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Combinatorial Polynomial Hirsch Conjecture

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Abstract

The Hirsch Conjecture states that for a d-dimensional polytope with n facets, the diameter of the graph of the polytope is at most $n-d$. This conjecture was disproven in 2010 by Francisco Santos Leal. However, a polynomial bound in n and d on the diameter of a polytope may still exist. Finding a polynomial bound would provide a worst-case scenario runtime for the Simplex Method of Linear Programming. However working only with polytopes in higher dimensions can prove challenging, so other approaches are welcome. There are many equivalent formulations of the Hirsch Conjecture, one of which is the Combinatorial Polynomial Hirsch Conjecture, which turns the problem into a matter of counting sets.

This thesis explores the Combinatorial Polynomial Hirsch Conjecture.

Contents

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Finally, thank you Willie Zuniga for being a wonderful friend during your time here. May your soul find rest in the great gig in the sky. We miss you dearly.

"And I am not frightened of dying. Any time will do, I don't mind. Why should I be frightened of dying? There's no reason for it – you've got to go sometime." - Gerry O'Driscoll.

Chapter 1

An Introduction to the Hirsch Conjecture

In his 1963 book "Linear Programming and Extensions," Dantzig (1963) George Dantzig noted that:

"While the simplex method appears a natural one to try in the n-dimensional space of the variables, it might be expected *a priori*, to be inefficient, as there could be considerable wandering on the outside edges of the convex set of solutions before an optimal extreme point is reached. This certainly appears to be true when $n - m = k$ is small..."

He then proceeded to quote Warren Hirsch on a possible worst case scenario.

- It has been conjectured that, by proper choice of variables to enter the basic set, it is possible to pass from any basic feasible solution to any other in m or less pivot steps, where each basic solution generated along the way must be feasible. For the cases $m \leq 4$ the conjecture is known to be true. [W.M. Hirsch, 1957, verbal communication]

The conjecture is restated later on in geometric terms:

Conjecture 1.0.1. Hirsch Conjecture

Does there exist a sequence of m or less pivot operations, each generating a new basic feasible solution which starts with some given basic feasible solution and ends at some other basic feasible solution, where m is the number of equations? Expressed *geometrically*: in a convex region in $n - m$ dimensional space defined by n halfplanes, is m an upper bound for the minimum length chain of the adjacent vertices joining two given vertices?

Ziegler (2012)

Thus the Hirsch Conjecture was born. Of course to understand the motivations and implications of the Hirsch conjecture, we need to dissect what Dantzig is saying.

The Simplex Method, originally created by Dantzig, is an optimization algorithm used in linear programming, a field of mathematics and computer science involving optimizing linear functions subject to linear constraints. Suppose we want to optimize a linear equation f with d variables (so we are working in d-dimensional space). Additionally, we have n constraint equations, which when combined, form a d-dimensional polytope. Then, it can be shown that the maximum value of f occurs at one of the vertices of the polyhedron formed, so long as there is an optimum. It can also be shown that if a vertex does not contain the maximum value of f, then an edge exists such that the adjacent vertex contains a larger value of f. The Simplex Method follows edges until the maximum vertex is found. Dantzig (1963)

The Hirsch Conjecture essentially states that in considering the vertex-edge graph of a polytope, the graph's diameter should be bounded by $n - d$. The conjecture would have provided an upper bound in the number of steps, or as Hirsch calls it, "pivot operations." The bound would have been not only polynomial time, but linear with respect to the number of facets and dimension of a polytope. There are certain situations where the Hirsch conjecture has been proven to be true - for example, when $d \leq 3$, and as stated above, when $n - d \leq 4$. Ziegler (2012) Dantzig (1963)

Unfortunately in 2010, the conjecture as stated by Hirsch was proven to be untrue by Francisco Santos Leal, who provided a 43-dimensional counterexample. While this is a setback on finding a worst-case bound on the Simplex Method, the possibility that the Simplex Method is still polynomial time has not been ruled out, and a polynomial bound is still being searched for. The ultimate question the Hirsch Conjecture now asks is if a polynomial bound on the maximum diameter exists. Santos (2012)

Now we come to a different, but equivalent formulation of the Hirsch Conjecture, the Combinatorial Polynomial Hirsch Conjecture:

Conjecture 1.0.2. Combinatorial Polynomial Hirsch Conjecture: Consider t non-empty families of subsets $F_1, ..., F_t$ of $\{1, ..., n\}$ that are disjoint (i.e. no set S can belong to two of the families F_i, F_j). Suppose that for every $i < j < k$ and every $S \in F_i$ and $T \in F_k$, then there exists $R \in F_j$ such that $S \cap T \subset R$. Let $f(n)$ be the largest value of t for which this is possible. Then $f(n)$ is of polynomial size in n.

Kalai (2010a)

Though this may initially seem relatively unrelated to the Hirsch Conjecture, the next section shall demonstrate how the two are connected, and ultimately equivalent.

Chapter 2

Base Abstractions and Connected Layer Families

2.1 Base Abstractions

In this section, we will build up some definitions which will allow us to demonstrate that the Combinatorial Polynomial Hirsch Conjecture implies the Hirsch Conjecture.

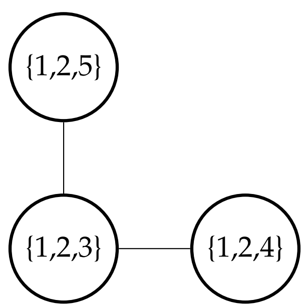
Definition 2.1.1. Fix a finite set S of cardinality n, called the *symbol set*. Let $\mathcal{A} \subseteq \binom{S}{d}$ where $\binom{S}{d}$ is the set of all d-element subsets of S. We consider connected graphs of the form $= (\mathcal{A}, E)$ with vertex set $\mathcal{A}$ and edge set E satisfying:

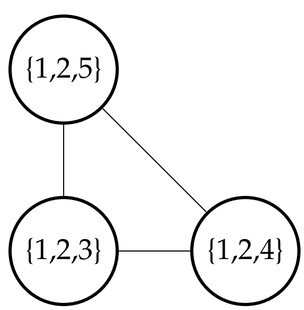
- for each $A, A' \subseteq \mathcal{A}$, there is a path from A to A' in the graph G using only vertices that contain $A \cap A'$.

