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Linear ODE systems having a fundamental matrix of the form $f(Mt)$

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**Keywords:** Differential Equations, Linear Algebra, Functional Calculus, Dynamic Slope Field, Fundamental Matrix, Non-autonomous Differential Equations, Factorial Function

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**Abstract:** We interweave scaffolded problem statements with exposition and examples to support the reader as they explore specific linear systems of differential equations with variable coefficients, $\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t)$ and initial value $\mathbf{x}(0)$. We begin with a constant $n \times n$ matrix $M$ and a real or complex-valued function $f$, analytic at the eigenvalues of $M$ with $f(0) = 1$, and construct a linear system of differential equations with solutions $x(t) = f(Mt)\mathbf{x}(0)$, where $t$ is a parameter in some interval including zero. In general, the solutions to the resulting non-autonomous system are more difficult to analyze than solutions to the constant coefficient case. However, some parts of the analysis of the constant coefficient case can be applied in some examples. We then use the system to explore various applications. This approach highlights numerous connections between linear algebra, elementary calculus, functional calculus and differential equations. For example, the coefficient matrix to the newly constructed system can be used to construct a dynamic slope field for the solutions to the initial value problem, as well as linear and quadratic approximations to its solutions formed by $f(Mt)$. To facilitate this approach, we introduce the reader to functional calculus beyond that normally taught in an elementary differential equations course or first course in linear algebra.

1 Introduction

When first encountering a system of linear differential equation in an introductory differential equations course or first course in linear algebra, we often begin with the linear system and re-formulate it using the matrix equation $\frac{d\mathbf{x}}{dt} = M\mathbf{x}$. This re-formulation allows us to explore the effects of the eigenvalues of $M$ on the behavior of the system. For example, when studying the two-dimensional system for which $M$ has distinct, real eigenvalues $\lambda_1$ and $\lambda_2$ with corresponding linearly independent eigenvectors $\eta_1$ and $\eta_2$, etc.
we learn \( \tilde{x}(t) = c_1 e^{\lambda_1 t} \tilde{\eta}_1 + c_2 e^{\lambda_2 t} \tilde{\eta}_2 \) solves this system and can re-formulate this solution as \( \tilde{x}(t) = e^{Mt} \tilde{x}_0 \) where \( e^{Mt} \) is the matrix exponential and \( \tilde{x}_0 = (x_1(0), x_2(0))^T \) consists of the initial values. Viewing the solution in this way connects us to the one-dimensional analogue from elementary calculus, \( \frac{dx}{dt} = kx \) with solution \( x(t) = \tilde{x}(0)e^{kt} \). In fact, we later learn that \( e^{Mt} \) can be more involved in these situations. Furthermore, in this two-dimensional case phase portraits allow us to visualize the behavior of the solutions by plotting multiple solutions from different initial values, or equivalently, showing the direction field.

Similar to the case above, this paper explores linear systems of differential equations with solutions of the form \( \tilde{x}(t) = f(Mt) \tilde{x}(0) \) where where \( t \) is a parameter in some interval including zero, \( M \) is a constant matrix, and \( f \) is a function analytic at the eigenvalues of \( M \) with \( f(0) = 1 \). The matrix \( f(Mt) \) is commonly referred to as the fundamental matrix of the system, which we define more formally in section 2.3. All such systems, except when \( f(x) = e^x \), will be non-autonomous, namely their coefficients depend in general on time: \( \tilde{x}'(t) = A(t) \tilde{x}(t) \), as derived in (3.1). We begin by forming the matrix \( f(Mt) \) and using it to construct the underlying system. While the solutions to the resulting non-autonomous system are more difficult to analyze, some parts of the analysis for the constant-coefficient case can be applied in specific examples. We then use the system to explore various applications, including constructing an alternative representation for the matrix \( f(Mt) \). The examples and problems throughout this paper introduce the reader to numerous connections between linear algebra, elementary calculus, functional calculus and differential equations. For example, the coefficient matrix to the newly constructed system can be used to construct linear and quadratic approximations to its solutions formed by \( f(Mt) \).

To facilitate this approach, we introduce the reader to functional calculus beyond that normally taught in an elementary differential equations course or first course in linear algebra. With its mix of exposition and scaffolded problems, this paper could serve to motivate a senior undergraduate research project, provide resources for an advanced undergraduate guided independent study, or serve as a resource for an instructor to use in a differential equations course. While we strongly encourage students to work through the problems on their own as they read the paper, we also provide solutions in Appendix A. We begin with a simple example.

**Example 1.1.** Consider the decoupled linear system of differential equations

\[
\begin{align*}
\frac{d}{dt} x_1(t) &= -a \frac{\sin(at)}{\cos(at)} x_1(t) \\
\frac{d}{dt} x_2(t) &= -b \frac{\sin(bt)}{\cos(bt)} x_2(t)
\end{align*}
\]

(1.1)

with initial value \( \tilde{x}_0 = (x_1(0), x_2(0))^T \). We can simply integrate each equation in the decoupled system and solve each one-dimensional initial value problem (IVP) separately, leading to the solution \( \tilde{x}(t) = (\cos(at), \cos(bt))^T \), which is easily verified. We will see in
Proposition 3.1 that using functional calculus the system can be expressed as
\[
\frac{d}{dt} \tilde{x}(t) = \frac{d}{dt} \log(\cos(Mt)) \cdot \tilde{x}(t) \quad \text{for} \quad M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}
\]  
with solution
\[
\tilde{x}(t) = \cos(Mt)\tilde{x}(0).
\]  

We notice several things about this simple example. In the remainder of the paper, we explore how our observations can generalize to non-diagonal systems.

- When \(Mt\) is a diagonal matrix, we form the matrices by applying the analytic functions to the diagonal elements of \(Mt\). For example,
\[
\log(\cos(Mt)) = \begin{pmatrix} \log(\cos(at)) & 0 \\ 0 & \log(\cos(bt)) \end{pmatrix}.
\]

- The discussion immediately preceding Proposition 3.1 and summarized in that proposition shows in the more general case why \(\frac{d}{dt} \log(\cos(Mt))\) yields the system coefficients for the system of differential equations whose solutions are expressed by \(\tilde{x}(t) = \cos(Mt)\tilde{x}(0)\).

- Once we determine the system of equations, we can obtain the slope field by using the slopes, \(\frac{dx_2}{dx_1} = \frac{dx_2/dt}{dx_1/dt}\), at each coordinate point \((x_1, x_2)\), see https://bit.ly/3YUA4Z5 to view solutions of (1.1) in the dynamic slope-field. Notice the coefficients are now functions of time and the slopes change through time. Plotting the slope-fields in three dimensions rather than two provides a static view, as demonstrated in figure 1 for \(a = -3\) and \(b = 9\). Each plot also includes two solutions, one corresponding to the initial values \(\tilde{x}_0 = (1, 1)^T\) (solid) and \(\tilde{x}_0 = (-1, 1)^T\) (dashed).

2 Functional Calculus

2.1 Analytic Functions applied to Matrices

First consider matrices with a complete set of eigenvectors. Such matrices can be diagonalized (i.e. \(M = P\text{diag}(\lambda_1, ..., \lambda_n)P^{-1}\) for an invertible matrix \(P\)) and provided that \(f\) is analytic at the eigenvalues of \(M\), as seen in [4], have
\[
f(M) = P\text{diag}(f(\lambda_1), ..., f(\lambda_n))P^{-1}.
\]  

We can analyze \(f(M)\) for matrices that do not have a complete set of eigenvectors by generalizing the decomposition using the Jordan normal form. Recall, \(M = PJP^{-1}\) where
Figure 1: Exploring the slope field for the dynamical system given in (1.1) with $a = -3$ and $b = 9$. Each images includes the solution corresponding to initial value $(1, 1)^T$ (solid) and the solution corresponding to initial value $(-1, 1)^T$ (dashed).
$J$, the Jordan Form of the matrix $M$, is the block-diagonal matrix $J = \text{diag}(J_1, \ldots, J_k)$ and each $J_i$ is a Jordan block associated with the eigenvalue $\lambda_i$,

$$J_i = \begin{bmatrix} \lambda_i & 1 \\ & \lambda_i & \ddots \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix}.$$  

For example, the Jordan normal form for the upper-triangular $2 \times 2$ matrix with eigenvalue $a$ and nonzero entry $b$ is

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/b \end{bmatrix} \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}. \quad (2.2)$$

If $f$ is an analytic at $\lambda_i$ and $J_i$ is a Jordan block corresponding to $\lambda_i$, then Ikebe [5] applies concepts from functional calculus to illustrate

$$f(J_i) = \begin{pmatrix} f(\lambda_i) & f'(\lambda_i) & f''(\lambda_i)/2! & \cdots & f^{(n-1)}(\lambda_i)/(n-1)! \\ 0 & f(\lambda_i) & f'(\lambda_i) & \cdots & f^{(n-2)}(\lambda_i)/(n-2)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f(\lambda_i) \end{pmatrix} \quad (2.3)$$

Furthermore, $(2.3)$ leads naturally to a general expression for $f(M)$.

