indicates that if \( 0 < p(0) < 1 \), then \( p \) is increasing and if \( -1 < p(0) < 0 \) or \( p(0) > 1 \), \( p \) is decreasing. This implies that if \( p(0) > 0 \), then \( p \) will be going toward \( p_E = 1 \), which is called a stable equilibrium value. We also note that if \( p(0) \) is close to, but not equal to zero, then \( p \) goes away from zero. We say \( p_E = 0 \) is an unstable equilibrium.

Remark 3.1. In our model, it does not make sense for \( p(0) < 0 \), but for a differential equation, we can mathematically analyze negative solutions. In fact, since \( y = -1 \) is a vertical asymptote for \( g \), deeper analysis shows that if \( -1 < p(0) < 0 \), then \( p(t) \) goes to negative infinity in finite time.
3.2 Approximate solutions

Another approach to studying differential equations is to graph approximate solutions using a method similar to (1.1), that is, use

\[ p(t + h) = p(t) + h \cdot r(p(t)) \cdot p(t) \]  
(3.2)

for a sufficiently small increment \( h \). This is known as Euler’s method.

For differential equation (3.1),

\[ \frac{dp}{dt} = \frac{p - p^2}{1 + p} \]

we use \( p(0) = 0.1 \) and \( h = 0.1 \). From this, we found that \( p(0.1) = 0.108 \) and \( p(0.2) = 0.117 \). Going forward, we find \( p(3) = 0.530 \). This approximate solution is graphed as the orange function, \( p_1 \) in Figure (4). Note that it goes to \( p_E = 1 \), as predicted. Red solution, \( p_2 \) started at \( p_2(0) = 2 \) while blue solution, \( p_3 \), started at \( p_3(0) = -0.1 \). Note that these solutions behave as predicted by the phase-line analysis.

![Figure 4: Approximate solutions to differential equation (3.1) with \( p_1(0) = 0.1 \) (orange), \( p_2(0) = 2 \) (red), and \( p_3(0) = -0.1 \) (blue).](image)

3.3 Phase-line analysis for a depensation model

Consider the depensation per capita growth rate (2.9) with \( j = 1 \), \( m = 0.7 \) and \( b = 0.8 \),

\[ r(p) = 0.8 \left( \frac{1 - \left( \frac{p-0.7}{0.3} \right)^2}{1 + 0.8 \left( \frac{p-0.7}{0.3} \right)^2} \right), \]

whose graph is seen in Figure (5 a). Note that the minimum viable population size is \( v = 0.4 \). From this graph, we graph

\[ \frac{dp}{dt} = 0.8p \left( \frac{1 - \left( \frac{p-0.7}{0.3} \right)^2}{1 + 0.8 \left( \frac{p-0.7}{0.3} \right)^2} \right) \]
as a function of $p$ in Figure (5 b). Using phase-line analysis, we conclude that there are three equilibrium values, the stable $p_0 = 0$, the unstable $p_u = v$, and the stable $p_s = 1$. The population size, $p_u = v$, was called the minimum viable population and if the population drops below this value, it will go to extinction.

Figure 5: Graph (a) is depensation per capita growth rate and graph (b) is the corresponding growth function.

4 Fishing and Harvesting

In this section, we will model the management of a population of species that is fished or hunted. Models similar to these have been applied to the fishing industry for years, in an attempt to understand how to preserve a population of fish while maintaining a high yield of fish from year to year.

We now consider harvesting models which can be applied to any resource modeled with a density dependent population model where the dynamical system is of the form

$$\frac{dp}{dt} = r(p)p$$

where $r$ might be one of the functions developed in Section (2) with intrinsic per capita growth rate $b$.

It might seem reasonable to assume the same harvest every time period, but this is unrealistic. As the size of the population decreases, it is harder to harvest the same amount. In fact, if the total population is smaller than our constant harvest, it is impossible to
harvest the same amount. Therefore, we will make what is called the **constant effort assumption**, that is, we harvest the same percent of the population, \( h_0 \rho \), where \( h_0 \) is the fixed constant proportion of the population harvested each time period. The differential equation that models this situation is

\[
\frac{dp}{dt} = r(p)\rho - h_0 \rho. \tag{4.1}
\]

Using phase-line analysis, we note that the equilibrium population sizes are solutions to

\[
(r(p) - h_0)p = 0
\]

which are \( p_0 = 0 \) and solutions to

\[
r(p) = h_0.
\]

### 4.1 Harvesting a compensation population

If our population satisfies a compensation model, then \( r \) is a decreasing function and there is one solution to \( r(p) = h_0 \) when \( 0 < h_0 < b \), say \( p_s \), as seen in Figure (6). This means that there are at most two equilibrium values, \( p_0 = 0 \) and maybe \( p_s \). From Figure (6), we also see that for there to be two equilibrium values for model (4.1), we must have

\[
0 \leq h_0 < b.
\]

If \( h_0 \geq b \), then \( p = 0 \) is the only equilibrium population size.

