Incorporating Pólya's Problem Solving Method in Remedial Math

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Synopsis

György Pólya's problem solving method has influenced generations of mathematicians and non-mathematicians alike. Though almost all math teachers have come across Pólya's problem solving method, his ideas are not regularly implemented in the classroom. Few studies have examined the effectiveness of his approach in teaching remedial math. In this article we revisit this once well-known teaching method and show how it can be used in basic skills math classes to ease student fears of math, and potentially change their common misconceptions of the subject.

Introduction

Ask a student the fastest way to get to school, and you'll be sure to get a detailed, thought out, and accurate answer. Ask the same student to find a common denominator, and all you may get is a shrug. To answer either question, students must engage in problem solving. How can they be taught that problem solving in math is the same as any other kind of problem solving?

Community colleges across the country face the daunting task of teaching remedial math to students with a history of math troubles [5, 6, 19]. It's a thorny situation for math instructors like myself: The very students who most need to learn basic math are also those who have the greatest difficulty learning it. Although there's been a deluge of research and projects that experiment with new teaching methods, most have been met with limited success. The statistics are disheartening, especially for community colleges.

In 2008 nearly one-third of all incoming freshmen nationwide required some form of remedial instruction in reading, writing or math. But among community college enrollees, 44 percent required remedial instruction in mathematics. And, unfortunately, less than a third of students placed into "basic skills" math courses ever move beyond them [2].

The students that walk into such a course know how to think. They make logical formulations and solve problems all day, every day. But they have little experience applying that everyday reasoning—that they clearly use outside the classroom—to math problems. A reflexive phobia of math has trained them to think of "common sense" reasoning and the kind of reasoning used in math as separate. Any instructor in such a class has seen what I've seen again and again: When faced with math, students abandon basic reasoning and prefer to drop numbers into formulas.

It's worth pointing out that this preference for rules and formulas over common sense likely comes from teaching methods used in K-12 education. It's been studied and well documented in the literature (see for instance [7, 18]) that U.S. teaching methods emphasize step-by-step technique and rote memorization. In general, instructors don't reinforce the basic concepts behind the methods. Nor do they show students how to connect what understanding they might already have to the procedures they are taught. Countries with higher performing students tend to spend a greater amount of time emphasizing the basic concepts.

The most basic concept of all is that math is merely a problem solving technique. If students can learn to see math problems as just a formal, codified version of any other kind of problem, perhaps their phobias will disappear. But how can a math instructor get students to make this connection? To answer that question I turned to a Hungarian mathematician who was himself obsessed with teaching problem solving.

Problem-Solving: György Pólya's Approach

Finding the answer is not the answer. Most students just want to find the right answer, never mind how, and wash their hands of any further discussion. They are satisfied to perform a rote set of calculations as long as they wind up with the correct number. They're like the individual in philosopher John Searle's "Chinese Room" thought experiment (see [17]). Given the right instructions, a person in a box could change Chinese symbols coming in on

one side to English ones to be sent out the other, all without understanding Chinese at all. Although the right translation might come out of the box, the individual inside has no real knowledge of the language.

A student who solves an equation without understanding what the equation is or how to construct it has not truly learned the language of mathematics. That student is far less likely, once the course is over, to be able to solve real-life problems that need math.

So just how do we teach math problem solving as any-old-problem solving? György Pólya, "[t]he father of the modern focus on problem solving in mathematics education" [11] had much to say on the matter.

Pólya was a Hungarian mathematician who taught at the Swiss Federal Institute of Technology Zurich and later at Stanford. He was a serious scholar in many fields of mathematics, such as number theory, combinatorics, and numerical analysis, and is responsible for many well-known conjectures and theorems. He spent much of his later years writing about basic problem solving methods—ones that applied to math problems as well as any other.