**Definition 2.1.** Given a function $f(x)$ that is analytic at the eigenvalues of $M = PJP^{-1}$,

$$f(M) = Pf(J)P^{-1}$$

with $f(J) = \text{diag}(f(J_1), \ldots, f(J_k))$ and $f(J_i)$ as defined in $(2.3)$.

**Example 2.2.** We will apply Definition 2.1 to determine $e^A$ for

$$A = \begin{bmatrix} -2 & 0 & 2 \\ -2 & -2 & 6 \\ 0 & -2 & 4 \end{bmatrix} = P \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} \text{ with } P = \begin{bmatrix} 4 & -2 & 1 \\ 8 & -2 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$

$$e^A = P \begin{bmatrix} f(0) & f'(0) & f''(0) \\ 0 & f(0) & f'(0) \\ 0 & 0 & f(0) \end{bmatrix} P^{-1} = P \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -5 & 10 \\ 2 & -4 & 7 \end{bmatrix}.$$

### 2.2 Differentiating Matrices $A = A(t)$

We differentiate $A = A(t)$ with respect to the parameter $t$ by taking componentwise derivatives. Rules, such as the product rule for differentiation, hold respecting the original order of the matrix multiplication. For example, if matrices $A(t)$ and $B(t)$ are of the size so that $C(t) = A(t)B(t)$ is defined then,

$$\frac{d}{dt}C(t) = A(t)\frac{dB}{dt} + \frac{dA}{dt}B(t). \quad (2.4)$$

The non-commutative nature of matrix multiplication forces us to take particular care before applying familiar differentiation rules. Those rules that rely on commutativity will not necessarily hold. Higham [3] provides additional examples.
2.2.1 The Exponential of a Matrix

Consider the exponential function, \( f(x) = e^x \). From elementary calculus we know

\[
e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!}.
\]

For any \( m \times m \) matrix \( A \), the exponential of \( A \) can be written,

\[
e^A = \sum_{k=0}^{\infty} \frac{A^n}{n!}.
\] (2.5)

The expression in (2.5) provides an alternative method for determining \( e^A \) when a closed form is easily obtained.

**Example 2.3.** Let us revisit the calculation in Example 2.2, this time applying (2.5) to the same matrix \( A = \begin{bmatrix} -2 & 0 & 2 \\ -2 & -2 & 6 \\ 0 & -2 & 4 \end{bmatrix} \). This matrix is nilpotent (\( A^3 = [0] \)) so that

\[
e^A = I + \frac{A^1}{1!} + \frac{A^2}{2!} = \begin{bmatrix} 1 & -2 & 4 \\ 2 & -5 & 10 \\ 2 & -4 & 7 \end{bmatrix}.
\]

Combining properties of the exponential function and the matrix \( A \) gives rise to different representations for \( e^A \). We refer to Moler and Van Loan for a more complete discussion on this topic [9].

Some familiar algebraic identities depend on commutativity of multiplication and thus special care must be taken in the functional calculus setting. For example,

\[
e^A e^B = e^{A+B} \text{ holds if } AB = BA.\] (2.6)

See [1] to find more general conditions under which \( e^A e^B = e^{A+B} \).

Recall, in a first course in differential equations we express the solution to \( \frac{dx}{dt} = Mx \) in exponential form \( \tilde{x}(t) = e^{Mt} \tilde{x}(0) \). This representation of the solution is usually justified by differentiating the power series representation for \( e^{Mt} \) and factoring the constant matrix \( M \) to the right in the resulting expression. The more general result in functional calculus is summarized in Theorem 2.4.

**Theorem 2.4.** If \( A(t) \) is a complex \( n \times n \) matrix with no real negative or zero eigenvalues that depend on a parameter \( t \), and \( A \) commutes with \( \frac{dA}{dt} \), then

\[
\frac{d}{dt}e^{A(t)} = e^{A(t)} \frac{dA}{dt} = \frac{dA}{dt} e^{A(t)}.
\]

Although Theorem 2.4 is special case of a more general result in functional calculus, [8], an elementary discussion can be found in [7].
Problem 2.5. Consider the diagonalizable matrix

\[ A(t) = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}. \]

1. Show \( A(t) \) does not commute with \( \frac{d}{dt} A(t) \)

2. Calculate the three expressions in the conclusion of Theorem 2.4.

Calculate the three expressions in the conclusion of Theorem 2.4.

2.2.2 The Factorial of a Matrix

Being able to construct the exponential of a matrix proves quite useful in many applications of functional calculus. Hence, its focus in the literature and direct computation in computer algebra systems such as Maple\textsuperscript{©} and Mathematica\textsuperscript{©}. For example, see Moler and Van Loan [9]. We can now construct other representations for functions defined in terms of the exponential. For example, one of the more well known representations for the factorial function, \( z! \), comes from the translated Euler integral,

\[ z! = \int_{0}^{\infty} e^{-t} t^z dt, \quad (2.7) \]

with Re\((z) > -1\). Applying (2.7) to a matrix \( A \) with all eigenvalues \( \lambda \) having Re\((\lambda) > -1\), we obtain a representation for the Matrix factorial,

\[ A! = \int_{0}^{\infty} e^{-t} t^A dt, \quad (2.8) \]

where the matrix \( t^A = e^{A\ln(t)} \) for \( t > 0 \). In (2.8), after the matrix in the integrand \( e^{-t} t^A \) is formed, the improper integral is taken component-wise. If the matrix \( A \) can be diagonalized, then as with any analytic function the factorial can be applied to the diagonal matrix in the factorization to obtain,

\[ A! = Q\text{diag}(\lambda_1!, \ldots, \lambda_m!) Q^{-1} \quad (2.9) \]

Problem 2.6. Consider the matrix \( M = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \).

1. Calculate \( M! \).

2. Does this example motivate a result that you can state and prove for diagonalizable \( n \times n \) matrices?
2.2.3 The Logarithm of a Matrix

Less well known and discussed in linear algebra courses is the matrix logarithm. We will need its representation and properties. In elementary calculus, the natural logarithm, \( y = \ln(x) \) is normally defined to be the inverse function of the natural exponential. In complex variables for non-zero \( z \), \( z = re^{i\phi} \) in polar form, \( \log(z) = \ln(r) + i\phi \). Notice several important properties. The complex \( \log(\cdot) \) is multi-valued with the principle value taken to be \(-\pi < \phi \leq \pi \) and it is still the inverse of the complex exponential with some restriction. \( \log(e^z) = z \) in the strip \(-\pi < \text{Im}(z) < \pi \), whereas, the \( e^{\log(z)} = z \) holds for all \( z \).

To understand the definition of the matrix logarithm we begin by substituting \( z = 1 + x \) into the power series for \( \ln(1 + x) \) to obtain,

\[
\ln(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z - 1)^k}{k}, \quad |z - 1| < 1 \tag{2.10}
\]

(2.10) holds for complex \( z \), where \( |\cdot| \) is the complex modulus and \( |z - 1| < 1 \) is the open disk of radius 1 centered at \( z = 1 \) on the real axis. We can now apply the power series to the matrix \( A \), where the modulus becomes a matrix norm.

\[
\log(A) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(A-I)^k}{k}, \quad \|A - I\| < 1 \tag{2.11}
\]

This definition can be extended to an \( n \times n \) matrix with no negative real or zero eigenvalues.

The Inverse property of the matrix logarithm is summarized in Theorem 2.7. Discussion and proofs of these results can be found in [3].

**Theorem 2.7.** If \( A \) is a complex \( n \times n \) matrix with no negative real or zero eigenvalues, then

1. \( \exp(\log(A)) = A \);
2. \( \log(\exp(A)) = A \) if and only if \( |\text{Im}(\lambda_i)| < \pi \) for every eigenvalue \( \lambda_i \) of \( A \).

Functional analysts have extended the laws of logarithms to the the matrix log as summarized in Theorem 2.8.

**Theorem 2.8.** Let \( A \) and \( B \) be \( n \times n \) complex matrices with no negative real or zero eigenvalues,

1. if \( |p| \leq 1 \), then \( \log(A^p) = p \log(A) \);
2. if \( AB = BA \) and \( |\arg \lambda_j + \arg \mu_j| < \pi \) for each corresponding pair of eigenvalues \( \lambda_j \) of \( A \) and \( \mu_j \) of \( B \), then

\[
\log(AB) = \log(A) + \log(B).
\]

**Example 2.9.** Consider a \( 3 \times 3 \) matrix in the form of a Jordan block with eigenvalue \( \lambda \),

\[
J = \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{bmatrix}
\]

and define \( B = \begin{bmatrix}
0 & \lambda^{-1} & 0 \\
0 & 0 & \lambda^{-1} \\
0 & 0 & 0
\end{bmatrix} \).
We observe

\[ J = \lambda I_3 + \lambda B = \lambda I_3 (I_3 + B) \, . \]

Given \( I_3 \) commutes with \( I_3 + B \) and \( B^3 = B^4 = \ldots = 0 \), by Theorem 2.8 and application of (2.11) to \( I_3 + B \) we have,

\[
\log(J) = \log(\lambda)I_3 + \log(I_3 + B) = \log(\lambda)I_3 + B - \frac{1}{2}B^2 \\
= \begin{bmatrix} \log(\lambda) & \lambda^{-1} & -\frac{1}{2}\lambda^{-2} \\ 0 & \log(\lambda) & \lambda^{-1} \\ 0 & 0 & \log(\lambda) \end{bmatrix}.
\]

We will often have multiple methods to do the same calculation in functional calculus as well as multiple representations of the matrix of interest. In Example 2.9, we chose to use the property of the matrix logarithm in Theorem 2.8 and the nilpotent property of the matrix to construct the matrix logarithm of the Jordan block.