![Figure 6: Graph of compensation per capita growth rate \( r \) (blue) with harvest rate \( h_0 \) (purple), \( 0 < h_0 < b \). It shows equilibrium population size, \( p_s \).](image)

Figure (7) shows the graph of the growth function \( r(p)\rho \) (blue) and the harvest line \( H = h_0 \rho \) (purple) with \( 0 < h_0 < b \). From that, we can use phase-line analysis by mentally imagining the difference between the growth function and the harvest line. We then see that:

- the only solutions to \( r(p)\rho - h_0 \rho = 0 \) are \( p_0 = 0 \) and \( p_s \), the solution to \( r(p) = h_0 \);
- when \( 0 < p < p_s \), then \( r(p)\rho - h_0 \rho > 0 \) so \( p \) will be increasing on that interval;
- when \( p < 0 \) or \( p > p_s \), \( r(p)p - h_0p < 0 \) so \( p \) will be decreasing on those intervals,
- thus \( p_0 = 0 \) is an unstable equilibrium and \( p_s \) is a stable equilibrium population size, and
- the stable sustainable harvest is \( H_s = h_0 p_s \).

\[ \text{Figure 7: Graph of } r(p)p \text{ and harvest line, } H = h_0 p \text{ for compensation per capita growth rate with } 0 < h_0 < b. \text{ This graph shows } p_s \text{ is a stable equilibrium and } p_0 = 0 \text{ is an unstable equilibrium.} \]

If we visualize increasing the harvest rate in Figure (7), we see that the sustainable harvest, \( H_s \), increases up to a maximum, and then decreases toward zero as \( h_0 \) approaches \( b \). Thus, there is a maximum sustainable harvest, as seen in Figure (8). To find this rate, we find the maximum of \( r(p)p \), which we designate as \( (p_m, H_m) \). Then the harvest rate for which \( H = h_0 p \) goes through this point is

\[ h_0 = \frac{H_m}{p_m}. \]

\[ \text{Figure 8: Harvest line, } H = h_0 p, \text{ goes through maximum of the function } r(p)p, \text{ resulting in maximum sustainable harvest.} \]

On the other hand, if \( h_0 > b \), then we obtain Figure (9). From this graph, we can use phase-line analysis to see that:
• \( r(p)p - h_0 p = 0 \) only when \( p = 0 \),

• if \( p > 0 \), then \( r(p)p - h_0 p < 0 \), so \( p \) will be decreasing on that interval, and

• if \( p < 0 \), then \( r(p)p - h_0 p > 0 \) so \( p \) will be increasing on that interval.

Figure 9: Graph of \( r(p)p \) and harvest line, \( h_0 p \) for compensation per capita growth rate with \( h_0 > b \). This graph shows \( p_0 = 0 \) is a stable equilibrium.

We can summarize these results as general principles.

**Principles 4.1.** Suppose we have a species that satisfies a compensation population growth model with intrinsic per capita growth rate, \( b \). We harvest the population at the constant rate, \( h_0 \).

• If \( 0 < h_0 < b \), then there is an equilibrium population size, \( p_s \), which is the solution to
  \[
  r(p) = h_0.
  \]

• The equilibrium sustainable harvest is then
  \[
  H_s = p_s h_0.
  \]

• The maximum sustainable harvest is obtained by harvesting at the rate
  \[
  h_0 = \frac{H_m}{p_m}
  \]

  where \((p_m, H_m)\) is the maximum of the function \( r(p)p \).