However important his contributions to mathematics, his most famous book is *How to Solve It*. In it he suggests that problems should be solved in four steps [12, pages *xvi-xvii*]: 1) Understand the problem. You must know what is known, what is not known, and the ultimate goal. 2) Devise a plan. See how what you know connects to what you are trying to determine. If there are no connections, find other problems that fill in the gaps. You should understand precisely how you will proceed. 3) Carry out the plan. Perform the mechanics of solving and check each step. Do you have a way of proving that your solution is correct? 4) Look back. Be sure to examine your solution to learn from what you have accomplished.

This method has influenced mathematicians and math educators alike at all levels. These four stages of problem solving are, in general, intuitive enough. But having them delineated as Pólya does allows educators to steer student thoughts through what should be a natural process. (In fact the method has been applied to a variety of contexts, even on the pages of this *Journal*. See for instance [20] where VanHattum describes how the method shaped her own math solving strategies in and out of the classroom. In another paper of 2011 [4], Susan D'Agostino urges teachers of all fields to apply Pólya's method to problem solving of any kind—and says it's a "crying shame" that his name and method are not more widely known.)

Most people follow these steps naturally, when faced with a typical daily problem. Drop and shatter a glass of water, and you instantly know that the problem is the glass on the floor. What you need to find out is exactly where all the shards went. The goal is to remove them so that no one gets cut. Then you move on to making a plan: Everyone puts shoes on and someone locates a broom or vacuum cleaner. Executing the plan involves sweeping or vacuuming the glass. Finally, you look over the floor to be sure that the glass is gone.

That's pretty much what everyone does with a case of broken glass. But when it comes to mathematics, many people become befuddled and do not follow a similar course of action. Anyone who has taught a basic skills math course has seen that, when it comes to solving mathematical problems, the majority of students do not follow Pólya's procedure at all. Instead, students jump directly to the third step, "carrying out a plan," without first "understanding the problem" or "devising a plan." They take whatever numbers they have been handed and attempt to put them through the mechanics of whatever equation or method has been taught last. And often, because they are blindly trying to plug numbers into a random formula or arithmetic procedure, they fail to solve the problem and end up feeling completely frustrated, helpless, and discouraged from trying the problem again. The failure reinforces the idea that mathematics is mysterious, intangible, and essentially beyond their abilities. Occasionally, the right numbers are put into the right formula and the problem is, more or less "accidentally" solved. Even though they might be pleased to have come up with the right answer, the occurrence only further ingrains the notion that math is strangely abstract—and that a true understanding is beyond their own grasp.

If not checked, an individual student's growing feeling that math is daunting and beyond reach is harmful, with long-term, life-affecting consequences for the student involved. Solving a math problem is, in essence, not unlike solving any other problem. The numbers and symbols are only the tools—like the broom or the vacuum cleaner—for the specific task. They allow us to solve problems that are really just elaborate problems of counting. If our students can come to understand how solving a math problem requires following each step of Pólya's problem-solving method in sequence, they may come to see that math, as a whole, is not impossible to master, and, ultimately, they will be able to use what they have learned long after they leave the classroom.

At LaGuardia Community College, where I teach, there has been a major push to link math to everyday experience and everyday reasoning. The college created *Project Quantum Leap* (PQL) in 2007 to shape lessons with compelling issues and timely, real-world examples [14]. Funded by a U.S. Department of Education grant, this program uses the approach developed by SENCER, a National Science Foundation initiative that aims to improve education in engineering and mathematics by "connecting learning to critical civic questions" [1]. The most recent results, from 2010, demonstrate that students in PQL courses have a greater passing rate—by 20 percent—than those in conventional courses [3].

My experience with PQL showed me that students were more apt to use their own common sense when the problems before them were engaging and real. They turned more readily to their own reasoning power instead of half-remembered procedural rules from long ago. The following lessons show my attempts to teach with the topical, using Pólya's method.

Lesson One: Guiding Students to Make the Connection Between Abstract and Concrete.