**Problem 2.10.** Use (2.3) to calculate the \( \log(J) \) in Example 2.9.

**Problem 2.11.** Consider the 90° rotation matrix \( A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \).

1. Factor the matrix \( A \) to diagonal form.

2. Determine the \( \log(A) \). Recall that \( \log(\pm i) = \pm i(\pi/2 + 2n\pi) \). The principle value of \( \log(A) \) is obtained when we let \( n = 0 \).

3. What role does the matrix \( A \) in our example seem to play in matrix algebra?

Next we explore \( \frac{d}{dt} \log(A(t)) \). In the following section this will allow us to apply a functional calculus version of logarithmic differentiation to construct a non-autonomous system of differential equations that have \( f(M_t) \) as a fundamental matrix.

**Theorem 2.12.** If \( A(t) \) is a complex \( n \times n \) matrix with no real negative or zero eigenvalues that depend on a parameter \( t \), and \( A \) commutes with \( dA/dt \), then

\[
\frac{d}{dt} \log(A(t)) = A^{-1} \frac{dA}{dt} = \frac{dA}{dt} A^{-1}.
\]

A derivation and discussion of Theorem 2.12 is found in [8]. Combining Theorem 2.12 with the results of Theorem 2.8 allows for a form of logarithmic differentiation of products and powers of matrices, but care must be taken to confirm the assumptions.

### 2.3 Summary from non-autonomous linear differential equations

A linear system of differential equations with time-dependent coefficients is called a non-autonomous system of differential equations. These systems are more complicated than autonomous systems and have a well developed theory. The existence and uniqueness theorem for a homogeneous first-order linear system of differential equations is given in Theorem 2.13, with a full derivation in [6].
Theorem 2.13. Let $A(t)$ be a time dependent and continuous matrix on some interval $I \in \mathbb{R}$. The linear system,

$$\frac{d}{dt} \eta(t) = A(t)\eta(t) \quad (2.12)$$

with initial condition

$$\eta(t_0) = \eta_0, \quad t_0 \in I \quad (2.13)$$

then there is a unique continuous solution to the IVP given in (2.12) and (2.13).

Due to linearity, the solution can be constructed using the principle of superposition, that is there is a fundamental matrix, $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ whose columns form a linear independent set of solutions such that a general solution to the equation in (2.12) can be written,

$$\eta(t) = X(t)\bar{c} \quad (2.14)$$

where $\bar{c} = (c_1, c_2, \ldots, c_n)^T$ is a column vector of arbitrary constants. The product of $X(t)$ with any nonsingular matrix $Y(t)$ is also a fundamental matrix. To form the solution to the IVP given in (2.12) and (2.13) we solve the system of equations,

$$X(t_0)\bar{c} = \eta_0, \quad \bar{c} = X^{-1}(t_0)\eta_0$$

and write

$$\eta(t) = \Phi(t, t_0)\eta_0 = X(t)X^{-1}(t_0)\eta_0 \quad (2.15)$$

$\Phi(t)$ is called the state transition matrix. If $A(t)$ is an analytic function of $t$, then one can assume a power series form for the solution $\eta(t)$ and construct a power series solution to (2.12) and (2.13) using the power series of $A(t)$. These are power series with matrix and vector coefficients and, in general, are difficult to construct. Another possible representation is analogous to the constant coefficient case. $\bar{x}(t) = e^{Mt}\bar{x}_0$ solves $\frac{d\bar{x}}{dt} = M\bar{x}$. For the system in (2.12) and (2.13) we have the following result from the theory of non-autonomous systems. A proof can be found using condition 2 in [6], and in [2] Campbell shows the equivalence of conditions 1. and 2.

Theorem 2.14. Assume one of the following conditions hold,

1. $A(t)$ commutes with $\int_{t_0}^t A(\tau) \, d\tau$ for all $t$,

2. $A(t_1)$ commutes with $A(t_2)$ for all $t_1, t_2$.

Then

$$\Phi(t, t_0) = \exp\left( \int_{t_0}^t A(\tau) \, d\tau \right).$$
Example 2.15. Apply Theorem 2.14 to find the solution to the system

\[
\begin{align*}
\frac{d}{dt}x_1(t) &= \frac{t}{1 + t^2}x_1(t) + \frac{-1}{1 + t^2}x_2(t) \\
\frac{d}{dt}x_2(t) &= \frac{1}{1 + t^2}x_1(t) + \frac{t}{1 + t^2}x_2(t)
\end{align*}
\]

with initial value \(\vec{x}_0 = (x_1(0), x_2(0))^T\).

Solution. Since \(A(t_1)\) commutes with \(A(t_2)\) for all \(t_1, t_2\), we can apply the result of Theorem 2.14,

\[
\int_0^t A(\tau) d\tau = \begin{pmatrix}
\int_0^t \frac{\tau}{1 + \tau^2} d\tau & \int_0^t \frac{-1}{1 + \tau^2} d\tau \\
\int_0^t \frac{1}{1 + \tau^2} d\tau & \int_0^t \frac{\tau}{1 + \tau^2} d\tau
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} \ln(1 + \tau^2) - \tan^{-1}(t) \\
\tan^{-1}(t) & \frac{1}{2} \ln(1 + \tau^2)
\end{pmatrix}.
\]

The reader may verify,

\[
\Phi(t, 0)\vec{x}(0) = \exp\left(\int_0^t A(\tau) d\tau\right)\vec{x}(0) = \begin{pmatrix}
1 & -t \\
t & 1
\end{pmatrix} \begin{pmatrix}
x_1(0) \\
x_2(0)
\end{pmatrix}.
\] (2.16)

We highlight several thoughts concerning Theorem 2.14 and Example 2.15.

- The exponential taken in the final step can be accomplished using a computer algebra system.

- If \(A(t) = M\), then Theorem 2.14 applies and gives the usual representation for the solution to the constant coefficient system, \(\vec{x}(t) = e^{Mt}\vec{x}_0\).

- If \(A(t)\) satisfies the hypothesis to Theorem 2.14 and has eigenvalues \(\lambda_1(t)\) and \(\lambda_2(t)\) with corresponding constant eigenvectors \(\vec{\eta}_1\) and \(\vec{\eta}_2\), respectively, then the general solution for (2.12) can be represented

\[
c_1 \exp\left(\int_{t_0}^t \lambda_1(\tau) d\tau\right)\vec{\eta}_1 + c_2 \exp\left(\int_{t_0}^t \lambda_2(\tau) d\tau\right)\vec{\eta}_2.
\] (2.17)

- The link to view the trajectories given in the solution to Example 2.15 in this Desmos demonstration https://bit.ly/3YSOGbc.

We are now able to apply the functional calculus introduced in this section to explore systems of differential equations with solution of the form \(\vec{x}(t) = f(Mt)\vec{x}(0)\) as first discussed in section 1.
3 Systems with the matrix $f(Mt)$ as a fundamental matrix

3.1 Introduction

Given an analytic function $f(\cdot)$ such that $f(0) = I$ and a square matrix $M$, we can construct a non-autonomous system of differential equations that have $f(Mt)$ as a fundamental matrix. First we will discuss a more general result that relies on the properties of the matrix logarithm outlined in Theorems 2.8 and 2.12.

\[
\frac{d}{dt} \log(B(t)) = B(T)^{-1} \frac{d}{dt} B(t)
\]

\[
\frac{d}{dt} \log(B(t))B(t) = \frac{d}{dt} B(t)
\]

\[
\frac{d}{dt} \log(B(t))B(t)\tilde{c} = \frac{d}{dt} B(t)\tilde{c}.
\]

When

\[
A(t) = \frac{d}{dt} \log(B(t)),
\]

$\tilde{x}(t) = B(t)\tilde{c}$ is a solution to the system $\frac{d}{dt} \tilde{x}(t) = A(t)\tilde{x}(t)$ and thus $B(t)$ is a fundamental matrix for this system. We are particularly interested in the case where $B(t) = f(Mt)$. When $f(\cdot) = \exp(\cdot)$, $B(t) = e^{Mt}$ and we have $A(t) = \frac{d}{dt} \log(e^{Mt}) = M$, the familiar constant coefficient case.

We summarize our results in Proposition 3.1 and illustrate the method of construction in Example 3.2 using a linear function for $f(\cdot)$ and the $90^\circ$ rotation matrix for $M$.

**Proposition 3.1.** Let $M$ be a square matrix and $f$ an analytic function with $f(0) = 1$. Then the system of differential equations $\tilde{x}'(t) = A(t)\tilde{x}(t)$, with $A(t) = \frac{d}{dt}(\log(f(Mt)))$, has $f(Mt)$ as its fundamental matrix, i.e. the solution to the system is $\tilde{x}(t) = f(Mt)\tilde{x}(0)$.