If this occurs, we say that G is a d-dimensional *base abstraction* of $\mathcal{A}$ on the symbol set S. The *diameter* of the base abstraction is the diameter of the graph G.

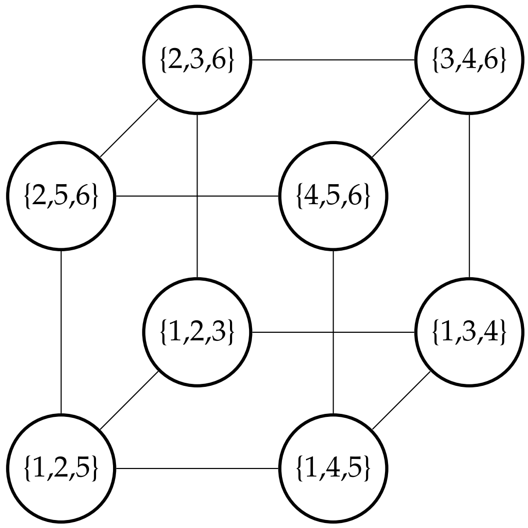
Kim (2012)

Example. Note that base abstractions are not necessarily well defined from the vertex set! For example, if we define $S = \{1,2,3,4,5\}$, $d = 3$, and $\mathcal{A} = \{\{1,2,3\}\{1,2,4\}\{1,2,5\}\}$, the following are valid base abstractions derived from the same vertex set.





Example. Note that graphs of simple d-dimensional polyhedra with n facets can be represented as base abstractions, with proper choice of $(\mathcal{A}, E)$. For example, the cube can be realized as a base abstraction as follows:



In general for d-dimensional polyhedra with n facets and vertices with degree d, we may create a base abstraction by the following algorithm. Let $S = \{1, 2, ..., n\}$. Assign each $i \in S$ to a facet, and for each vertex v_k, let its corresponding set $A_k \in \mathcal{A}$, be defined as such: A_k contains $i \in S$ if and only if v is on the border of facet i. Note then that $\mathcal{A} \subseteq \binom{S}{d}$.

2.2 Connected Layer Families

We now come to a new definition of a family of sets, which have similar properties to the properties outlined in the Combinatorial Polynomial Hirsch Conjecture.

Definition 2.2.1. A d-dimensional *connected layer family* of $\mathcal{A} \subseteq \binom{S}{d}$ on a set of $n = |S|$ symbols is a family $\mathcal{V} = \{\mathcal{V}_0, ..., \mathcal{V}_t\}$ of non-empty sets such that:

- **partition property:** $\mathcal{A} = \mathcal{V}_0 \cup ... \cup \mathcal{V}_t$
- **disjointness property:** $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ if $i \neq j$
- **connectivity property:** for all $i < j < k$ and $A \in \mathcal{V}_i, A' \in \mathcal{V}_k$, there is an $A'' \in \mathcal{V}_j$ such that $A \cap A' \subseteq A''$.

Each individual $\mathcal{V}_i$ is called a *layer*. The *diameter* of the connected layer family $\mathcal{V} = \{\mathcal{V}_0, \ldots, \mathcal{V}_t\}$ is t.

Kim (2012)

Example. A straightforward example of a connected layer family is:

- $\mathcal{V}_0 = \{\{1, 2, 5\}\}$
- $\mathcal{V}_1 = \{\{1, 4, 5\}, \{1, 2, 3\}, \{2, 5, 6\}\}$
- $\mathcal{V}_2 = \{\{1, 3, 4\}.\{4, 5, 6\}, \{2, 3, 6\}\}$
- $\mathcal{V}_3 = \{\{3, 4, 6\}\}$

It is clear here that the partition property and disjointness property are held. An example of the connectivity is setting $i = 1, j = 2, k = 3$. Then note that for $A \in \mathcal{V}_1$ and $A' \in \mathcal{V}_3$, $A \cap A'$ is only ever 3, 4, or 6, all of which are in sets in $\mathcal{V}_2$.

This connected layer family was obtained from the cube in the previous example: we took $\mathcal{V}_0$ to be one vertex, and placed each other subset in a layer corresponding its distance from the initial subset. In general, this process can be used to obtain a connected layer family from a base abstraction. This will be proved later.

Base abstractions and connected layer families are linked. Let $B(n, d)$ represent the maximum diameter of a d-dimensional base abstraction generated from a symbol set of size n, and $C(n, d)$ represent the maximum size of $\mathcal{V}$ of a d-dimensional connected layer family generated from a symbol set of size n.

Theorem 2.2.1. Let $G = (\mathcal{A}, E)$ be a d-dimensional base abstraction and fix a particular d-subset $Z \in \mathcal{A}$. Then, let $\mathcal{V}_i := \{A \in \mathcal{A} : \text{dist}_G(A, Z) = i\}$. Then $\mathcal{V} = \{\mathcal{V}_0, \ldots, \mathcal{V}_t\}$ is a connected layer family.

Proof. The partition property and disjointness properties of $\mathcal{V}$ follow from our construction: each d-subset has a unique distance away from Z, and thus falls into one and only one set $\mathcal{V}_i$. Thus $\mathcal{A} = \mathcal{V}_0 \cup \ldots \cup \mathcal{V}_t$ and $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ if $i \neq j$.

To prove the connectivity property, we will begin with the case where $i = 0$. Any set $Z' \in \mathcal{V}_k$ must be distance k away from Z in G. By the base abstraction property, any vertex $Z'' \in \mathcal{A}$ on a path P connecting Z and Z'

must contain $Z \cup Z'$. Since P contains vertices in all $\mathcal{V}_l$ for $0 \leq l \leq k$, we have demonstrated the connectivity property for $i = 0$.

