Claremont Colleges
Scholarship @ Claremont

HMC Senior Theses

HMC Student Scholarship

2017

Sudoku Variants on the Torus

Kira A. Wyld
Harvey Mudd College

Recommended Citation
Wyld, Kira A., "Sudoku Variants on the Torus" (2017). *HMC Senior Theses*. 103.
https://scholarship.claremont.edu/hmc_theses/103

This Open Access Senior Thesis is brought to you for free and open access by the HMC Student Scholarship at Scholarship @ Claremont. It has been accepted for inclusion in HMC Senior Theses by an authorized administrator of Scholarship @ Claremont. For more information, please contact scholarship@cuc.claremont.edu.

Sudoku Variants on the Torus

Kira Wyld

Francis Su, Advisor

Kenji Kozai, Reader

HARVEY MUDD COLLEGE

Department of Mathematics

May, 2017

Copyright © 2017 Kira Wyld.

The author grants Harvey Mudd College and the Claremont Colleges Library the nonexclusive right to make this work available for noncommercial, educational purposes, provided that this copyright statement appears on the reproduced materials and notice is given that the copying is by permission of the author. To disseminate otherwise or to republish requires written permission from the author.

Abstract

This paper examines the mathematical properties of Sudoku puzzles defined on a Torus. We seek to answer the questions for these variants that have been explored for the traditional Sudoku. We do this process with two such embeddings. The end result of this paper is a deeper mathematical understanding of logic puzzles of this type, as well as a fun new puzzle which could be played.

Contents

List of Figures

Acknowledgments

My thanks to Professor Su, for working with me on this project, and to Professor Kozai for agreeing to be my second reader. Additional thanks, of course, to my mother for her patience, my father for apparently predicting my thesis 12 years ago, all of my friends for their loving support, but especially Jonathan for remembering the word 'diagonal' when no one else was logged onto Facebook.

Chapter 1

Introduction

Pick up any newspaper in the country and flip through it, and eventually you will find the puzzle section. Though this section will likely not catch you up on the happenings of the world, it is a useful way to check one's ability to make logical deductions or know what cultural cues are relevant. A Sudoku puzzle is one such logic puzzle which is typically found here in the paper, as well as in airplane magazines, puzzle books, and numerous other places intended to help brains stay active. In essence, what a Sudoku puzzle asks you to do is to use a set of numbers ('clues') to deduce where the rest of the numbers must be placed. It is a fun activity using numerical symbols to understand how one element can affect another, and a gentle introduction to logic.

Sudoku were first created in 1979 Hayes (2006), but we will later see that they are related to a mathematical tiling question that has been studied since at least the 18th century. Originally named 'Number Place', the puzzles eventually took on the moniker Sudoku, Japanese for 'digit single', indicating that each digit must be placed once and only once (a single time) per row, column, or box. They have since spread across the globe as a fun way to exercise one's brain and logic skills.

My first experience with Sudoku was when I was 9 (the same age as the dimension of the typical Sudoku square!). It was at this time that my father began to figure out coding so that he could generate his own puzzles. Thus, Sudoku became a part of my family and are still something I turn to as a way to relax while checking my brain's capabilities. While I've had people tell me that Sudoku stress them out because they're "too math-y" I always felt the opposite way, that Sudoku puzzles were a fun way to interact with numerical symbols without a chance to make algebra mistakes. It was only recently that I began to look at Sudoku as a mathematical construct, and thus this thesis was born.

Sudoku are fascinating to mathematicians not just because they're a fun logic puzzle, but because of the sometimes puzzling nature of the game itself. Not only is it a useful tool for helping students gain comfort with logic and math, as discussed in the introduction to *Taking Sudoku Seriously* Taalman (2011), but there are interesting questions to be asked about the puzzles themselves. While we understand how to design puzzles, in a sense, it is much harder to find out how many puzzles there are, and what rules the clues must follow. Thus, something that at first seems mathematically trivial is, rather, a complex example of how using mathematics to invent tools doesn't always give us all of the mathematical details of the tool itself.

Many puzzlers have found their familiarity with the Sudoku structure allows them to make certain logic jumps in how they do the puzzles, making finding solutions less of a challenge. As such, variants are also in popular demand, either by increasing the size of the puzzle, modifying an existing rule, or adding a new rule. Such variants allow users to feel some level of comfort with the puzzle, as the rules are familiar, but the individual logic steps change enough to present a challenge. One goal of this thesis is to find interesting variants of Sudoku, motivating topological intuitions.