**Example 3.2.** Use Proposition 3.1 to determine the system of differential equations whose solutions are given by $f(Mt)\tilde{x}(0)$, where $M$ is the $90^\circ$ rotation matrix as shown in Problem 2.11 and $f(x) = x + 1$.

**Solution.** First, notice $\log(f(x)) = \log(x + 1)$ and we form $\log(f(Mt))$,

\[
\log(f(Mt)) = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \log(-it + 1) & 0 \\ 0 & \log(it + 1) \end{bmatrix} = \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix}
\]

Second, we take the derivative,

\[
\frac{d}{dt} \log(f(Mt)) = \frac{1}{2} \begin{bmatrix} \frac{d}{dt}(\log(-it + 1) + \log(it + 1)) & -i \frac{d}{dt}(\log(-it + 1) - \log(it + 1)) \\ i \frac{d}{dt}(\log(-it + 1) - \log(it + 1)) & \frac{d}{dt}(\log(-it + 1) + \log(it + 1)) \end{bmatrix}
\]

\[
= \begin{bmatrix} \frac{t}{1 + t^2} & -\frac{1}{1 + t^2} \\ \frac{1}{1 + t^2} & \frac{t}{1 + t^2} \end{bmatrix}
\]
Finally, we form the system of equations

\[
\begin{bmatrix}
  x'_1(t) \\
  x'_2(t)
\end{bmatrix} = \frac{d}{dt} \log(f(Mt)) \vec{x}(0) = \begin{bmatrix}
  \frac{t}{1+t^2} & \frac{-1}{1+t^2} \\
  \frac{1}{1+t^2} & \frac{-t}{1+t^2}
\end{bmatrix} \begin{bmatrix}
  x_1(0) \\
  x_2(0)
\end{bmatrix}
\]  

(3.2)

and the solution to this system is given by

\[
\begin{bmatrix}
  x_1(t) \\
  x_2(t)
\end{bmatrix} = f(Mt) \vec{x}(0) = \begin{bmatrix}
  -i & i \\
  1 & 1
\end{bmatrix} \begin{bmatrix}
  -it + 1 & 0 \\
  0 & it + 1
\end{bmatrix} \begin{bmatrix}
  x_1(0) \\
  x_2(0)
\end{bmatrix} = \begin{bmatrix}
  1 & -t \\
  t & 1
\end{bmatrix} \begin{bmatrix}
  x_1(0) \\
  x_2(0)
\end{bmatrix}
\]  

(3.3)

We notice that the system derived in Example 3.2 is the system whose solution is illustrated in Example 2.15. Indeed, the matrix \( f(Mt) \) is used to generate the system using (3.1) for which it is a fundamental matrix.

**Problem 3.3.** Consider the matrix \( M = \begin{bmatrix} a & 1 \\ 0 & d \end{bmatrix} \). Use the steps below to determine the system of differential equation for which \( f(Mt) \vec{x}(0) \) is a solution for \( f(x) = 1 + x^2 \).

1. Form \( f(Mt) = I + (Mt)^2 \) and simplify.
2. Determine the matrix \( A(t) = \frac{d}{dt} \log(f(Mt)) \).

We end this section by considering the the factorial function and work out a few of the details for the representation given (2.8),

\[ M! = \int_0^\infty e^{-u} u^M dt, \]

where \( M \) is the 90° rotation matrix as shown in Problem 2.11. In this case

\[ \exp(M\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}. \]

Hence,

\[ u^{Mt} = \exp(Mt \log(u)) = \begin{bmatrix} \cos(t \log(u)) & -\sin(t \log(u)) \\ \sin(t \log(u)) & \cos(t \log(u)) \end{bmatrix} \]

so that

\[
(Mt)! \vec{x}(0) = \int_0^\infty e^{-u} u^{Mt} du \vec{x}(0) = \begin{bmatrix}
  \int_0^\infty e^{-u} \cos(t \log(u)) du \cdot x_1(0) & -\int_0^\infty e^{-u} \sin(t \log(u)) du \cdot x_2(0) \\
  \int_0^\infty e^{-u} \sin(t \log(u)) du \cdot x_1(0) & \int_0^\infty e^{-u} \cos(t \log(u)) du \cdot x_2(0)
\end{bmatrix}
\]  

(3.4)
Using the matrix components of (3.4) in Desmos, we show trajectories of \((Mt)!\) with various initial values in Figure 2. We leave it as a problem for the reader to determine the system of equations for which \((Mt)!\) is a fundamental matrix and from it determine the slope-field.

Figure 2: Trajectories showing solutions for \(\vec{x}(t) = (Mt)!\vec{x}(0)\), where \(M\) is the 90° rotation matrix with eigenvalues \(-i, i\).


Figure 3: Exploring the trajectories from Figure 2 together with the slope field for the dynamical system plotted over time.


**Problem 3.4.** Consider the 90° rotation matrix \(M\) as shown in Problem 2.11.

1. Determine the linear system of differential equations for which \((Mt)!\) is a fundamental matrix. The reader may find the digamma function [12] helpful in expressing the coefficients.

2. View the dynamic slope field associated with the system found in 1. Compare your solution of the slope field with the approximations for the expressions used in Desmos: https://bit.ly/46Efkri.

**3.2 Related systems and an application to Calculus.**

We begin this section with an observation. Consider the system of differential equations with constant coefficients whose coefficients are given by the 90° rotation matrix \(M\) as shown in Problem 2.11. This system of differential equations

\[
\begin{align*}
\frac{d}{dt}x_1(t) &= -x_2(t) \\
\frac{d}{dt}x_2(t) &= x_1(t)
\end{align*}
\]  

(3.5)
has solutions given by

\[
\tilde{x}(t) = e^{Mt}\tilde{x}(0)
\]

\[
= \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}
\]

(3.6)

The graph of the solution passing through the initial value \(\tilde{x}(0)^T = (x_1(0), x_2(0))\) is a circle centered at the origin as shown in Figure 4 for different initial values.

In Example 3.2, the fundamental matrix in (3.3) is \(f(Mt)\), where \(f(x) = 1 + x\) and \(M\) is the 90° rotation matrix shown in Problem 2.11. The two systems of differential equations have the relationship that the fundamental matrix for one system produces the tangent lines to the trajectories produced by the fundamental matrix of the other system as illustrated in Figure 5. Additionally in Figure 6 we illustrate a similar idea for generating the quadratic curves of best fit. See demonstration in \url{https://bit.ly/3YXBVw8} to view trajectories in Figure 6. In fact, given a particular solution to (3.5), any solution to (3.2) with initial value on the particular solution to (3.5) will be tangent to the trajectory. We will formalize this observation to any \(2 \times 2\) system of differential equations with constant coefficients in Corollary 3.6. First, we provide motivation from Calculus for this result.

Consider the Taylor series for \(e^{ax}\) centered at \(x_0\),

\[
e^{ax} = \left(1 + a(x - x_0) + \frac{1}{2}a^2(x - x_0)^2 + \frac{1}{3!}a^3(x - x_0)^3 + ... \right) e^{ax_0}
\]

From the power series we see the tangent line and the quadratic of best fit at \(x = x_0\), respectively, is

\[
\ell(x) = (1 + a(x - x_0)) e^{ax_0},
q(x) = \left(1 + a(x - x_0) + \frac{1}{2}a^2(x - x_0)^2 \right) e^{ax_0}.
\]
Figure 6: Graph showing trajectories generated by 
\[ \ddot{x}(t) = (I + M(t-t_0) + \frac{1}{2} M^2(t-t_0)^2) \dot{x}(t_0) \]
with initial value on a solution curve to (3.6).

For a curve in parametric form, we will use the term quadratic of best fit at \( \ddot{x}(t_0) \) to refer to the curve, \( \ddot{q}(t) \), quadratic in \( t \) that agrees in slope and curvature to \( \ddot{x}(t) \) at \( t_0 \)

**Theorem 3.5.** Suppose \( \ddot{x}(t) \) solves the linear system of differential equations \( \ddot{x}'(t) = A(t) \ddot{x}(t) \) with initial value \( \ddot{x}(0) \). Then,

1. the line \( \ddot{\ell}(t) = (I + A(t_0)(t-t_0)) \ddot{x}(t_0) \), is tangent to \( \ddot{x}(t) \) at \( \ddot{x}(t_0) \).
2. the curve \( \ddot{q}(t) = (I + A(t_0)(t-t_0) + \frac{1}{2} (A(t_0)^2 + A'(t_0))(t-t_0)^2) \ddot{x}(t_0) \), has the same slope and curvature as \( \ddot{x}(t) \) at \( \ddot{x}(t_0) \).