For $i > 0$ the proof follows similarly. Given any $A \in \mathcal{V}_i, A' \in \mathcal{V}_k$, by the base abstraction property, every $v \in \mathcal{A}$ on a path P connecting A and A' contains $A \cap A'$. We may show P must move through a point in every $\mathcal{V}_j, ..., \mathcal{V}_k$. Suppose to the contrary that P does not contain a vertex in $\mathcal{V}_l$ for $j < l < k$. In that case, P must jump at some point from a vertex $v \in \mathcal{V}_m$ to a vertex $v' \in \mathcal{V}_n$ for $m < l < n$. However, this would imply that v' has distance $m + 1$ from Z and thus $v' \in \mathcal{V}_{m+1}$. Therefore $m + 1 = n$, a contradiction since $m < l < n$ and thus $|m - n| > 1$. Thus P passes through each $\mathcal{V}_l$ for $i < l < k$ and we have demonstrated the connectivity property for $i > 0$.

Thus the layering process forms a connected layer family from a base abstraction. □

Corollary 2.2.1.1. $B(n, d) = C(n, d)$.

Proof. The proof comes from the fact that we may generate a connected layer family from a base abstraction's $\mathcal{A}$-set, and vice versa, based on the distances of the subsets in $\mathcal{A}$. To show $B(n, d) \leq C(n, d)$, we use the previous theorem, thus demonstrating that the diameter of a base abstraction is less than or equal to the maximum size of $\mathcal{V}$.

To show $C(n, d) \leq B(n, d)$, we use a reverse process: if $A \in \mathcal{V}_i$ and $A' \in \mathcal{V}_j$, then connect A and A' if and only if $|i - j| \leq 1$. It is clear that this will produce a base abstraction, from the connectivity property of connected layer families, and thus the maximum size of $\mathcal{V}$ is less than or equal to the diameter of a base abstraction generated from the same set.

□

So we note that given any simple polytope, we can generate not only a base abstraction, but a connected layer family corresponding to it as well! Then, if the Combinatorial Polynomial Hirsch Conjecture is true, the Hirsch Conjecture will be as well. Additionally, Connected Layer Families and Base Abstractions are considerably easier to work with than d-dimension polytopes, so we have a nicer environment to work with in an attempt to prove (or disprove) the Hirsch Conjecture.

It is worth noting that we can slighlty weaken the definition of Connected Layer Families and Base Abstractions to apply them to non-simple polynomials, polynomials where vertices have variable degrees. To do so, we remove the restriction that $\mathcal{A} \subseteq \binom{S}{d}$, and instead let $\mathcal{A} \subseteq 2^S$. In other words, the subsets of S may have variable size. The properties we have proven still hold, as none of our proofs utilized the fact that the subsets all had size d.

However, additional properties must be considered if we are to attempt to equivocate the conjectures, as not every Connected Layer Family belonging to $C(n, d)$ may correspond directly to a d-dimensional polytopes with n facets, that is, the Connected Layer Family may not have polygonal representations. We may introduce a generalized form of connected layer families to work with instead, and introduce restrictions which may help us find a direct bijection between the two conjectures.

Chapter 3

Subset Partition Graphs

Now that we have motivated the connection between the Hirsch Conjecture and the Combinatorial Polynomial variant, we may introduce a generalization of base abstractions, **subset partition graphs**.

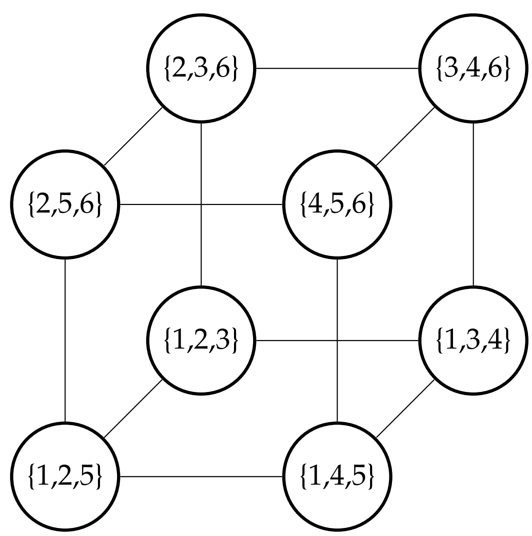
Definition 3.0.1. Fix a finite set S of cardinality n and a set $\mathcal{A} \subseteq \binom{S}{d}$ of subsets. Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph with vertex set $\mathcal{V} = \{\mathcal{V}_0, ..., \mathcal{V}_t\}$. If $\mathcal{V}$ is a partition of $\mathcal{A}$ in the sense that:

- $\mathcal{A} = \mathcal{V}_0 \cup ... \cup \mathcal{V}_t$,
- $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ if $i \neq j$, and
- $\mathcal{V}_i \neq \emptyset$ for all i.

Then we say that G is a d-dimensional subset partition graph of A on the symbol set S.

Kim (2012)

Example. Recall the cube from before:



Assign each vertex to a unique $\mathcal{V}_i$, then we have a 3-dimensional subset partition graph.

We may note that subset partition graphs are a weaker version of connected layer families - though they share the same structure, the subset partition graph contains no connectivity property, other than requiring that the graph be connected. However, we may also note that a d-dimensional subset partition graph is a combinatorial abstraction of a simple d-dimensional polyhedra with n facets. Each of the n facets of a d-dimensional polyhedron P corresponds to a symbol $s \in S$ and a vertex of P corresponds to a d-set $A \in \mathcal{A}$ given by the incident facets. However note that subset partition graphs alone do not yield much interesting information - we must define additional properties.