**Example 4.2.** Suppose we have the per capita growth rate given by (2.2) with intrinsic growth rate \( b = 0.8 \) and \( j = 1 \), that is,
  \[
  r = \frac{0.8 - 0.8p}{1 + 0.8p}.
  \]
Suppose we harvest at the rate of \( h_0 = 0.2 \) or 20\% of the population harvested each time period. Solving
\[
\frac{0.8 - 0.8p}{1 + 0.8p} = h_0 = 0.2
\]
gives that
\[
p_s = \frac{0.8 - 0.2}{0.8(0.2) + 0.8} = 0.625.
\]
The sustainable harvest is then
\[
H_s = p_s h_0 = 0.125,
\]
or 12.5\% of the carrying capacity.

The maximum of the function
\[
g(p) = r(p)p = \frac{0.8p - 0.8p^2}{1 + 0.8p}
\]
can be found by finding where \( g'(p) = 0 \) which is
\[
p_m = -1 + \sqrt{1.8}/0.8 \approx 0.427.
\]
The maximum occurs at \( g(0.427) \approx 0.146 = H_m \). This sustainable harvest results by harvesting at the rate of
\[
h_0 = \frac{0.146}{0.427} \approx 0.342
\]
or 34.2\%.

We note that in the differential equation
\[
dp/dt = r(p)p - h_0p = \frac{0.8p - 0.8p^2}{1 + 0.8p} - h_0p,
\]
for each value, \( 0 < h_0 < 0.8 \), there are two equilibrium values, \( p_0 = 0 \) and
\[
p_s = \frac{0.8 - h_0}{0.8(1 + h_0)}
\]
which is found by solving \( r(p_s) = h_0 \) for \( p_s \). Phase-line analysis shows \( p_s \) is stable and \( p_0 \) is unstable. But for \( h_0 \geq 0.8 \), \( p_0 \) is stable. This is summarized in what is called a bifurcation diagram, seen in Figure (10). This is a graph of the equilibrium population sizes, \( p_s \), as a function of the harvest rates, \( h_0 \). For each value of \( h_0 \) on the horizontal axis, the arrows go in the vertical direction that \( p(t) \) will go for \( p(0) \) in the interval indicated by the extent of the arrow. Thus, this figure describes the behavior for a collection of differential equations.

Remark 4.3. In Figure (10), we say that a transcritical bifurcation occurs at the red/green point \((0.8, 0)\), and that \( p_0 = 0 \) is semistable when \( h_0 = 0.8 \). This will be discussed in more detail later.
In Example (4.2), fishing at a rate higher than 34.2 percent of the carrying capacity results in a smaller sustainable harvest. Unfortunately, fisherman often harvest at a rate higher than the optimal rate, resulting in a partial collapse of the fish stock.

There is an alternate way of visualizing harvesting. We were given a harvesting rate, $h_0$ and solved
\[ r(p) = h_0 \]
to find the positive stable equilibrium population size, $p_s$. The sustainable harvest is then $p_s h_0$. Thus, if we consider the harvest rate as the independent variable, we can get the sustainable harvest, $H_s$, as a function of $h_0$.

For the rational compensation model
\[ r(p) = \frac{b - bp^j}{1 + bp^j}, \]
solving $r(p) = h_0$ for $p$ gives
\[ p_s = \left( \frac{b - h_0}{b + bh_0} \right)^{1/j}. \]
We now have a function for the sustainable harvest in terms of the harvest rate
\[ H_s = h_0 p_s = h_0 \left( \frac{b - h_0}{b + bh_0} \right)^{1/j}. \]

For Example (4.2), this would give
\[ H_s = h_0 p_s = h_0 \left( \frac{0.8 - h_0}{0.8 + 0.8h_0} \right). \]

We could then find the maximum directly at $(h_0, H_s) = (0.342, 0.146)$, as seen in Figure (11). This figure also shows that for there to be a positive sustainable harvest, we must have $0 < h_0 < 0.8 = b$. 

---

Figure 10: Bifurcation diagram: graph of $p_s$ represents a stable equilibrium, so is in green. Horizontal axis is red on interval $(0, 0.8)$, where $p_0 = 0$ is unstable and is green on interval $(0.8, \infty)$ where $p_0 = 0$ is stable.
Figure 11: Sustainable harvest, $H_s$, as a function of harvest rate, $h_0$, for Example (4.2).

Figure (11) is the clearest explanation that as the harvest rate increases, the sustainable harvest will increase, but then decrease. Thus, we have to be careful to maximize our sustainable harvest.