The phrase "least common multiple" is, admittedly, none too warm and fuzzy. Utter the words and many students erect an instant barrier. But its meaning is simple. In the lesson below I used Pólya's four steps to draw students to an understanding of the concept. The example is from an *Introduction to Algebra* course (LaGuardia offers two sequential, remedial math courses: *Introduction to Algebra* and *Beginning Algebra*). The focus of the course is on basic arithmetic, integers, fractions, and decimals.

The students were first presented with the following setup:

> Two different bus lines stop at the Union Square bus stop. The M14D bus comes every 15 minutes (at 8:00 a.m., 8:15 a.m. and 8:30 a.m., etc). The M14A bus comes every 12 minutes (at 8:00 a.m., 8:12 a.m., and 8:24 a.m., etc).

> The schedules of both buses continue at the same rate throughout the day. What is the first time after 8:00 a.m. that they are scheduled to arrive together at the Union Square bus stop?

I asked students to first read the question carefully and to be sure they understand what it was asking (step 1). The phrase "common multiples" was

not mentioned at all. I left the idea of numbers with multiples that coincide for later. Instead, I directed students to think of actual objects moving at different rates, with arrivals that coincide.

To answer the question, the students simply wrote down the actual bus schedules and figured out when the buses first stopped together (step 2). Here is one such completed chart:

M14D	8:00	8:15	8:30	8:45	9:00	9:15	9:30
M14A	8:00	8:12	8:24	8:36	8:48	9:00	9:12

It's easy to see that 9:00 is the earliest time that the buses will arrive together at the bus stop after 8:00 (step 3). Few students had any trouble writing such a chart. They found the matching time and came up with the correct answer without any trouble.

In my previous attempts to teach the same concept I had started with the abstract idea—and terminology—of finding the common multiples of 12 and 15. The students did find the answer (60) without much trouble. But they subsequently had great difficulty applying the idea to real world problems such as the one above. By starting with the concrete before moving to the abstract, they more easily made the connection between the two on their own.

After they found the right answer I asked the class to look back at the original question and think again about its meaning (step 4). Instructors often underemphasize this step, but it's crucial for making the link between the abstract and the concrete. And there's research to back me up, at least for middle school students (see [8]).

Having built up their confidence, I asked them to consider the following question:

> Is there a *faster* way to find when the two buses will reach the bus stop at the same time?

I then guided them through the following reasoning, allowing them to answer on their own as much as possible:

> Since the M14D bus comes every 15 minutes, the intervals between arrival times for the M14D bus will be **multiples of 15**. Similarly, the intervals for the M14A will be **multiples of 12**.

> Thus, the buses will arrive **together** at an interval that is a **common multiple** of both 12 and 15!
>
> Since the **least common multiple** of 12 and 15 is 60, they will first meet again after 60 minutes (i.e., at 9:00 a.m.).

Essentially, this is, or was, a least common multiple question in disguise. When the phrase was finally introduced, its meaning was perfectly clear. Though Pólya's method might seem the obvious, unconscious, path for solving everyday problems, when it comes to solving math problems, guiding basic skills students through each step clearly increased their understanding.

Lesson Two, The Power of Math in the Real World

Another way to motivate students is to give them a car. Not a real one, I'm afraid (though I'm sure such a reward would help them put their noses to the grindstone). I asked students to choose between a Mercedes and a Honda as a way of securing their interest, and showing them the power of math—all in keeping with Pólya's thinking.

Pólya felt that the problems that are offered to students should have merit beyond the math concept. The problems should be "not merely routine problems but problems requiring some degree of independence, judgment, originality, creativity" [13, page *xi*]. What better way to engage students than to offer them a little fantasy auto acquisition?

In this lesson, I wanted to introduce percentages with a simple concrete example. I also wanted that example to show students how basic skills are needed to make purchasing decisions—and how those decisions might even affect our planet.