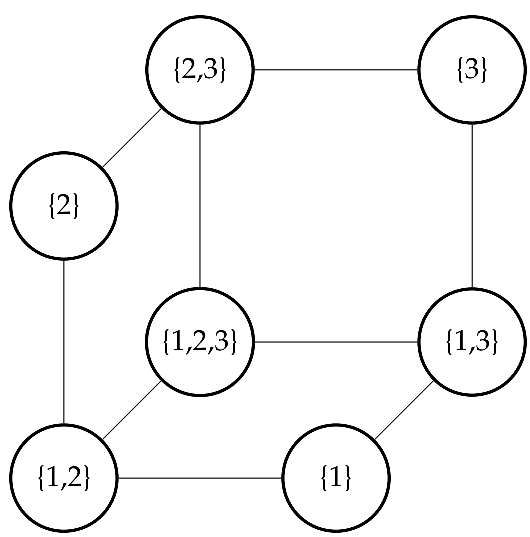
3.1 Restriction

Definition 3.1.1. (Restriction) Let $G = (\mathcal{V}, E)$ be a subset partition graph of $\mathcal{A}$ on the symbol set S, and let $F \subseteq S$ be a collection of symbols. We define a new subset partition graph $G_F = (\mathcal{V}_F, E_F)$ of $\mathcal{A}_F$ on the symbol set $S' := S$.

We define $\mathcal{A}_F := \{A \in \mathcal{A} : F \subseteq A\}$. That is to say, $\mathcal{A}_F$ is obtained from deleting from $\mathcal{A}$ (and the containing $\mathcal{V}_i$) any d-set A which does not contain F. This deletion from the vertices in $\mathcal{V}$ which are still non-empty, and two vertices in $\mathcal{V}_F$ are connected by an edge in E_F exactly when the associated vertices were connected in E. The subset partition graph G_F is called the restriction of G with respect to F.

Kim (2012)

Example. For example, let us consider the cube from before, but restricted with $F = \{1, 2, 3\}$. Then, G_F appears as such:



In terms of polytopes, we can think of restriction as removing certain faces, and the corresponding edges and vertices from the vertex-edge graph. This gives us an interesting method of observing only certain aspects of polytopes or sets, and potentially allows us to reduce dimensions. If we were to let $F = \{1\}$ in the previous example, we would have returned a square.

3.2 Applicable Properties to Subset Partition Graphs

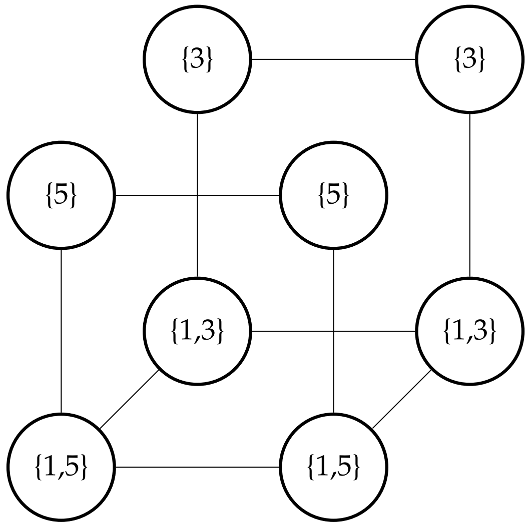
Edward Kim outlines four main properties which can be considered for subset partition graphs which may produce interesting results:

- **dimension reduction**: if $F \subseteq S$ such that $|F| \leq d$ then the underlying graph of the trestriction G_F is a connected graph.
- **adjacency**: if $A, A' \in \mathcal{A}$ and $|A \cap A'| = d - 1$, then A and A' are in the same or adjacent vertices of G.
- **strong adjacency**: adjacency holds and if two vertices $\mathcal{V}_i$ and $\mathcal{V}_j$ are adjacent in G then there are d-sets $A \in \mathcal{V}_i$ and $A' \in \mathcal{V}_j$ such that $|A \cap A'| = d - 1$.
- **endpoint-count**: if $F \in \binom{S}{d-1,}$ then $|\{A \in \mathcal{A} : F \subset A\}| \leq 2$.

Kim (2012)

Kim states that these properties are defined in a manner such that we may flip these properties "on" and "off" as we want a flexible framework where we consider certain collections of properties at a time. Each one of these properties has its own set of implications.

Example. Note that the cube follows the dimension reduction property! There are only two unique ways (up to symmetry) to remove three faces, one such instance is demonstrated in the previous example, and the other can be seen as a rectangle bent along two lines.



However in this reduction, since no vertices have been omitted, all edges are still present; some have been removed in the diagram simply to demonstrate the full structure of the reduction.

Example. We may note as well that our cube follows all four properties. Endpoint-count is satisfied in that the intersection of $d-1$ facets is an edge, which connects two vertices, the only two which touch all of those facets. In the case of a cube, the intersection of 2 faces is an edge, which corresponds to the two vertices connected by the edge. Adjacency and strong adjacency follow similarly.

Note that the class of subset partition graphs with the dimension reduction property with the additional condition that the underlying graph of G is a path is exactly the class of connected layer families.

There are a few additional combinatorial properties of polytopes which translate into natural properties to consider for subset partition graphs.

- d-**connectedness**: the graph G is d-connected.
- d-**regularity**: the graph G is d-regular.
- d-**neighbors**: for every $A \in \mathcal{A}$, $|\{A' \in \mathcal{A}\backslash\{A\} : |A \cap A'| = d-1\}| = d$.
- **one-subset**: $|\mathcal{V}_i| = 1$ for each $i = 0, \ldots, t$.

Example. Again, note that the cube obeys all of these properties. The cube is 3-connected, that is, we must remove a minimum of 3 vertices to disconnect the graph. It is 3-regular, that is, each vertex has degree three, also showing the d-neighbors property. Lastly, we construct the cube graph by putting each A in its own $\mathcal{V}$-set, so the one-subset property is fulfilled.