For completeness, we note that if we model a compensation population with an exponential $r$, then solving for $p_s$ in terms of $h_0$, then computing $H_s = p_s h_0$, we get

$$H_s = h_0 \left(1 - \frac{\ln(1 + h_0)}{\ln(1 + b)}\right)^{1/j}.$$ 

In general, an expression for the per capita growth rate $r$ cannot be found, but at least we have some idea about issues that can occur and that we should watch for.

4.2 Harvesting a depensation population

When harvesting a population modeled by a depensation per capita growth rate, more serious problems can occur. Let’s begin with the rational depensation per capita growth rate (2.9),

$$r(p) = b \left(1 - \frac{1 - (\frac{p-m}{1-m})^{2j}}{1 + b \left(\frac{p-m}{1-m}\right)^{2j}}\right).$$

Figure (12) gives the graph of a possible depensation per capita growth rate with $m < 0.5$.

From Figure (12), and remembering that equilibrium values for different harvesting rates are solutions to $r(p) = h_0$, we have that:

- for small $h_0$-values (red line), there will be one positive equilibrium value, that is solution to $r(p) = h_0$, (red circle);
- for larger values of $h_0$ (green line) there will be two positive solutions (green circles);
- when $h_0 = b$ (black line), there is again only one positive solution (black circle); and
- when $h_0 > b$ (purple line), there are no positive solutions.
Figure 12: Graph of a depensation per capita growth rate with positive $r$ for $0 < p < 1$. The four horizontal lines correspond to four different values for $h_0$.

The critical depensation per capita growth rate in Figure (13) is negative for $p < v$, which is called the minimum viable population size. For $0 < h_0 < b$ (green line), there will be two equilibrium solutions (green circles), when $h_0 = b$ (black line), there is only one solution (black circle), and when $h_0 > b$ (purple line), there are no solutions.

Figure 13: Graph of a critical depensation per capita growth rate with positive $r$ for $v < p < 1$. The three horizontal lines correspond to three different values for $h_0$.

In Figure (14) is the function, $g(p) = pr(p)$, and the harvest line $H = h_0p$ for the critical depensation per capita growth rate in Figure (13). Applying phase-line analysis, we see that there are three equilibrium population sizes, the stable $p_0 = 0$, the unstable $p_u$ and the stable $p_s$. The unstable population size, $p_u$, is called the minimal viable population for this harvesting strategy: If the population drops below that value, it will begin dying out
until $h_0$ is sufficiently lowered.

Figure 14: Depensation function, $pr(p)$, and harvest line, $h_0p$. The three equilibrium population sizes are the three points of intersection.

There are two unusual behaviors that can occur with depensation models. It is easiest to explain them with examples.

**Example 4.4.** Let’s consider the depensation species modeled by a rational $r$ with $j = 1$, $m = 1/3$ and $b = 0.5$ that is harvested at a rate of $h_0$, that is,

$$p' = 0.5p \left( \frac{1 - \left( \frac{3p-1}{2} \right)^2}{1 + 0.5 \left( \frac{3p-1}{2} \right)^2} \right) - h_0p. \quad (4.2)$$

Let’s assume a harvest rate of $h_0 = 1/3$. The graph of $g(p) = pr(p)$ and the harvest line, $H = p/3$ are seen in Figure (15 a).

Figure 15: Equation (4.2) with $h_0 = 1/3$: Graph (a) uses phase-line analysis with function $pr(p)$ and harvest line to show $p_0 = 0$ is semistable from below and $p_s = 2/3$ is stable. Graph (b) gives three approximate solutions to support these conclusions.

From Figure (15 a), we can see that if the slope of the harvest line is less than $h_0 = 1/3$, there is only one positive equilibrium value, but if the slope is between $1/3$ and $0.5$, there will be two positive equilibrium values, a stable one and an unstable one. The change in behavior as $h_0$ increases past $1/3$ results from the harvest line $H = (1/3)p$ being tangent to $pr(p)$ at the origin.
The second unusual feature occurs when the harvest rate equals the intrinsic per capita growth rate, $h_0 = b = 0.5$ in this example. Figure (16 a) shows $pr(p)$ and harvest line $H = 0.5p$. Again, there are only two equilibrium population sizes, $p_s = 1/3$ which is semistable from above, and $p_0 = 0$ which is stable. This occurs because the harvest line $H = 0.5p$ is tangent to the growth function at the point $(1/3, 1/6)$. This conclusion is supported from the approximate solutions in Figure (16 b). Note that if $0 < p(0) < 1/3$, then $p(t)$ goes to zero, but very slowly (orange curve).