**Proof.**

1. We are given \( \ddot{x}(t) \) solves the system

\[ \ddot{x}'(t) = A(t) \ddot{x}(t). \]

Hence the tangent to the curve \( \ddot{x}(t) \) at \( \ddot{x} = \ddot{x}(t_0) \) is given by

\[
\ddot{\ell}(t) = \ddot{x}(t_0) + \ddot{x}'(t_0)(t-t_0) \\
= \ddot{x}(t_0) + A(t_0) \ddot{x}(t_0)(t-t_0) \\
= [I + A(t_0)(t-t_0)] \ddot{x}(t_0).
\]  

2. Differentiating

\[ \ddot{x}'(t) = A(t) \ddot{x}(t), \]

we have

\[
\ddot{x}''(t) = A(t) \ddot{x}'(t) + A'(t) \ddot{x}(t) \\
= A(t)(A(t) \ddot{x}(t)) + A'(t) \ddot{x}(t) \\
= (A(t)^2 + A'(t)) \ddot{x}(t).
\]

Consider \( \ddot{q}(t) = \left(I + A(t_0)(t-t_0) + \frac{1}{2} (A(t_0)^2 + A'(t_0))(t-t_0)^2 \right) \ddot{x}(t_0) \). We have

\[ \ddot{q}'(t_0) = A(t_0) \ddot{x}(t_0) \]  

(3.8)
and
\[
\tilde{q}''(t_0) = (A(t_0)^2 + A'(t_0)) \tilde{x}(t_0)
\]  \hspace{1cm} (3.9)

By (3.8), \( \tilde{q}'(t_0) \) agrees with \( \tilde{x}'(t) \) at the point \( \tilde{x}(t_0) \) and by (3.9) \( \tilde{q}''(t_0) \) agrees with \( \tilde{x}''(t) \) at \( \tilde{x}(t_0) \). Therefore, \( \tilde{q}(t) \) agrees in curvature with \( \tilde{x}(t) \) at the point \( \tilde{x}(t_0) \).

\[\square\]

Corollary 3.6. Consider \( \tilde{x}(t) = e^{Mt}\tilde{x}(0) \) where \( M \) is a \( 2 \times 2 \) matrix with real coefficients.

1. The line \( \tilde{f}(t) = (I + (t - t_0) \cdot A(t_0))\tilde{x}(t_0) \) with values \( \tilde{x}(t) \) on solution \( \tilde{x}(t) = B(t)\tilde{x}(0) \), for \( B(t) \) given in Problem 3.7.


2. The curve \( \tilde{q}(t) = (I + M(t - t_0) + \frac{1}{2}M^2(t - t_0)^2)\tilde{x}(t_0) \) has the same slope and curvature as \( \tilde{x}(t) \) at \( \tilde{x}(t_0) \).


Proof. Recall \( \tilde{x}(t) = e^{Mt}\tilde{x}(0) \) solves the constant coefficient system, \( \tilde{x}'(t) = M\tilde{x}(t) \) with initial value \( \tilde{x}(0) \). Therefore, \( A(t) = M \) and (i) follows. (ii) follows since, \( A'(t) = \frac{d}{dt}M = 0 \) and \( A(t)^2 = M^2 \).

\[\square\]

Problem 3.7. Demonstrate the mathematics shown in the result in Figure 7.

1. Consider
\[
B(t) = \begin{bmatrix}
1 + a^2t^2 & 2at \\
0 & 1 + a^2t^2
\end{bmatrix}.
\]

Determine the matrix \( \log(B(t)) \). The reader may recall Formula (2.2)
2. Determine the matrix $A(t) = \frac{d}{dt} \log(B(t))$. Note this is the coefficient matrix for the system of equations $\ddot{x}(t) = A(t)x(t)$, where the solution is given by $\ddot{x}(t) = B(t)x(0)$.

3. Form the expression for the tangent lines using (3.7) and the coefficients of $A(t)$ considered in 2. Compare your result to the expressions in the Desmos demonstration (https://bit.ly/4bfpB0f) folder labelled, "Tangents and points on tangents."

**Problem 3.8.** Demonstrate the mathematics shown in the result in Figure 8. First consider the matrix $M = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$.

1. Form the matrix $Mt$ and determine its Jordan normal form factorization using (2.2).

2. Form the two matrices $\cos(Mt)$ and $\log(\cos(Mt))$.

3. Determine the matrix $A(t) = \frac{d}{dt} \log(\cos(Mt))$. Note this is the coefficient matrix for the system of equations $\ddot{x}(t) = A(t)x(t)$, where the solution is given by $\ddot{x}(t) = \cos(Mt)x(0)$.

4. Compare your solutions to the inputs in the Desmos demonstration in Figure 8.

5. Observe the solution and quadratics of best fit when you change the eigenvalue of the system with the slider for $a$ in Desmos.

**Problem 3.9.** Let’s revisit the constant coefficient case. Consider the coefficient matrix $M$ in Jordan form shown in Problem 3.8.

1. Determine the solution to the system $\ddot{x}(t) = Mt\ddot{x}(t)$ with initial value $\ddot{x}_0$ using this coefficient matrix. Recall $\ddot{x}(t) = e^{Mt}\ddot{x}(0)$ solves the constant coefficient system.

2. Determine the expression for the tangent line using either Theorem 3.5 or Corollary 3.6 at the point $\ddot{x}(t_0)$ on the solution.

3. Determine the expression for the quadratic of best fit using either Theorem 3.5 or Corollary 3.6 at the point $\ddot{x}(t_0)$ on the solution.

4. Modify the Desmos demonstration in Figure 8 to show the results of your solution to this problem. Including the solution curve corresponding to the initial value, tangent approximations at different times along the trajectory and quadratic approximations at those same times.

3.3 Describing the dynamical system for $f(Mt)$

In this section, we will explore some systems produced by the non-autonomous system of differential equations when $f(Mt)$ is the fundamental matrix for the system where $f(\cdot)$ is analytic in an interval containing 0 and $f(0) = 1$. In particular, we will use $f(x) = \cos(ax)$ and compare the analysis of this system to the constant coefficient case produced by $f(x) = e^{ax}$. To illustrate, let’s recall the analysis of solutions for the homogeneous
system of two linear differential equations with constant coefficients. The solutions to
the initial value problem are represented by \( \mathbf{x}(t) = e^{M t} \mathbf{x}_0 \) solve \( \frac{d \mathbf{x}}{dt} = M \mathbf{x} \) where \( \mathbf{x}(0) = \mathbf{x}_0 \).

The properties of these trajectories are determined by the eigenvalues, \( \lambda_1 \) and \( \lambda_2 \), of the
coefficient matrix \( M \) as well as the corresponding eigenvector(s). The equilibrium solution
to the system is \( x_1(t) = 0, x_2(t) = 0 \) which is unique exactly when \( \det(M) \neq 0 \). The
phase portrait showing a collection of representative trajectories for a number of initial
values, provides an appealing visual account of the asymptotic behavior of the system.

For example, consider the diagonalized system,

\[
\begin{bmatrix}
x'_1(t) \\
x'_2(t)
\end{bmatrix} =
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\tag{3.10}
\]

with \( \lambda_1 \neq \lambda_2 \). The general solution is

\[
\mathbf{x}(t) = e^{M t} \mathbf{c} =
\begin{bmatrix}
e^{\lambda_1 t} & 0 \\
e^{\lambda_2 t}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{\lambda_2 t}
\tag{3.11}
\]

where the last expression in (3.11) is in eigenvalue/eigenvector form. One can then analyze
the solution behavior by considering the different cases for \( \lambda_1 \) and \( \lambda_2 \). Consider \( \lambda_1 < 0 < \lambda_2 \).
If we let \( c_2 = 0 \), then the solution lies on the \( x_1 \)-axis and \( \lim_{t \to \infty} \mathbf{x}(t) = 0 \). If \( c_1 \neq 0 \) and
\( c_2 \neq 0 \) then \( x_1(t) \to 0 \) and \( x_2(t) \to \infty \) as \( t \to \infty \). The behavior is reversed for \( t \to -\infty \).
Additionally, by eliminating the parameter \( t \),

\[
x_2 = b x_1^{\frac{\lambda_2}{\lambda_1}},
\tag{3.12}
\]

where \( b = \frac{c_2}{c_1^{\frac{\lambda_2}{\lambda_1}}} \). Notice (3.12) shows the trajectories for \( \frac{\lambda_2}{\lambda_1} < 0 \) lie on a hyperbolic type
curve as shown in Figure 9. This type of analysis is used to describe the solutions in the
other constant coefficient cases. For example, if \( \frac{\lambda_2}{\lambda_1} = 2 \), then the trajectories would form
a family of parabolas. The reader can review a complete analysis for solutions in the
constant coefficient case in most introductory texts in differential equations. For example,
see [11].

In each case for the constant coefficient system, the properties of the exponential
function play a key role in the solution properties. We expect the same drivers for
determining characteristics to solutions to the systems whose fundamental matrix is given
by \( f(Mt) \). Indeed, consider again Example 1.1. In this case, a comparison of the phase
plane is shown for the constant coefficient case and the case for which \( \cos(Mt)\mathbf{x}(0) \) forms
the solutions. Each phase portrait is shown using the same initial values in the upper-half
plane with \( 0 \leq t \leq 2\pi/3 \), \( \lambda_1 = -3 \) and \( \lambda_2 = 9 \).
Figure 9: Phase portrait showing solutions for $\ddot{x}(t) = e^{Mt}\dot{x}(0)$, where $M$ is diagonal with eigenvalues $\lambda_1 = -3, \lambda_2 = 9$.