Note that the d-connectedness property for subset partition graphs is desirable, since the graph of a d-dimensional polytope is d-connected by Balinski's Theorem. It may be observed that d-regularity and d-neighbors hold for simple d-polytopes (where each vertex has the same degree). However, note that these properties do not hold for unbounded polyhedra. THe one-subset property holds for the graphs of polytopes, as each vertex in the subset partition graph should contain the d-set of incident facets. Kim (2012)

We may lastly introduce two graph operations. Let $\mathcal{V}_i$ and $\mathcal{V}_j$ be two vertices in $\mathcal{V}$. Then:

- **Contraction:** If $\mathcal{V}_i$ and $\mathcal{V}_j$ are connected by an edge in E, contraction on the edge produces a new subset partition graph with one less vertex: the two vertices $\mathcal{V}_i$ and $\mathcal{V}_j$ are replaced with a new vertex which contains all of the d-sets which were in $\mathcal{V}_i$ and $\mathcal{V}_j$.

- **Edge addition:** If $\mathcal{V}_i$ and $\mathcal{V}_j$ were not connected by an edge in E, edge addition makes the two vertices adjacent. The resulting subset partition graph has one more edge than the original partition graph G does.

Note that dimension reduction, adjacency, and endpoint-count properties for subset partition graphs are preserved under the outlined graph operations. Additionally, after a sufficient number of contractions and edge additions, the resulting graph will be a complete graph, and thus the dimension reduction and adjacency conditions will hold. Kim (2012)

Chapter 4

Moving Towards Sets

4.1 Returning to Connected Sets

Now, we shift focus from establishing a solid background on the Hirsch Conjecture and polytopes, and the duality of connected layer families, base abstractions, and polytopes, and will begin to play with sets with the Combinatorial Polynomial Hirsch Conjecture properties given. Our goal is to gain a better understanding of how the sets behave at lower dimensions or n-values, with the hope that we can generalize to higher dimensions or more complex examples. In addition, we look to understand which sets can correspond to polytopes and which do not, as a long term goal to attempt to draw equivalency between the conjectures.

Recall that the Combinatorial Polynomial Hirsch Conjecture states:

Conjecture 4.1.1. Combinatorial Polynomial Hirsch Conjecture Consider t non-empty families of subsets $F_1, ..., F_t$ of $\{1, ..., n\}$ that are disjoint (i.e. no set S can belong to two of the families F_i, F_j). Suppose that for every $i < j < k$ and every $S \in F_i$ and $T \in F_k$, then there exists $R \in F_j$ such that $S \cap T \subset R$. Let $f(n)$ be the largest value of t for which this is possible. Then $f(n)$ is of polynomial size in n.

Kalai (2010a)

Example. The family of sets: $\{\}, \{1\}, \{12\}, \{123\}, \{23\}, \{3\}$ is a valid family according to the connectivity property above.

The family of sets: $\{\}, \{12\}, \{1, 2\}, \{123\}, \{3\}$ is *not* a valid family, here we see that $\{12\} \cap \{123\} = \{12\}$, which is not contained in $\{1, 2\}$.

The family of sets $\{\}, \{1\}, \{1, 2\}, \{23\}, \{2, 3\}$ is *not* a valid family, as the subset $\{2\}$ explicitly falls in both $\{1, 2\}$ and $\{2, 3\}$.

We look to seek bounds on $f(n)$, both upper and lower. Trivially $f(n) \leq 2^n$ since there are 2^n subsets of S_n. We can also show that in general, $f(n) \geq 2n$, by the following construction: $\{\}, \{1\}, \{12\}, \{2\}, \{23\}, ...,$ $\{(n-1)n\}, \{n\}$ Kalai (2010a).

An alternative construction is: $\{\}, \{1\}, \{12\}, ...\ \{12...(n-1)n\}, \{2...(n-1)n\}, ...\ \{(n-1)n\}, \{n\}$. Note that if the set $\{12...(n-1)n\}$ is contained in a family (and we are working with n-sets), the rest of the family *must* be of the above form, although other arbitrary subsets may be thrown in the collections if desired, and some of the other families may be missing, although this would render the family non-maximal (if maximality was an option).

4.2 f(n) for Very Small n

In fact, for $n \leq 4$, it is true that $f(n) = 2n$!

Theorem 4.2.1. For $n \leq 4$, $f(n) = 2n$.

Kalai (2010a)

Proof. First note that if a sequence of families contains $\{12\ldots n\}$, then it contains an ascending chain to the left of this set and a descending chain to the right, and thus has length at most $2n$, as we see in the above example.

If $12\ldots n$ is not used and two subsets A,B of cardinality $n-1$ appear in families F_i, F_j then $|i-j|\ eq2$ - this is because only one set contains the intersection of A and B, that intersection set itself.

With these two properties we prove $f(3) = 6$ and $f(4) = 8$ as follows:

For $f(3)$, suppose to the contrary that $f(3) > 6$ - then the maximal family cannot contain 123 and thus $f(3) = 7$, as there are only 7 subsets of 123 remaining. However note that the three subsets 12, 23, and 13 cannot all fall in different families, since this would violate the connectivity property, no matter how we order them, a contradiction.

For $f(4)$, as before, we can assume no F_i contains 1234, as this set would have length 8. We do a case study according to how many of the triplets abc are used. That is, how many of the $I(abc)$'s are non-empty, where $I(x)$ is a restriction of subset x upon the family of sets (note that the restriction will return connected sets due to the connectivity property):

If three or four $I(abc)$ are non-empty, then they are confined to an interval of length 3 by the second property above. It follows that the $I(ab)$ are confined in an interval of length 5 (because any of them is contained in one of the used abc's), and the $I(a)$ are confined in an interval of length 7. We are done because $I(0)$ contains at most one more element, and therefore the family has 8 collections of sets.

The second case is when exactly two $I(abc)$ are non-empty, say $I(123)$ and $I(124)$ without loss of generality. If their values differ by 2 (which is maximum possible), then $I(13)$ and $I(23)$ are confined in an interval E of length 3, $I(14)$ and $I(24)$ are confined in an interval F of length 3, and F intersects E in only one point (between $I(123)$ and $I(124)$). Now $I(34)$ can be empty, in which case the $I(ab)$ are confined in an interval $E \cup F$ of length 5, or $I(34)$ is a singleton and we only have to show that it is in that same interval.