**Figure 16:** Equation (4.2) with $h_0 = 0.5$: Graph (a) uses phase-line analysis with growth function and harvest line to show $p_0 = 0$ is stable and $p_s = 1/3$ is semistable from above. Graph (b) gives three approximate solutions to support these conclusions.

Graph (16 a) points out a serious problem for species satisfying a depensation model. For a compensation model, as the harvest rate approaches the intrinsic per capita growth rate, the equilibrium population size decreases toward zero, making it clear there is a problem. For this depensation model, as the harvest rate approached the intrinsic per capita growth rate, the equilibrium population size remained relatively large, above $1/3$. Then as the harvest rate increases past 0.5, the only stable equilibrium is zero and the population starts decreasing toward extinction. In other words, for a depensation model, we do not have a warning as we do for compensation models. A summary of this information is seen in the bifurcation diagram (17) in which the stable, semistable, and unstable equilibrium values are plotted for each harvest rate. This graph has two bifurcation points, $(1/3, 0)$ which is called a transcritical bifurcation, and $(0.5, 1/3)$ which is called a saddle-node bifurcation.

Behavior similar to that found in Example (4.4), in which a species can be driven to extinction through excessive harvesting, is called the **Allee effect** [1].

Figure (18) is a bifurcation diagram for a critical depensation model, such as graphed in Figure (13).

The bifurcation graphs are found by solving for $p_s$ and $p_u$ in terms of $h_0$ for our rational critical depensation model (2.9), giving

\[
p_s = m + (1 - m) \left( \frac{b - h_0}{(1 + b)h_0} \right)^{1/2j}
\]

and

\[
p_u = m + (m - 1) \left( \frac{b - h_0}{(1 + b)h_0} \right)^{1/2j}.
\]
Figure 17: Bifurcation diagram for rational depensation model with $j = 1$, $b = 0.5$, and $m = 1/3$. Unstable equilibria are graphed in red, stable in green, and semistable are green/red points.

Figure 18: Bifurcation diagram for critical depensation model. Unstable equilibria are graphed in red, stable in green, and semistable is a green/red point. Note there is only a saddle-node bifurcation at $(b, p_s)$.

We can then give the equation for the stable sustainable harvest, $H_s$, in terms of the harvest rate, $H_s = h_0 p_s$. In this case, using the rational depensation per capita growth rate, we get

$$H_s = p_s h_0 = \left( m + (1 - m) \left( \frac{b - h_0}{(1 + b) h_0} \right)^{1/2j} \right) h_0,$$

whose graph is seen in Figure (19). In order to take the roots, this function only exists for $0 \leq h_0 \leq b$.

The graph in Figure (19) reinforces our concerns with harvesting a species that obeys a depensation model. Note that as the harvest rate increases up to $h_m$, the sustainable harvest increases to a maximum, $H_m$. After this point, the sustainable harvest decreases, then suddenly, there is no sustainable harvest as we are driving the population to extinction. We also note that the rate which maximizes sustainable harvest appears to be close to the
intrinsic per capita growth rate meaning that in trying to maximize our harvest, we are putting the species at risk in that, a small error on our part might result in $h_0 > b$. 

5 Maximizing Profit

Let’s assume that there is some fixed price, $q$ dollars per unit, at which we can sell the fish. This assumption might seem a little artificial in that if there are more fish for sale, we might assume that the price offered per fish would go down. On the other hand, there are many different types of fish available for world consumption, and if the price of one fish increases, other fish may replace it in the marketplace. We are also looking at this problem from the point of view of one country or one individual. This one country or individual may have only a small portion of the world market for this fish. So considering these aspects, assuming a fixed price per fish, no matter what our harvest, is not that unreasonable. What this assumption implies is that the rate of harvest that results in maximum revenue is the same as the rate that gives maximum harvest, since the revenue function, $R$, is a constant multiple of the harvest function,

$$R(h_0) = qH(h_0).$$

We call the maximum sustainable harvest the social maximum. Maximizing harvest, and therefore revenue, results in society having the largest possible supply of the species over an indefinite period of time.

The social maximum is not always the goal in harvesting. Another goal to consider is maximizing profit. We could be interested in maximizing the profit for an entire country, or for just one individual.