In keeping with Pólya's ideas (and as part of PQL), I designed a lesson that compared the fuel efficiency of two cars. At the beginning of the class I asked the students which car they would choose if someone offered them a Honda Civic Hybrid or a Mercedes Mercedes-Benz E63. As I expected, most of the students answered that they would take the Mercedes. Then I asked if they were aware of the current price of gas, as well as which car they thought would save them money on gas, and which car they thought would pollute the air less.

Before jumping into the problem solving, I gave them some hypothetical facts: They should assume they would drive 15,000 miles this year; of this,

45% would be city driving and 55% highway driving. Also, they should assume the price of gas would be $3.75 per gallon. Now they had to imagine driving the Honda, which gets 40 miles per gallon in the city and 43 miles per gallon on the highway.

In order to guide students towards figuring out the annual cost of gas for the Civic, I broke the problem down into more fundamental questions:

1. How many gallons of gasoline would you use for highway driving this year?
2. How many gallons of gasoline would you use for city driving this year?
3. How many total gallons of gasoline would you use this year?
4. How much would you spend on gasoline this year?

Then I asked students to answer the same questions but this time with the Mercedes in mind. That meant 13 miles per gallon in the city and 20 on the highway. Finally, I posed this question: How much money would you save if you were to drive a Honda Civic Hybrid instead of a Mercedes-Benz E63 AMG?

Together we discovered that, with the assumptions made above, driving a Mercedes would cost more than $2000 more in gas, annually, than driving the Honda Civic. Some students were surprised by the extent to which making a few calculations could affect both their pocket and the environment. When they discovered the price difference, many started to vocally change their choice. The calculations sparked a discussion about the results. One student commented, "Who wants to use a Mercedes?" Another student remarked, "I guess you wouldn't care if you're rich." Then I asked the students to put the environmental impact of the two cars into consideration. How would the difference in gas usage affect the atmosphere? If everyone who had a Honda Civic switched to a Mercedes, would there be a discernable effect on the environment?

My fundamental belief is that when students understand the solving power of mathematics and see that math problems are not so different from other life problems, including problems that have an impact their lives and the world, they can get beyond the blocks that hold them back. Of course, if the blocks came tumbling down so easily, they probably would not be there in the first place. Even some of the stronger students had difficulty transforming the problem about the Honda Civic versus the Mercedes into

mathematical terms, even though they seemed to grasp the nature of the problem. In particular, the percentages tripped them up.

Regardless, the students were more engaged than in previous classes and understood exactly what it was we were trying to solve. No one jumped straight to the third step of Pólya problem-solving procedure. Seeing the power of percentages on their pocketbooks kept their attention. And the surprising conclusion we reached—that the Civic might be the "better" car—seemed to give students the kind of satisfaction one gets from using the right tool to get a job done. I was encouraged to see that the students were truly engaged in solving a real-world problem. Math was becoming, I would like to think, a tool for their use.

Conclusion and discussion

My experience confirmed the potential of Pólya's method. The four steps help students keep in mind the common sense nature of math and mathematical problem solving. With them, students were able to use the reasoning abilities they already have to leap hurdles they might have previously thought insurmountable.

However effective the teaching strategy, there is no shortcut for immersion and practice. Pólya and his method have faced some criticism. For example, Alan Schoenfeld, professor of cognition and development at the University of California Berkeley argues that different situations may require different strategies [15, 16]. Frank Lester, professor of education at Indiana University, writes that lectures that discuss problem solving are inferior to actual problem solving [9]. Pólya certainly thought that in-class discussion of problem solving should be accompanied by carefully gradated and engaging problems. Still others have found his four steps too general to be useful in any specific context. In their book *Thinking Mathematically* [10], Mason, Burton and Stacey expand his four steps into what they think is a more useful seven.

In my own assessment I found Pólya's four steps effective when combined with enough repetition. Students can be taught to see the usefulness of math in problems that affect them personally as well as problems of great magnitude. But, in order to be able to use those skills on their own, outside of class, they must learn to move from the concrete to the abstract and back again. This requires a link to common sense problem solving, but also requires that long proven method: practice.

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