But if it where, e.g., on the right of this interval, then $I(3)$ would have at least 2 points not lying in any of $I(13), I(23), I(123)$ nor $I(34)$, a contradiction. Then in this case, the maximum length of the collection is 8.

If $I(123)$ and $I(124)$ are adjacent or equal, then all of $I(ab)$ where ab is not 34 are confined in a interval G of length 4, which is a union of two intervals of length 3, the first one containing $I(13)$ and $I(23)$, the second one $I(14)$ and $I(24)$. A similar argument than above proves that $I(34)$ must be adjacent to G. This case is suboptimal compared to the above case, so we may disregard it.

The third case is when exactly one of the $I(abc)$ is not empty, $I(123)$ say. Then $I(12), I(23)$ and $I(13)$ are confined in a interval E of length 3 around $I(123)$, and $I(14)$, $I(24)$, $I(34)$ are either empty or singletons. Looking at $I(1)$ shows that (if not empty) $I(14)$ is in the 2-neighborhood of E. But looking at $I(4)$ shows that all non-empty $I(a4)$ are at distance at most 2, so that all $I(ab)$ are confined in some interval of length 5. Some of them could be empty, but in any case it is easily checked that $I(0)$ is confined in an interval of length at most 8.

Kalai (2010a)

For the fourth case, assume that all $I(abc)$ are empty but some $I(ab)$ are not. Note that $I(a)$ can only have length at most 4, and that it must intersect with all other $I(b)$ in that span. If $I(a)$ has length 4 then the maximum size of the set is 8. Additionally if $I(a)$ has length 3 and this is maximum for all elements, it will not intersect with some $I(b)$, rendering $I(b)$ to length at most 2. Additionally the $I(c)$ potentially connecting $I(b)$ may only have length up to 3 since it cannot use element d - $I(d)$ must also have length 2. The situation described here is exactly the construction described to show $f(4) \geq 8$, above.

Last, consider the case when all $I(ab)$ are empty. Then we have at most 4 points covered by the $I(a)$, and at most 5 by $I(0)$, and we are (finally) done. □

One last quick lemma which gives us a framework for recursively trying to build up $(n+1)$-families from n-families is as follows:

Theorem 4.2.2. If $f(n) = k$, $f(n+1) \geq k+2$.

Proof. We prove so with an explicit construction. The collection, if it is maximal, either begins or ends with an empty set. Let the non-empty tail set be of the form $\{a_1, \ldots, a_k\}$. Then append the following two sets on:

$\{a_1 \cap \{n+1\}, \ldots, a_k \cap \{n+1\}\}$, $\{n+1\}$. This set is not guaranteed to be maximal but it has length 2 longer. □

In initial constructions of the above lemma, it was considered that perhaps the last set is a singleton of the form $\{a\}$. Though no counter-examples have been discovered, no clear proof of this has been realized as of yet.

Conjecture 4.2.1. Every maximal set begins on an empty collection and ends on a singleton set of the form $\{a\}$, or vice versa.

Until recently a proof idea existed, but one case persisted to exist - the case where we end on a set of the form $\{ij\}$ but have a set in the middle including the subset $\{i\}$ and another of the form $\{j\}$. At present, it is not believed that such a maximal set could exist however. A proof of this conjecture would be extremely helpful, as this would greatly simplify the process of finding maximal sets.

While it may seem intuitive that a maximal set should begin and end on a singleton (excluding the null set), observe the following set, which helps build up a 5-set demonstrating $f(5) \geq 11$:

Example. $\{\}, \{12\}, \{1,2\}, \{13,24\}, \{14,23\}, \{3,4\}, \{34\}$

This set is one away from maximality, and additionally has the interesting property that any $I(ab)$ has length 1. Hence, it is impossible to insert any subset of size 3 into a new family.

A last quick bound for $f(n)$ is provided on polymath, this time a recursive upper bound on $f(n)$.

Theorem 4.2.3. $f(n) \leq f(n-1) + f(\lfloor n/2 \rfloor) + f(\lfloor (n-1)/2 \rfloor) - 1$

Kalai (2010a)

The proof of this follows similarly to the proof that $f(5) < 13$, which will be presented in the next chapter.

Chapter 5

f(5): A Challenging Task

While $f(3)$ and $f(4)$ were *relatively* easy to find and prove, $f(5)$ is more challenging to harness. We note this is the first instance where $f(n) \neq 2n$.

Theorem 5.0.1. $f(5) = 11$.

Two constructions from polymath are as follows:

- $\{\}, \{1\}, \{15\}, \{14, 5\}, \{12, 35, 4\}, \{13, 25, 45\}, \{245, 3\},$
  $\{24, 34\}, \{234\}, \{23\}, \{2\}$
- $\{\}, \{1\}, \{12\}, \{125\}, \{15, 25\}, \{135, 245\}, \{145, 235\},$
  $\{35, 45\}, \{345\}, \{34\}, \{4\}$

Note that in the previous chapter, we witnessed the example of a close to maximal 4-family, $\{\}, \{12\}, \{1, 2\}, \{13, 24\}, \{14, 23\}, \{3, 4\}, \{34\}$. Note that the second set was built up from this set by removing the first collection, then adding a 5 to each subset, then completed by extending the two ends naturally. In the future, this method could provide a framework for recursively building maximal sets.

Proving $f(5) < 13$ is fairly straightforward.

Proof. ($f(5) < 13$) Suppose to the contrary that $f(5) = 13$. Then there are 12 non-empty sets, without loss of generality say $S_0 = \{\}$. Additionally S_1, S_2, S_3 must contain 3 elements, since otherwise if they contain two they contain some isomorphic copy of $\{1\}, \{12\}, \{2\}$ which implies the other 9 sets must contain only 4 elements, a contradiction since $f(4) = 8$. The same argument applies for S_{10}, S_{11}, S_{12}. Then there is some element i that spans sets S_3 through S_{10}. Restricting those sets to the other four elements implies there is a 4-set collection of length 8. However, since the first or last element of a maximal 4-set is empty, this implies S_3 or S_{10} is the set $\{i\}$, which brings us back to the first case, a contradiction. □

Similarly we can show for any odd $n = 2k+1$, $f(2k+1) < f(2k)+2(f(k)-1)$, the proof follows the same steps as above generalized to k.