Figure 10: Graph showing the phase portrait for $\ddot{x}(t) = \cos(Mt)\dot{x}(0)$, where $M$ is diagonal with eigenvalues $\lambda_1 = -3, \lambda_2 = 9$.

Just as the trajectories depicted in Figure 9 lie on the curves given in (3.12), trajectories in Figure 10 also lie on easily described curves. See https://bit.ly/3IutkK6 for Demos demonstration. Recall the triple angle identity for cosine, $\cos(3x) = 4\cos^3(x) - 3\cos(x)$. By substituting $x = -3t$ into the triple angle identity, we eliminate the parameter $t$ in the solution to the system $x_1(t) = c_1\cos(-3t)$ and $x_2(t) = c_2\cos(9t)$ to obtain the expression for the cubic polynomial that our solutions lie on,

$$x_2 = 4(c_2/c_1^3)x_1^3 - 3(c_2/c_1)x_1$$

where $c_i = x_i(0)$ for $i = 1, 2$. In general, whenever one eigenvalue is an integral multiple of the other, $\lambda_1 = n \cdot \lambda_2$, the trajectories for $\cos(Mt)\ddot{x}(0)$ lie on a particular polynomial of degree $n$. This polynomial comes from the identity for $\cos(nx)$ which can be derived from DeMoivre’s identity from complex variables. We can observe graphically in the Desmos demonstration, that our solutions are periodic.

Problem 3.10. Exploring trajectories $\ddot{x}(t) = \cos(Mt)\dot{x}(0)$, where $M$ is diagonal with eigenvalues $\lambda_1$ and $\lambda_2$.

1. In the Demos demonstration, https://bit.ly/3IutkK6, change $\lambda_1 = c = 2$ and $\lambda_2 = d = -8$ and observe the trajectories and confirm and observe the period graphically.

2. Determine the analytic expression for the polynomial on which the trajectories lie, as well as the period.

3. What properties about the trajectories can you determine or observe for the case that $\lambda_2 = \frac{p}{q}\lambda_1$, for integers $p$ and $q$, $\gcd(p, q) = 1$. 


While our current analysis is for a diagonal system, it can be extended to systems with a diagonalizable fundamental matrix by first changing to the basis of eigenvectors, applying the diagonal analysis, and then moving back to the standard basis. This type of analysis is sometimes illustrated in elementary linear algebra texts for the constant coefficient case; see, for example, [10].

4 Conclusion and Summary

In this paper, we draw the reader’s attention to functional calculus beyond that normally studied in introductory courses. Beginning with the properties of matrix decomposition in linear algebra, we review the literature necessary to understand the calculations highlighted in the paper. We have also chosen to use both elementary functions and special functions as illustrative examples as well as to include links to the Maple© code used in generating some illustrations and links to the illustrations generated by the online calculator Desmos. We use the linear constant coefficient case in which solutions are represented by the matrix exponential, $\tilde{x}(t) = e^{Mt}\tilde{x}(0)$ where $M$ is the $n \times n$ matrix containing the coefficients to the system, to motivate the more general case where solutions are represented by $\tilde{x}(t) = f(Mt)\tilde{x}(0)$, for the function $f(\cdot)$ analytic at the eigenvalues of $M$. We use functional calculus to construct the system for which $f(Mt)$ is a fundamental matrix. By making this connection with the system of differential equations we can determine expressions used for generating a dynamic slope field for the phase portrait. Moreover, connections between solutions to these systems of differential equations can be illustrated, such as solutions to one system generating all of the tangent lines or quadratics of best fit to solutions of another system.

The authors aim to provide the tools and background material necessary for exploring topics in differential equations that lie outside the traditional undergraduate curriculum, particularly topics that connect with functional calculus and linear algebra. This includes a discussion of the theory involved together with visualization tools that allow the reader to explore the examples and problems presented in the paper. These tools may also serve as a template for further study. Students with prior knowledge of linear algebra may find the problems in this paper interesting to explore as a final project in their differential equations course. The problems and open ended exploration below also provide a nice introduction to undergraduate research for students interested in linear algebra and differential equations. Additionally, the paper and problem set might make an interesting individual study or reading course for faculty and students wishing to explore extensions of the material traditionally introduced in an introductory differential equation course.

Project 4.1. Choose an analytic function $y = f(x)$ with interesting properties such that $f(0) = 1$ and a $2 \times 2$ Matrix $M$. Determine the non-autonomous system of differential equations $\tilde{x}'(t) = A(t)\tilde{x}(t)$ for which $f(Mt)$ is a fundamental matrix. Use this system to construct the dynamic slope field for the solutions to the system. Explore the properties of the trajectories formed from $\tilde{x}(t) = f(Mt)\tilde{x}(0)$. Also, use the coefficient matrix to the system to determine the tangent lines and quadratics of best fit for the trajectories that form the solutions to this system. Use technology to further visualize your results for
various coefficient matrices $M$. Notice the role that the eigenvalues and corresponding eigenvectors of $M$ play in your analysis.

References


A Problem solutions

Problem 2.5 Consider the diagonalizable matrix

\[ A(t) = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}. \]

1. Show \( A(t) \) does not commute with \( \frac{d}{dt}A(t) \)

2. Calculate the three expressions in the conclusion of Theorem 2.4.

Solution. 1. We differential term-by-term to obtain \( \frac{d}{dt}A(t) \),

\[ \frac{d}{dt}A(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \]

and calculate \( A(t) \cdot \frac{d}{dt}A(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( \frac{d}{dt}A(t) \cdot A(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \).

2. We first diagonalize the matrix \( A(t) \)

\[ A(t) = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -t & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix}. \]

To calculate \( e^{A(t)} \), apply the exponential to the diagonal and multiply the matrices,

\[ e^{A(t)} = \begin{bmatrix} -t & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} e & t(e - 1) \\ 0 & 1 \end{bmatrix}. \]

Calculating the three expressions in Theorem 2.4 we have

\[ \frac{d}{dt}e^{A(t)} = \begin{bmatrix} 0 & e - 1 \\ 0 & 0 \end{bmatrix}, \quad e^{A(t)} \frac{dA}{dt} = \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{dA}{dt}e^{A(t)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \]

\[ \square \]

Problem 2.6 Consider the diagonalizable matrix \( M = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \).

1. Calculate \( M! \).

2. Does this example motivate a result that you can state and prove for \( n \times n \) matrices?

Solution. 1. We first diagonalize the matrix \( M \)

\[ M = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}. \]

Therefore,

\[ M! = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1! & 0 \\ 0 & 2! \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} = M. \]
2. Result. Let \( M \) be a diagonalizable \( n \times n \) matrix. If each eigenvalue of \( M \) is either 1 or 2 then \( M! = M \).

**Proof** The matrix \( M = Q \text{diag} (\lambda_1, ..., \lambda_m) Q^{-1} \). \( \lambda_i! = \lambda_i \) for \( i = 1, ..., n \) since each \( \lambda_i \) is either a 1 or 2. Therefore,

\[
M! = Q \text{diag} (\lambda_1!, ..., \lambda_m!) Q^{-1} = M. \tag{A.1}
\]

\( \square \)

**Problem 2.10** Use (2.3) to calculate the \( \log(J) \) in Example 2.9.

**Solution.** The matrix in Example 2.9 is a \( 3 \times 3 \) matrix in the form of a Jordan block with eigenvalue \( \lambda \),

\[
J = \begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda \\
\end{bmatrix}.
\]

The result from Ikebe in (2.3) for a function \( f \) analytic at the eigenvalue \( \lambda \) in the Jordan block \( J \)

\[
f(J) = \begin{bmatrix}
f(\lambda) & f'(\lambda) & f''(\lambda)/2! & \cdots & f^{(n-1)}(\lambda)/(n-1)! \\
0 & f(\lambda) & f'(\lambda) & \cdots & f^{(n-2)}(\lambda)/(n-2)! \\
0 & 0 & f(\lambda) & \cdots & f'(\lambda) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & f(\lambda) \\
\end{bmatrix}. \tag{A.2}
\]

Taking \( f(z) = \log(z) \), \( f'(z) = 1/z \), and \( f''(z) = -1/z^2 \) we have,

\[
\log(J) = \begin{bmatrix}
\log(\lambda) & \lambda^{-1} & -\frac{1}{2}\lambda^{-2} \\
0 & \log(\lambda) & \lambda^{-1} \\
0 & 0 & \log(\lambda) \\
\end{bmatrix}.
\]

\( \square \)

**Problem 2.11** Consider the \( 90^\circ \) rotation matrix \( A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \).

1. Factor the matrix \( A \) to diagonal form.

2. Determine the \( \log(A) \). Recall that \( \log(\pm i) = \pm i(\pi/2 + 2n\pi) \). The principle value of \( \log(A) \) is obtained when we let \( n = 0 \).