To review, the growth in a population we are harvesting satisfies the differential equation

$$p' = r(p)p - h_0p,$$

where $r$ is the per capita growth rate and $h_0$ is the rate of harvest. We have developed several different functions $r$, depending on the characteristics of our population. Using phase-line analysis, we have determined the stable equilibrium population size, $p_s$, as
a function of the harvest rate, \( h_0 \). This gives us a function for the sustainable harvest, 
\( H_s = p_h(h_0) h_0 \). We were then able to find the harvest rate, \( h_m \) that maximized the sustainable harvest, \( H_m \).

Assume that the cost, \( C \), of harvesting our fish is proportional to the fraction of fish caught, that is 
\[
C(h_0) = c_1 h_0.
\]
for some constant \( c_1 \). This should make sense. To double the fraction of fish caught, we would have to double the amount of time spent harvesting. This would double the labor costs, and might double the number of boats used. While a proportional cost function is not exact, it is a good first approximation for our study.

Remark 5.1. There are some cost models in which the cost function is discontinuous in that, to increase the harvesting past some level, we might need more boats, so there would be a constant jump, the capital costs in buying more boats. We will ignore this aspect.

Recalling that 
\[
\text{Profit} = \text{Revenue} - \text{Cost},
\]
we now have a function for profit, 
\[
P(h_0) = R(h_0) - C(h_0) = qH_s(h_0) - c_1 h_0.
\]

We are now going to study \( P(h_0) \) under different assumptions about the characteristics of fish species.

5.1 Maximizing profit for a compensation species

Consider a species of fish that satisfies a compensation growth model. Assuming the constant effort harvesting, \( H = h_0 p \), if the harvest rate is \( h_0 = 0 \), then there is obviously zero harvest, that is, \( H_s = 0 \) and our revenue will be zero. As we learned, if the harvest rate equals the intrinsic per capita growth rate, \( h_0 = b \), then the population goes to extinction and again, the sustainable harvest is \( H_s = 0 \) so the revenue is zero. Thus, the sustainable harvest function, \( H_s \) has a parabolic shape, such as in Figure (11), where the intrinsic per capita growth rate was \( b = 0.8 \). Our assumption is that our revenue function has the same shape as our sustainable harvest function.

Figure (20) shows graphs of a revenue function (blue) and a cost function (red line through the origin). Note that maximum revenue occurs at the harvest rate of \( h_R \) and that zero profit occurs at the harvest rate of \( h_z \). Maximum profit occurs where there is a maximum distance between the revenue and costs functions. This is the value \( h_P \) which is at the point where the slope of the revenue function equals the slope of the cost function, \( R'(h_0) = c_1 \), seen as where the red line that is parallel to the cost line intersects the revenue curve. The reason is that before this point, the revenue is increasing faster than the cost and after this point, the cost is increasing faster than the revenue. This point is found by solving where 
\[
R'(h_0) = qH'_s(h_0) = c_1.
\]

Note that if the cost rate, \( c_1 \), increases, then the point on the revenue curve with the same slope decreases, that is, the harvest rate that maximizes profit, \( h_P \), decreases. If \( c_1 \)
Figure 20: Revenue and cost functions for compensation model. Points are marked where maximum profit, maximum revenue, and zero profit occur, $h_P$, $h_R$, and $h_z$, respectively.

decreases, due to efficiencies in the industry, then the rate of harvest that maximizes the profit increases, but the rate of harvest that maximizes profit is always less than the rate that maximizes revenue, $h_P < h_R$, since the slope of the revenue function at its maximum is zero and the cost rate, $c_1 > 0$.

**Example 5.2.** Suppose we are studying a population of fish with a rational per capita growth rate with $b = 0.7$ and $j = 1$, that is,

$$r(p) = 0.7 \left( \frac{1 - p}{1 + 0.7p} \right).$$

We know that for a harvest rate, $h_0 < 0.7$, the equilibrium population size is the solution, $p_s$, to $r(p) = h_0$, that is,

$$p_s = \frac{0.7 - h_0}{0.7 + 0.7h_0}.$$

Thus, the equilibrium sustainable harvest is $p_s h_0$, which is

$$H_s = \frac{0.7h_0 - h_0^2}{0.7 + 0.7h_0}.$$

Let’s assume the fish sell for $q = 1$ and the cost function is $C = 0.2h_0$ so $c_1 = 0.2$. Solving for maximum revenue (where $H'_s = 0$) gives

$$h_0 = -1 + \sqrt{1.7} \approx 0.304.$$

Maximizing the profit,

$$P(h_0) = \frac{0.7h_0 - h_0^2}{0.7 + 0.7h_0} - 0.2h_0,$$

(where $P'(h_0) = 0$) gives

$$h_0 = -1 + \sqrt{\frac{85}{57}} \approx 0.221.$$

These results can be seen in Figure (21).
Remark 5.3. We note that solving for maximum profit using rational $r$ and $j \neq 1$ or using exponential $r$ is quite difficult or impossible, but can be estimated using technology.