The next proof that $f(5) < 12$ comes from Polymath, so it has not been properly peer reviewed and may be flawed. However it appears to be valid. Unfortunately it is proven entirely by casework and is difficult for one to verify.

Proof. ($f(5) < 12$) Suppose to the contrary that $f(5) = 12$.

Suppose we have a counterexample that under restriction to sets containing the element 5 contains a set of length 7 which does not contain the set containing only the element 5 but contains a set which under restriction contains all 4 elements. Then we know that the original set must contain a set which contains all the elements and hence has length 10. So we can eliminate this case.

For the next case assume that in the restriction to sets containing the the element 5 we don't have a set with four elements and the element 5, we don't have a set containing the element 5 only, and we have three sets containing the element 5 and three other elements. We note 3 or more sets of three elements and the element 5 contains all combinations of two elements and the element 5.

Now the first family and the second family must contain a set containing at least a pair of elements. If not, there are only single elements in the first two elements. Then one element will only appear in one single set which cause the entire case to have at most 9 families and we are done. The last and the second to last family must also contain a set containing a pair of elements by similar reasoning.

Since all pairs are contained in the three sets containing three or more elements and the element 5, the pairs mentioned at the second and first families and those at the other end must be in the sets of three and the element 5. This means that the third and third to last families must contain sets containing three or more elements. Now if either of these families doesn't contain the element 5 then they will contain a common element and we continue this proof with that element replacing the element 5. If it is a previous case we use the previous proof and if it is one the cases to come we will deal with it then.

Next we note that the three sets of three elements and the element 5 must lie within three consecutive positions or else the extremal sets of three elements must contain a common pair which must appear four times which is not possible. Now all the sets of two elements and the element 5 which are contained in the sets of three elements and the element 5, which as we have noted are all the sets of two elements, and the element 5 must appear in the positions containing the the three element sets and the element 5 or the two other positions adjacent to these which is a total of five positions. But we have two sets of three elements at both ends of the sequence which correspond to a set of two elements and the element 5 which lie on a span of 7 elements and we have a contradiction and we are done.

For the next case assume that in the restriction to sets containing the the element 5 we don't have a set with four elements and the element 5 and we don't have a set containing the element 5 only and we have two sets containing the element 5 and three other elements. We note 2 sets of three elements and the element 5 contains all combinations of two elements and the element 5 except one pair. In this case we will assume that this pair of elements together with the element 5 is present.

Now the first family and the second family must contain a set containing at least a pair of elements. If not there are only single elements in the first two elements. Then one element will only appear in one single set which cause the entire case to have at most 9 families and we are done. The last and the second to last family must also contain a set containing a pair of elements by similar reasoning.

Since all pairs are contained in the 2 sets containing three or more elements and the element 5, or the set consisting of the remaining pair and the element 5, the pairs mentioned at the second and first families and those at the other end must be in the sets of three and the element 5. And this means that the third and third to last families must contain sets containing three or more elements. Now if either of these families doesn't contain the element 5 then they will contain a common element and then we continue this proof with that element replacing the element 5. If it is a previous case we use the previous proof and if it one the cases to come we will deal with it then.

Next we note that the two sets of three elements and the element 5 must lie within three consecutive positions or else the extremal sets of three elements must contain a common pair which must appear four times which is not possible. Now all the sets of two elements and the element 5 which are contained in the sets of three elements and the element 5 which as we have noted are all the sets of two elements except 1 and the element 5 must appear in the positions containing the the three element sets and the element 5 or the two other positions adjacent to these which is a total of five positions. But we have two sets of three elements at both ends of the sequence which means that we have the set of two elements not in the two triples together with the element 5 at one end of the sequence.

Now at the other end of the sequence there is a set of two or three elements together with the element 5. If there is a set of three elements there is one element in common with besides 5 in one set for each of the 7 elements and we get a contradiction.

So there must be a two element set plus the element 5 at both ends and at one end there must be the two elements in the intersection of the two three element sets plus the element 5 and at the other the two elements in neither set of 3 plus the element 5.

Then the two sets of three elements plus the element 5 must lie in the two spaces following the set of two elements plus the element 5 as there is no remaining element to fill the space between them.

Then the remaining sets of two elements plus the element 5 can only go in the next consecutive slot.

This means that the two slots preceding the set of two elements not in either triple plus the element 5 can only be single elements plus the element 5. But this means that the elements in these sets are disjoint which means that there can be no common elements on either side but either set of three elements plus the element 5 has an element besides the element 5 in common with the set of two elements not in either set of three elements. We have a contradiction and we are done.

□

Kalai (2010a)

As we can see, even demonstrating that $f(5) = 11$ is an arduous task. It is safe to expect even more difficulty in verifying $f(n)$ for larger n. While computing small $f(n)$ may have merit, perhaps instead of finding an explicit formula it is wiser to look for ways to tighen upper bounds.

Chapter 6

f(6): A Potential Start

Considerable work has been put into finding the value of $f(6)$, which has proven challenging. From the previous section, there are a few things we can conclude immediately:

- $f(6) \geq 13$, since $f(n+1) >= f(n) + 2$.
- $f(6) \leq 20$ by our recursive bound.

Unfortunately, this leaves us with a lot of margin of error. One observation however (which was previously mentioned) is that one of the maximal size $n = 5$ collections can be considered as a recursive build from a smaller $n = 4$ set.