While Sudoku may be off putting to those who are concerned with their math abilities, they are not necessarily so. The phrase 'topological intuitions' however, is a hefty one, and needs unpacking. One issue with higher mathematics, such as topology, is finding ways to help students visualize the new spaces they'll be working within, which is usually done with a combination of words, graphics, and examples that students may already know. For example, when explaining how to visualize living in a torus, it is common to rely on students understanding of the game Pacman, and how the characters in that game move as comparable to how those living on a torus would move. Placing puzzles on a torus, then, is a similar way to help motivate those interested in Sudoku into gaining insights which could help them with other mathematical topics.

The motivation of this thesis, then, is to use topological concepts to generate a new variant of Sudoku, both as a source of interest in itself, as well as to understand how answering questions about a particular variant can help us answer questions about Sudoku puzzles as a whole.

1.1 Sudoku

In order to proceed, we must mathematically define a Sudoku tiling. To do so, we first define a Latin Squares tiling.

Definition 1.1. *A* ***Latin Square*** *is an n x n square with each square labeled with a number $i \in \{1, \dots n\}$ so that the numbers $1 \dots n$ appear once and only once in each row and column.*

1	2	3
2	3	1
3	1	2

Figure 1.1 A 3x3 Latin Square Tiling

A Sudoku tiling, then, is a Latin Square tiling, but with the further constraint that there are n regions, that also must contain the numbers $1 \dots n$ once and only once. Traditionally, a Sudoku square is on a 9 x 9 grid, so that theseregions are 3 x 3 squares. An example is in Figure 1.2.

While finding any Sudoku tiling is tricky in and of itself, a *puzzle* is traditionally defined as a Sudoku tiling with most of the tiles missing, leaving only the necessary *clues* which can be used to find a unique completed tiling. A Sudoku puzzle, then, implies the existence of at least one tiling, which must be unique to that puzzle. An example is shown in Figure 1.3, and the reader is encouraged to attempt to solve the puzzle.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

Figure 1.2 A 9x9 Sudoku Tiling

			6		4			
6								7
3		4		7		9		2
7		1				6		8
	8						2	
2		6				3		4
8		5		3		7		1
9								3
			9		5			

© sudokuplace.com All rights reserved.

Figure 1.3 A 9x9 Sudoku Puzzle Danburg-Wyld (2005)

1.2 Literature Review and Questions

Jason Rosenhouse and Laura Taalman's book, *Taking Sudoku Seriously* Taalman (2011) is a novel-length exploration of how Sudoku and the mathematics can be used to help students engage in mathematics in a more meaningful way. By taking a 'game', and then using it as an example in mathematical queries and proofs, they hope to engage readers and show them how accessible mathematics can be. They further lay out the mathematical foundations of Sudoku, from the older problem of Latin Squares to the combinatorics, group theory, graph theory, and algebraic elements which are used in the puzzles. The 214 pages not only explore Sudoku, but also use Sudoku as an avenue to exploring deep mathematics.

Some of the questions which the book explores are things such as how many distinct Sudoku tilings are there? What is the minimal number of clues needed to generate a unique tiling? What is the maximal number of clues that can fail to generate a unique tiling? Other questions, such as how one can generate one tiling from another, are also of interest. The book uses mathematical proofs to answer some of these questions, but others are answered by computational power alone.

The concept of putting puzzles and games onto a torus in order to motivate topological understandings came to me from the book, *The Shape of Space* Weeks (1985). The book, which seeks to help readers develop topological intuitions, and become better able to visualize different spaces, uses tic-tac-toe and chess as examples of games which can be easily transferred to a torus via gluing the edges correctly. While Weeks' has further developed this idea, with an entire website of such games and puzzles at http://www.geometrygames.org/TorusGames/ , I found Sudoku to be a meaningful example which had yet to be explored, and so took it upon myself to do so.

1.3 Variants

There are 6,670,903,752,021,072,936,960 different Sudoku puzzles Taalman (2007), which should be more than enough to keep any game enthusiast busy. Unfortunately, many find that using the same logic steps over and over is unsatisfying and instead search out variants of the game to keep themselves occupied. Such variations often involve changing the size of the board, changing constraints, or adding additional ones.

One simple example is to change the shape of the boxes which are being filled. Another variant which brings to mind more topological ideas is to have a constraint requiring the tiling of diagonals as well, using 'wrapping' to make each diagonal the same size. Figures of both are included, and readers are encouraged to attempt both!

My goal is to use topological polygonal presentations in order to find satisfying variants of Sudoku, both to create new and delightful games, as well as to develop intuitions on which properties of a Sudoku board constrain answers to our questions in different ways.