For some compensation models with $j > 1$ (and so exhibiting scramble competition), the harvest rate that maximizes revenue is close to the intrinsic per capita growth rate, putting the population at risk. We can see in Figure (22) that maximizing profit results in a lower harvesting rate than maximizing revenue making it safer for the population.

A question we might have is, when is the cost for fishing too high so that we cannot have a profit at any harvesting rate, as seen in Figure (23). The answer is when $c_1 \geq R'(0)$. We recall that $R(h_0) = qps(h_0)h_0$. We find that

$$R'(h_0) = qp + qp'h_0.$$
so

\[ R'(0) = q \rho_s(0). \]

We recall that \( \rho_s(h_0) \) is the stable equilibrium population size for the harvest rate, \( h_0 \). If \( h_0 = 0 \), then the fish are not being harvested, so the equilibrium population size is the carrying capacity, \( \rho_s(0) = 1 \). This means there is no profit if \( c_1 \geq q \). This means there is no profit if the cost for harvesting one unit of fish, \( c_1 \), is greater than or equal to the sale price for one unit of fish, \( q \). This makes sense. We note that there are a lot of species of fish which are not popular, so that the price for selling them is small or zero. For these fish, \( c_1 > q \), so we just don’t fish for them.

Figure 23: Cost line that results in no profit for any harvest rate.

5.2 Maximizing profit for a depensation species

Recall from Section (4.2) that for a species satisfying a depensation model, the sustainable harvest, \( H_s \), as a function of the harvest rate, \( h_0 \), may be discontinuous at \( h_0 = b \), the intrinsic per capita growth rate, as seen in Figure (24). In such a case, maximizing harvest, and hence, revenue, can put the species at risk. As we noted in Figure (22), maximizing profit can make it safer for the species, which we see in Figure (24).

5.3 Principles for maximizing profit

We state some principles derived from this discussion.

**Principles 5.4.** Suppose a species satisfies a density dependent population model.

- Maximum revenue corresponds to maximum sustainable harvest.

- Maximum profit occurs at a lower harvest rate than that which maximizes revenue, resulting in a larger stable fish population but smaller sustainable harvest.
Figure 24: Revenue function and cost line for a depensation model. Maximizing profit makes it safer for the species.

- Lowering cost results in a higher harvest rate when maximizing profit, but still less than the harvest rate that maximizes revenue.

- To have a profit, the cost for harvesting a unit of fish, $c_1$, must be less than the price for a unit of fish, $q$.

See [10] (from which much of the material in this section was adapted), for more details about harvesting strategies, although in a discrete form.

6 What often happens

We now consider profit from the point of view of an individual. As discussed before, Figure (20) displays three harvest rates, $h_P$ that results in maximum sustainable profit, $h_R$ which results in maximum sustainable revenue, and $h_z$ which results in zero profit. Suppose currently that the harvest rate is $h_0 < h_z$ so that $R(h_0) > C(h_0)$, meaning that there is a profit for those individuals involved in fishing. Also assume that there are unemployed individuals. They observe that people are making a profit in fishing. If these unemployed have access to a boat, they will also begin fishing, viewing a small profit as better than no income at all. Thus, the value for $h_0$ will increase.

When a resource is open to all, it is called open access. The fishing industry has traditionally been open access. From the previous discussion, we see that whenever the value for $h_0$ is less than the value that results in no profit, $h_z$, then people enter the industry and the value for $h_0$ will increase since more people are fishing. This continues to happen until the value for $h_0$ is at or near the no profit value, $h_0 = h_z$ in Figure (20). The result is that no one who is harvesting the fish is making a profit, or if they are, it is very small. This also can result in a harvest rate that is relatively near the intrinsic per capita growth rate, particularly if it is relatively inexpensive to fish, $c_1$ is small, and the fish satisfy a compensation model.
Open access can be even more of a problem for a depensation species. In Figure (25) is a revenue function for a species with depensation where the intrinsic per capita growth rate, $b$, occurs at $0 < p = m < 1$. (See Equation 2.9.) We see three different cost lines. Let’s assume open access so the harvesting rate increases to where cost equals revenue. For cost $C_1$ (red), we have a harvest rate, $h_z$, that actually results in near maximum sustainable harvest and is a little away from the intrinsic per capita growth rate, $b$.