- $\{\}, \{1\}, \{12\}, \{125\}, \{15, 25\}, \{135, 245\}, \{145, 235\}, \{35, 45\}, \{345\}, \{34\}, \{4\}$

which is a maximal set, when restricted to to $\{1, 2, 3, 4\}$ produces a very nice looking set:

- $\{\}, \{1\}, \{12\}, \{12\}, \{1, 2\}, \{13, 24\}, \{14, 23\}, \{3, 4\}, \{34\}, \{34\}, \{4\}$.

This set was built up from taking the non-maximal set with the interesting property that it both starts and ends (excluding the empty set) on sets of size 2.

- $\{12\}, \{1, 2\}, \{13, 24\}, \{14, 23\}, \{3, 4\}, \{34\}$

(one short of maximality, since the empty set is not included), and adding a 5 to each subset of S, then finished by competing the trail of non-5 sets. We conjecture that a similar process could be used to create larger 6-sets.

Note that because the above non-maximal 4-set begins with a set of size 2, this enables 3 sets to be inserted before it when generating the 5-set, and similarly, the ending 2-set enables 2 additional sets to be inserted after. In general, if we have some length k n-set which begins with a set of size a_0 and ends with one of size a_1, then we can generate a length $k + a_0 + a_1 + 1$ $(n + 1)$-set. If a_0 and a_1 are 0 and 1 respectively, then we see this is the same as the recursive bound proven earlier.

Now we face a new question - what is $f(5)$ when we demand that the starting and/or ending set is of size 2? Though we lack evidence, it is believed that if we restrict both the start and end to be of size 2, $f(5) = 8$ which then gives us a construction of size 13 - unhelpful for recursively generating maximal sets. However, a set of size 9 of this form could exist - it simply has not been discovered yet.

A few sets generated are:

- $\{12\}, \{1,2\}, \{14,25\}, \{143,253\},$
  $\{153,243\}, \{15,24\}, \{4,5\}, \{45\}$
- $\{12\}, \{1,2\}, \{13,23\}, \{143,253\},$
  $\{153,243\}, \{45,34\}, \{4,5\}, \{45\}$

which then correspond to 6-sets of size 13:

- $\{\}, \{1\}, \{12\}\{126\}, \{16,26\}, \{146,256\},$
  $\{1436,2536\}, \{1536,2436\}, \{156,246\},$
  $\{46,56\}, \{456\}, \{45\}, \{4\}$
- $\{\}, \{1\}, \{12\}\{126\}, \{16,26\}, \{136,236\},$
  $\{1436,2536\}, \{1536,2436\}, \{456,346\},$
  $\{46,56\}, \{456\}, \{45\}, \{4\}$

We also have our set generated from the proof that $f(n+1) \geq f(n) + 2$:

- $\{\}, \{1\}, \{12\}, \{125\}, \{15,25\}, \{135,245\},$
  $\{145,235\}, \{35,45\}, \{345\}, \{34\}, \{4\}, \{46\}, \{6\}$

Generating a 6-set of size greater than 13 has not occurred as of yet. However, the implication that $f(6) = 13$ is strange to believe, as this implies that the rate of growth of f actually slows at points. Given the complex constraints of these sets, it is entirely plausible that this occurs. These sets behave in unpredictable ways, and it is certainly possible that some factors, such as the parity of n, could affect the growth rate of $f(n)$.

On the other hand, a construction that shows that $f(7) \geq f(6) + 3$ is more plausible, by using a similar construction going from $f(4)$ to $f(5)$. The convenience of working when n is even plays nicer - see the non-maximal set from earlier. We have much more symmetry, which plays a key role in creatinng these recursive sets.

Perhaps one direction to go from here is to show that $f(n+1) - f(n) \geq f(n) - f(n-1)$, to confirm that $f(6) = 13$ is false. However, a full proof cementing the value of $f(6)$ will be quite difficult, especially considering the fact that it will be considerably harder than proving $f(5) = 11$.

Chapter 7

Where Do We Go From Here?

A first step would like to be able to identify exactly when a subset partition graph is some form of d-dimensional polytope. Perhaps this is the easiest question to study.

Kim brings up a few open questions which are of interest. These questions are certain cases of the Combinatorial Polynomial Hirsch Conjecture, and while they may not immediately solve the conjecture, they are still steps towards a full solution. Kim (2012)

- **Problem 1**: Prove a non-trivial upper bound on the diameters of subset partition graphs with the strong adjacency and endpoint-count conditions.
- **Problem 2**: Construct a family of subset partition graphs with super-linear diameter satisfying all of the main properties.

Subset partition graphs satisfying strong adjacency and endpoint-count conditions have superlinear diameter (as noted in Kim's paper), which is evidence against the original Linear Hirsch conjecture. This motivates question 2, and Kim provides a possible approach for constructing a class of subset partition graphs with all three properties, which provides a method of disproving the Linear Hirsch Conjecture. Kim (2012)

- Start with a family of subset partition graphs satisfying at least the endpoint-count property with superlinear diameter growth, such as the family resulting from Theorem 4.4 or 4.5 (see Kim's paper).
- Gain the other main properties that do not yet hold with contraction and edge addition operations.
- If the resulting family of graphs still has superlinear diameter, realize the sequence of graphs as a sequence of polytopes.

Other mathematicians have worked towards this direction as well, and it is quite clear that this construction is quite difficult. Kalai (2010b)

To go about looking for answers in one of these questions, the first step is to further examine the various properties outlined in chapter 3. When is there a one to one correspondence between subset partition graphs and d-dimensional regular polytopes, for example?

Lastly, a question that deals with the correlation between the two conjectures is as follows:

- When does a collection of subsets of a set correlate to a polytope in d dimensions?

If we restrict ourselves to regular polytopes of d dimensions, then we already have been provided a framework for determining when sets correspond to polytopes and vice versa. However, if we remove regularity, then the question becomes much more difficult, as we could have subsets of multiple sizes corresponding to different vertices, which could then correspond to the number of facets that vertex touches. Though it is true that the Combinatorial Polynomial Hirsch Conjecture implies the Hirsch Conjecture, finding a stronger statement could be fruitful.

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