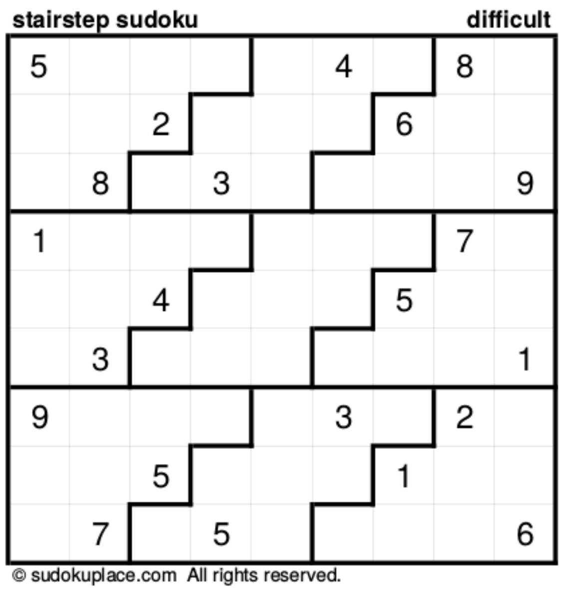


Figure 1.4 A Sudoku Variant With Changed Constraints Danburg-Wyld (2005)

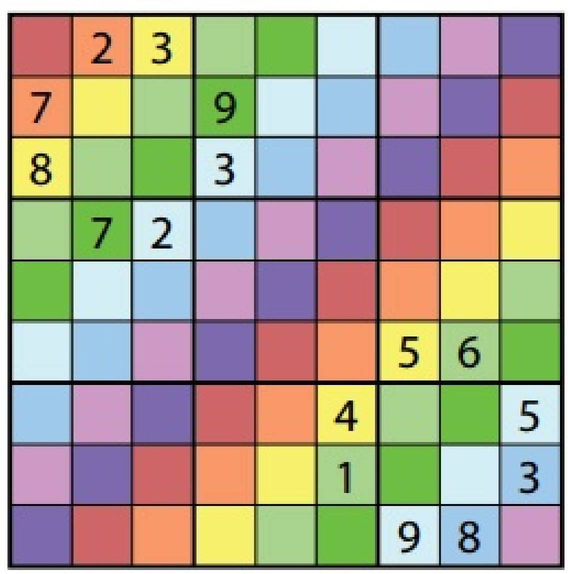


Figure 1.5 A Sudoku Variant With Additional Constraints Taalman (2007)

Chapter 2

Board Structure

2.1 Constructing a Variant

One of the most frequently cited manifolds of topology is the torus, colloquially known as the surface of a doughnut. When we're working with a Sudoku puzzle then, the question of how to embed it in this space can seem tricky, as bending a Sudoku puzzle into a coffee mug seems more annoying than placing it onto a doughnut. However, topological understandings of these spaces can be more easily viewed for puzzles by looking at their polygonal presentations. Since any topological manifold can be represented by a polygon with pairs of edges identified (Su, 2016 Draft), it now seems logical to embed puzzles onto these polygons, which we can more easily visualize, rather than to put puzzles arbitrarily into topological spaces. Since the torus is one of the easier topological manifolds to visualize, regardless of presentation, it seemed sensible to begin by placing puzzles there, such as in Figure 2.1.

If we are to claim that the puzzle is on a torus, however, we must be able to prove that the space it is on is indeed a torus. For example, a square with the proper edge identifications is topologically equivalent to a torus. This is shown in Figure 2.2

When trying to embed a Sudoku puzzle into a torus, I first began with the polygonal presentation discussed above, of the square. However, closer observance shows that doing so doesn't actually change the puzzle in the least, as rows and columns will remain adjacent only to themselves. This representation just shows how the Sudoku can naturally fit into this environment. Torus ideas can be used to help define additional constraints

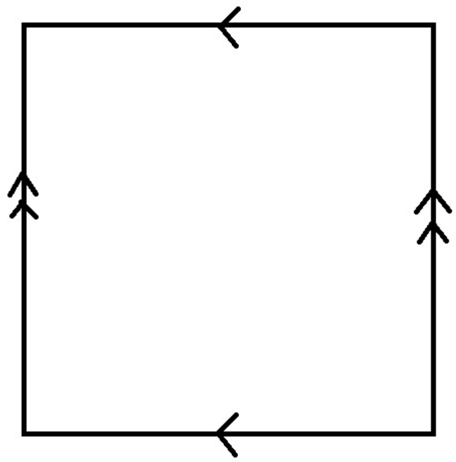


Figure 2.1 A Torus Represented by a Square

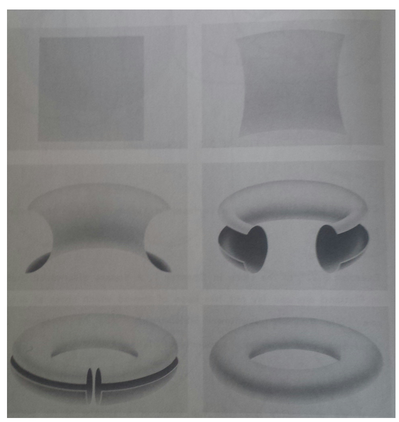


Figure 2.2 A Square Being Deformed into a Torus Weeks (1985)

that may exist to add interest to the puzzle, as seen in Figure 1.5.