![Figure 25: Revenue and three costs functions for depensation model.](image)

If efficiencies in fishing reduce the slope of the cost function, as cost line $C_2$ (brown) in Figure (25), then with open access policies, the harvest rate, $h_0$, will keep increasing until it passes the intrinsic per capita growth rate. At that point, the population will begin dying out until the harvest rate is reduced, hopefully before going below the minimum viable population size. This is another example of the Allee effect. This result is often called the tragedy of the commons. An interesting introduction is given in [4]. It is also described briefly in [11] as it refers to the fishing industry. The term originally referred to when people would graze their livestock on common fields, depleting those fields from overgrazing.

We might ask, what slope should the cost function be so that it intersects the revenue function? We use the revenue function, $R(h_0) = qp_s h_0$, and cost function, $C(h_0) = c_0 h_0$. Recall that for a depensation model, $p_s$ is the solution to $r(p) = h_0$, so when $h_0 = b$, $p_s = m$. (Again, see Equation 2.9.) This means the end-point of the revenue function is $(b, qb_m)$, as seen in Figure (25). Thus, the gray cost line $C_3$ has

$$\text{slope} = qm.$$  

Note that the larger $m$, $0 \leq m < 1$, the larger the slope of the cost line, $c_1$, has to be to intersect the revenue function, resulting in a harvest rate that produces zero profit instead of a harvest rate that results in extinction. Also note that this again implies $c_1 < q$.

**Principles 6.1.** Suppose a species satisfies a depensation growth model with intrinsic per capita growth rate $b$ occurring at $m$, $0 < m < 1$.

- If the cost rate, $c_1$, satisfies $q > c_1 > mq$, then open access results in a harvest rate $h_z < b$ so the population will not go to extinction, but may be at risk.
• If the cost rate satisfies $c_1 < mq$, then open access results in an increasing harvest rate until $h_0 > b$, so the population will go to extinction.

• Therefore, more efficient harvesting, lowering $c_1$, puts the population at risk.

7 Conclusion

There have been many occurrences of fishing stock collapses around the world for which open access policies have at least contributed. There have been different attempts to make fishing more sustainable, but most have problems associated with them.

There has been some success with an approach of using Individual Transferable Quotas or ITQ’s. Essentially, the government determines the optimal value for $h_0$, the fraction of the fish that can be caught. They conduct studies that indicate the amount of fish present that year. Combining these estimates, the government determines how many fish should be caught that year and issues permits allowing each fisherman to catch a certain part of that quota, based on their previous catches. Using this approach, it guarantees those in this fishing industry a decent profit each year, and a sustainable profit. There are many objections to this policy. Those that are not given permits feel that it is unfair to keep them out of what they believe should be an open access fishery. It also can lead to corruption, from people trying to payoff someone to obtain permits and to fishermen who catch more than their quota. It also opens the market to large conglomerates buying up all the ITQ’s and creating a monopoly. Finally, since there is a quota, fishermen may discard smaller fish which have less value, which results in more fish being caught (and some discarded) than were planned. There are advantages also, from more stable employment to a more optimal use of fishing boats.

Another attempt at limiting harvesting in an open access arena is having a hunting or fishing season. The downside it that for some species of fish, the fishing season is quite short, so fishermen and their boats may be idle for long periods of time. With a short season, fishermen will often exhibit risky behavior by going out in inclement weather during the short season, as was described in the movie, The Perfect Storm [6].

Some governments have implemented no fish (or no take) zones, such as in Australia’s Great Barrier Reef Marine Park. These zones allow for the recovery of fish species which have become rare in over-fished areas.

Maine is one example of a government attempting to make harvesting, lobsters in this case, sustainable. A lobster fisherman in Maine must complete an apprenticeship then apply for a license. There are a limited number of licenses to limit the number of lobsters caught. There is also a limit on the minimum and maximum size of lobsters caught as well as on females carrying eggs. Of course, there is the problem that those completing an apprenticeship must hurry up and wait to get their license.

We encourage the reader to look for other approaches to sustainable harvesting, and think about the pros and cons of those approaches.
References