3. What role does the matrix \( A \) in our example seem to play in matrix algebra?

**Solution.** \( A \) is diagonalized using its eigenvalues \(-i \) and \( i \) and corresponding eigenvectors \( \eta_1 = (-i, 1)^T \) and \( \eta_2 = (i, 1)^T \), respectively.

\[
A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix}.
\]
2. 

\[
\log(A) = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \log(-i) & 0 \\ 0 & \log(i) \end{bmatrix} \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix} \\
= \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -i(\pi/2 + 2n\pi) & 0 \\ 0 & i(\pi/2 + 2n\pi) \end{bmatrix} \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix} \\
= \begin{bmatrix} 0 & -(\pi/2 + 2n\pi) \\ (\pi/2 + 2n\pi) & 0 \end{bmatrix} \\
= A(\pi/2 + 2n\pi).
\]

3. In the algebra of complex variables, \(\log(i) = i \cdot (\pi/2 + 2n\pi)\). This provides the hint that the matrix \(A\) may serve a role similar to the imaginary unit in matrix algebra. Indeed, \((-i)^2 = i^2 = -1\), and we see that \((-A)^2 = A^2 = -I_2\).

Also, in the algebra of complex variables we have Euler’s formula, \(e^{i\pi} = \cos(z) + i \sin(z)\). In our case,

\[
\exp(Az) = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-iz} & 0 \\ 0 & e^{iz} \end{bmatrix} \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix} \\
= \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos(z) - i \sin(z) & 0 \\ 0 & \cos(z) + i \sin(z) \end{bmatrix} \begin{bmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{bmatrix} \\
= \begin{bmatrix} \cos(z) & -\sin(z) \\ \sin(z) & \cos(z) \end{bmatrix} = \cos(Iz) + A \sin(Iz).
\]

\(\square\)

**Problem 3.3** Consider the matrix \(M = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}\). Use the steps below to determine the system of differential equation for which \(f(Mt)\delta(0)\) is a solution for \(f(x) = 1 + x^2\).

1. Form \(f(Mt) = I + (Mt)^2\) and simplify.

2. Determine the matrix \(A(t) = \frac{d}{dt} \log(f(Mt))\).

**Solution.**

1. Using Matrix Algebra with the matrix \(Mt\) we obtain

\[
I + (Mt)^2 = \begin{bmatrix} 1 + a^2t^2 & 2at^2 \\ 0 & 1 + a^2t^2 \end{bmatrix}.
\]
2. Now using (2.2), factor the matrix \( f(Mt) \) into its Jordan normal form,

\[
\begin{bmatrix}
1 + a^2t^2 & 2at^2 \\
0 & 1 + a^2t^2
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 1 + a^2t^2 \\
0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
1 + a^2t^2 & 1 \\
0 & 1 + a^2t^2
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 2at^2
\end{bmatrix}.
\]

Form the \( \log(f(Mt)) \) using (2.3),

\[
\log(f(Mt)) = \begin{bmatrix}
1 & 0 \\
0 & 1/(2at^2)
\end{bmatrix} \begin{bmatrix}
\log(1 + a^2t^2) & 1/(1+a^2t^2) \\
0 & \log(1 + a^2t^2)
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 2at^2
\end{bmatrix}

= \begin{bmatrix}
\log(1 + a^2t^2) & 2at^2/(1+a^2t^2) \\
0 & \log(1 + a^2t^2)
\end{bmatrix}.
\]

Finally, we differentiate term by term to get the coefficient matrix \( A(t) \),

\[
A(t) = \frac{d}{dt} \log(f(Mt)) = \begin{bmatrix}
\frac{d}{dt} \log(1 + a^2t^2) & \frac{d}{dt} 2at^2/(1+a^2t^2) \\
0 & \frac{d}{dt} \log(1 + a^2t^2)
\end{bmatrix}

= \begin{bmatrix}
-2at^2/(1+a^2t^2) & 4at/(1+a^2t^2)^2 \\
0 & 2a^2t/(1+a^2t^2)
\end{bmatrix}.
\]

\[\Box\]

Problem 3.4 Consider the 90° rotation matrix \( M \) as shown in Problem 2.11

1. Determine the linear system of differential equations for which \( (Mt)! \) is a fundamental matrix. The reader may find the digamma function \([10]\) helpful in expressing the coefficients.

2. View the dynamic slope field associated with the system found in 1. Compare your solution of the slope field with the approximations for the expressions used in Desmos: \( \text{https://bit.ly/46Efkri} \).

Solution. Since our matrix \( Mt \) is diagonalizable, the eigenvalues \(-it\) and \( it\) with eigenvectors \( \eta_1 = (-i,1)^T \) and \( \eta_2 = (i,1)^T \), respectively.

\[
(Mt)! = \begin{bmatrix}
-1 & 1 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
(-it)! \\
0
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
(-it)! + (it)! & -i(-it)! + i(it)! \\
i(-it)! - i(it)! & (-it)! + (it)!
\end{bmatrix} = \begin{bmatrix}
\text{Re}(it)! & -\text{Im}(it)! \\
\text{Im}(it)! & \text{Re}(it)!
\end{bmatrix}.
\]

Recall from analytic number theory the digamma function \( \Psi^{(0)}(z) = \frac{d}{dz} \log(\Gamma(z)) \) which is the first of the higher-order log-derivatives of the gamma function, called the polygamma functions. This class of functions is used in the study of harmonic numbers.
They have integral and summation representations. For our purposes, we are using a translation of the gamma function, \( z! = \Gamma(z + 1) \). Hence, we use

\[
\psi^{(0)}(z + 1) = \frac{d}{dz} \log(\Gamma(z + 1)) = \frac{d}{dz} \log(z!) = -\gamma + \sum_{k=1}^{\infty} \frac{z}{k(k + z)}. \tag{A.4}
\]

For \((Mt)!\) as shown in (A.3), the eigenvalues are \((-it)\) and \((it)\). By (A.4),

\[
r(t) = \frac{d}{dt} \log((-it)!) = \psi^{(0)}(1 + (-it))(-i) = \gamma i + \sum_{k=1}^{\infty} \frac{-t}{k(k - it)}
\]

and

\[
s(t) = \frac{d}{dt} \log((it)!) = \psi^{(0)}(1 + (it))(i) = -\gamma i + \sum_{k=1}^{\infty} \frac{-t}{k(k + it)}.
\]

Hence,

\[
A(t) = \frac{d}{dt} \log((Mt)!) = \begin{bmatrix} 1 & i \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r(t) & 0 \\ 0 & s(t) \end{bmatrix} \begin{bmatrix} i/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}.
\]

Finally,

\[
A(t) = \begin{bmatrix} -h(t) & -f(t) \\ f(t) & -h(t) \end{bmatrix},
\]

where

\[
f(t) = -\frac{i}{2} (s(t) - r(t)) = -\gamma + \sum_{k=1}^{\infty} \frac{t^2/k}{k^2 + t^2}
\]

and

\[
h(t) = -\frac{1}{2} (r(t) + s(t)) = \sum_{k=1}^{\infty} \frac{t}{k^2 + t^2}.
\]

\[\Box\]
Problem 3.7 Demonstrate the mathematics shown in the result in Figure 7.

1. Consider

\[ B(t) = \begin{bmatrix} 1 + a^2 t^2 & 2at \\ 0 & 1 + a^2 t^2 \end{bmatrix}. \]

Determine the matrix \( \log(B(t)) \). The reader may wish to use Formulas (2.2) and (2.3).

2. Determine the matrix \( A(t) = \frac{d}{dt} \log(B(t)) \). Note this is the coefficient matrix for the system of equations \( \dot{x}'(t) = A(t)\dot{x}(t) \), where the solution is given by \( \dot{x}(t) = B(t)\dot{x}(0) \).

3. Form the expression for the tangent lines using (3.7) and the coefficients of \( A(t) \) considered in 1. Compare your result to the expressions in the Desmos demonstration folder labelled, “Tangents and points on tangents.”

Solution. 1.

\[ B(t) = \begin{bmatrix} 1 + a^2 t^2 & 2at \\ 0 & 1 + a^2 t^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]
To obtain the expression for \( \log(B(t)) \), we again use (2.2)

\[
\log(B(t)) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2at} \end{bmatrix} \begin{bmatrix} \log(1 + a^2t^2) \\ \frac{1}{1 + a^2t^2} \log(1 + a^2t^2) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2at \end{bmatrix}
\]

\[
= \begin{bmatrix} \log(1 + a^2t^2) & \frac{2at}{1 + a^2t^2} \\ 0 & \log(1 + a^2t^2) \end{bmatrix}
\]

2. We form \( A(t) \) taking the derivative term-by-term

\[
A(t) = \frac{d}{dt} \log(B(t)) = \begin{bmatrix} \frac{2a^2t}{1 + a^2t^2} & \frac{2a(1 - a^2t^2)}{(1 + a^2t^2)^2} \\ 0 & \frac{2a^2t}{1 + a^2t^2} \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}.
\]

3. In component form it is written,

\[
x_1(t) = (1 + a_{11}(t_0)(t - t_0))x_1(t_0) + a_{12}(t_0)(t - t_0)x_2(t_0)
\]

\[
x_2(t) = (1 + a_{22}(t_0)(t - t_0))x_2(t_0).
\]

\[\Box\]

**Problem 3.8** Demonstrate the mathematics shown in the result in Figure 8. First consider the matrix \( M = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \).

1. Form the matrix \( Mt \) and determine its Jordan normal form factorization using (2.2).

2. Form the two matrices \( \cos(Mt) \) and \( \log(\cos(Mt)) \).

3. Determine the matrix \( A(t) = \frac{d}{dt} \log(\cos(Mt)) \). Note this is the coefficient matrix for the system of equations \( \tilde{x}'(t) = A(t)\tilde{x}(t) \), where the solution is given by \( \tilde{x}(t) = \cos(Mt)\tilde{x}(0) \)

4. Compare your solutions to the inputs in the Desmos demonstration in Figure 8.

5. Observe the solution and quadratics of best fit when you change the eigenvalue of the system with the slider for \( a \) in Desmos.

**Solution.** 1. Applying (2.2) to \( Mt \) we have

\[
Mt = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} at & 1 \\ 0 & at \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}.
\]
2. Since \( \frac{d}{dz} \cos(z) = -\sin(z) \), we apply formula 2.3 to obtain,

\[
\cos(Mt) = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} \cos(at) & -\sin(at) \\ 0 & \cos(at) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} \cos(at) & -t \sin(at) \\ 0 & \cos(at) \end{bmatrix}.
\]

To obtain the second expression, we observe \( \frac{d}{dz} \log(\cos(z)) = -\tan(z) \) and apply formula 2.3 to obtain,

\[
\log(\cos(Mt)) = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} \log(\cos(at)) & -\tan(at) \\ 0 & \log(\cos(at)) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} \log(\cos(at)) & -t \tan(at) \\ 0 & \log(\cos(at)) \end{bmatrix}.
\]

3. We form \( A(t) \) taking the derivative term-by-term

\[
A(t) = \frac{d}{dt} \log(\cos(Mt)) = \begin{bmatrix} \frac{d}{dt} \log(\cos(at)) & \frac{d}{dt}(-t \tan(at)) \\ 0 & \frac{d}{dt} \log(\cos(at)) \end{bmatrix} = \begin{bmatrix} -a \tan(at) & -\tan(at) - at \sec^2(at) \\ 0 & -a \tan(at) \end{bmatrix}.
\]

4. The entries for the matrix \( \cos(Mt) \) are found in the Desmos folder "Solution for \( f(Mt) = \cos(Mt) \), \( \lambda \) Jordan Block, eigenvalue \( \lambda \)" and labelled using the notation \( b_{ij} \). The initial value to the system is labelled \( (a_1, b_1) \). \( A(t) \) is found in the Desmos demonstration in the folder labelled "Non-autonomous system of linear differential equations for \( f(Mt), f(x) = \cos(x) \). The matrix entries for \( A(t) \) are labelled using the notation \( f_{ij} \). \( \frac{d}{dt}A(t) \) is found in the Desmos demonstration in the folder labelled "\( A'(t) \) for quadratics of best fit". The matrix entries for \( \frac{d}{dt}A(t) \) are labelled using the notation \( g_{ij} \) and the derivative operation in Desmos is used \( g_{ij} = \frac{d}{dt}f_{ij} \).

\( \square \)

**Problem 3.9** Let’s revisit the constant coefficient case. Consider the coefficient matrix \( M \) in Jordan form shown in Problem 3.8.

1. Determine the solution to the system \( \ddot{x}(t) = M\dot{x}(t) \) with initial value \( \ddot{x}(0) \) using this coefficient matrix. Recall \( \dot{x}(t) = e^{Mt}\ddot{x}(0) \) solves the constant coefficient system.

2. Determine the expression for the tangent line using either Theorem 3.5 or Corollary 3.6 at the point \( \ddot{x}(t_0) \) on the solution.
3. Determine the expression for the quadratic of best fit using either Theorem 3.5 or Corollary 3.6 at the point $\bar{x}(t_0)$ on the solution.

4. Modify the Desmos demonstration in Figure 8 to show the results of your solution to this problem. Including the solution curve corresponding to the initial value, tangent approximations at different times along the trajectory and quadratic approximations at those same times.

**Solution.**

1. Since 
   \[ M_t = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} at & 1 \\ 0 & at \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \]
   and $\frac{d}{dz} e^z = e^z$, we apply formula 2.2 to obtain,
   \[ e^{M_t} = \begin{bmatrix} 1 & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} e^{at} & e^{at} \\ 0 & e^{at} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{at} & te^{at} \\ 0 & e^{at} \end{bmatrix}. \]

   Therefore, we can write the components of the solution $\bar{x}(t) = e^{M_t} \bar{x}(0)$,
   \[ x_1(t) = e^{at} x_1(0) + te^{at} x_2(0) \]
   \[ x_2(t) = e^{at} x_2(0) \]

2. Using Corollary 3.6 we form the tangent line
   \[ \bar{\ell}(t) = (I + M(t-t_0))\bar{x}(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} (t-t_0) \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}. \]

   In component form it is written,
   \[ x_1(t) = (1 + a(t-t_0))x_1(t_0) + (t-t_0)x_2(t_0) \]
   \[ x_2(t) = (1 + a(t-t_0))x_2(t_0). \]

3. Again, using Corollary 3.6 we form the quadratic of best fit
   \[ \bar{q}(t) = \left( I + M(t-t_0) + \frac{1}{2} M^2(t-t_0)^2 \right) \bar{x}(t_0) \]
   \[ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} (t-t_0) + \frac{1}{2} \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix} (t-t_0)^2 \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}. \]
In component form it is written,

\[
x_1(t) = \left(1 + a(t - t_0) + \frac{1}{2}a^2(t - t_0)^2\right) x_1(t_0) + \left((t - t_0) + a(t - t_0)^2\right) x_2(t_0)
\]

\[
x_2(t) = \left(1 + a(t - t_0) + \frac{1}{2}a^2(t - t_0)^2\right) x_2(t_0)
\]

4. Modify the Desmos demonstration shown in Figure 8. First, follow the link in the caption and save a copy. Then do the following:

(a) Change the label for the first folder to, "Non-autonomous system of linear differential equations for f(Mt), f(x)=exp(x)"

(b) Enter the coefficients for A(t)=M, \(f_{11}(r) = a, f_{12}(r) = 1, f_{21}(r) = 0, f_{22}(r) = a\)

(c) No change is necessary in the folder, "A'(t) for quadratics of best fit"

(d) Change the label for the third folder to, "Solution for f(Mt)=exp(Mt), M Jordan Block, eigenvalue a."

(e) Enter the coefficients for B(t)=f(Mt), \(b_{11}(r) = \exp(ar), b_{12}(r) = r \cdot \exp(ar), b_{21}(r) = 0, b_{22}(r) = \exp(ar)\).

(f) In the folder, "Initial Values for all trajectories" set the initial values of the solution curve used for calculating values for the tangents, to the same initial values for the displayed solution curve. That is \(a_1 = a_0\) and \(b_1 = b_0\). Now when the Desmos solution curve is dragged around, the tangent lines and quadratics of best fit will remain on the curve.

(g) The Desmos demo is now ready to use. You can find our solution at the link https://bit.ly/3K44LF8.

\[\square\]

**Problem 3.10** Exploring trajectories \(\ddot{x}(t) = \cos(Mt)\dot{x}(0)\), where \(M\) is diagonal with eigenvalues \(\lambda_1\) and \(\lambda_2\).

1. In the Desmos demonstration, https://bit.ly/3IutkK6, change \(\lambda_1 = c = 2\) and \(\lambda_2 = d = -8\) and observe the trajectories and confirm and observe the period graphically.

2. Determine the analytic expression for the polynomial on which the trajectories lie, as well as the period.

3. What properties about the trajectories can you determine or observe for the case that \(\lambda_2 = q\) and \(\lambda_1 = p\), for integers \(p\) and \(q\), \(\gcd(p, q) = 1\).

**Solution.** 1. After inserting the inputs \(c = 2\) and \(d = -8\), the reader will observe the period of the trajectories is \(\pi\), by moving the slider from zero to \(\pi\). The reader may observe that the \(\gcd(2, 8) = 2\) and the period of the trajectory is \(2\pi/2\). In the text, \(c = -3\) and \(d = 9\). Again, the \(\gcd(3, 9) = 3\) and the observed period is \(2\pi/3\).
2. Recall the multiple angle identity for cosine, \( \cos(4x) = 8 \cos^4(x) - 8 \cos^2(x) + 1 \). By substituting \( x = 2t \), we obtain,

\[
\cos(8t) = 8 \cos^4(2t) - 8 \cos^2(2t) + 1.
\]

Substituting \( \cos(-8t) = \frac{x_2(t)}{c_2} \) and \( \cos(2t) = \frac{x_1(t)}{c_1} \) and solving for \( x_2 \),

\[
x_2 = 8 \frac{c_2}{c_1^4} x_1^4 - 8 \frac{c_2}{c_1^2} x_1^2 + 1.
\]

3. First, the reader will recall part 1. The period of these trajectories with \( \gcd(p, q) = 1 \) is \( 2\pi \). The curves are quite complicated and are algebraic curves. When eliminating the parameter \( t \), one obtains a polynomial \( p \) such that \( p(x_1, x_2) = 0 \). The reader should verify the period and derive the polynomial equation, \( p(x_1, x_2) = 0 \), for the case \( \lambda_1 = 2 \) and \( \lambda_2 = 3 \).