In order to create a new variant, I thus looked at a completely different polygonal representation of the torus. I found the hexagonal presentation to be of particular interest, and so explored how we could use this to create a new puzzle. The representation in interest is shown in Figure 2.3, as well as how it may be deformed into a torus. We see the sides opposite each other are marked as having arrows on them that denote where they are to be 'glued' to each other, to complete the torus. This identification is shown later in this paper through color coding.

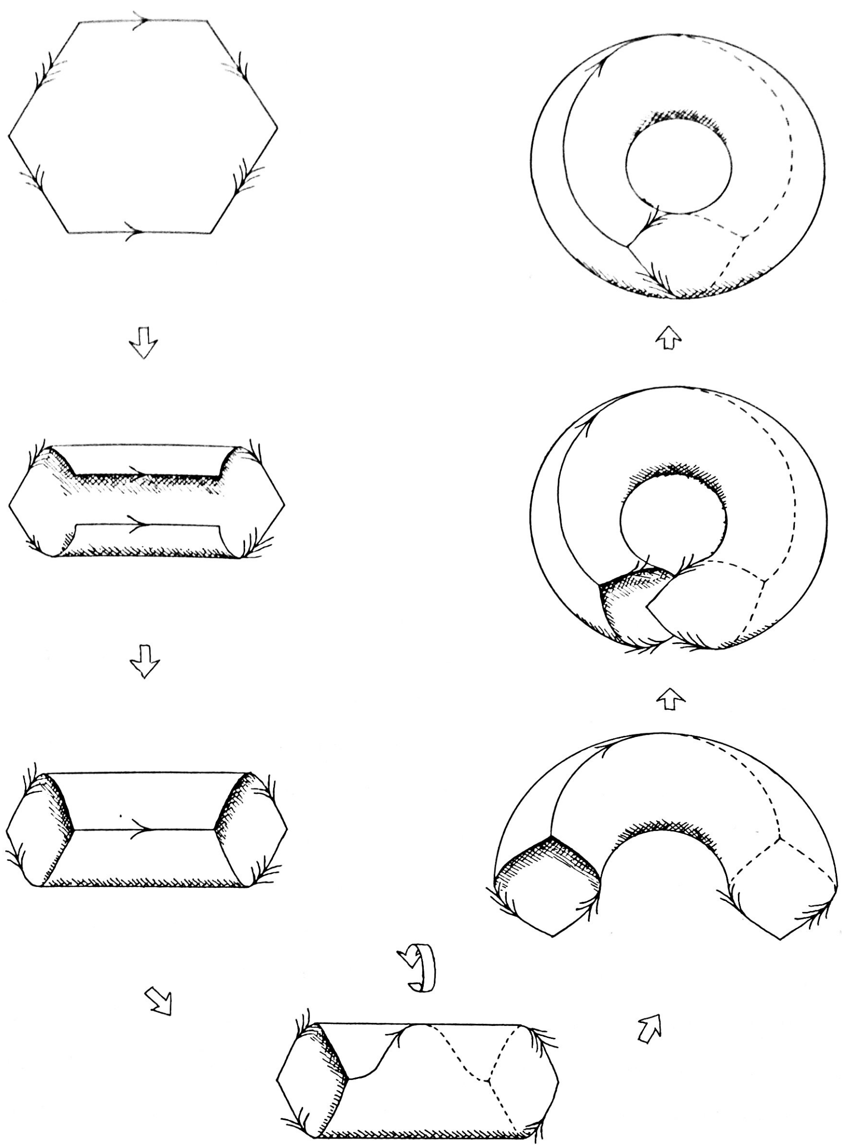


Figure 2.3 A Hexagon Being Glued into a Torus Weeks (1985)

2.2 Making the Hexagon a Sudoku

Now that we have a fun polygonal presentation of a torus, we have to find a way to turn it into a board. To do this, we first divide the hexagon into 6 parts, and then use barycentric subdivisions to make each 'big triangle' have six subtriangles, comparable to a traditional sudoku board. Big triangles are considered neighbors if they are either right next to each other, or across from each other, as understood from the gluing. See Figure 2.4.

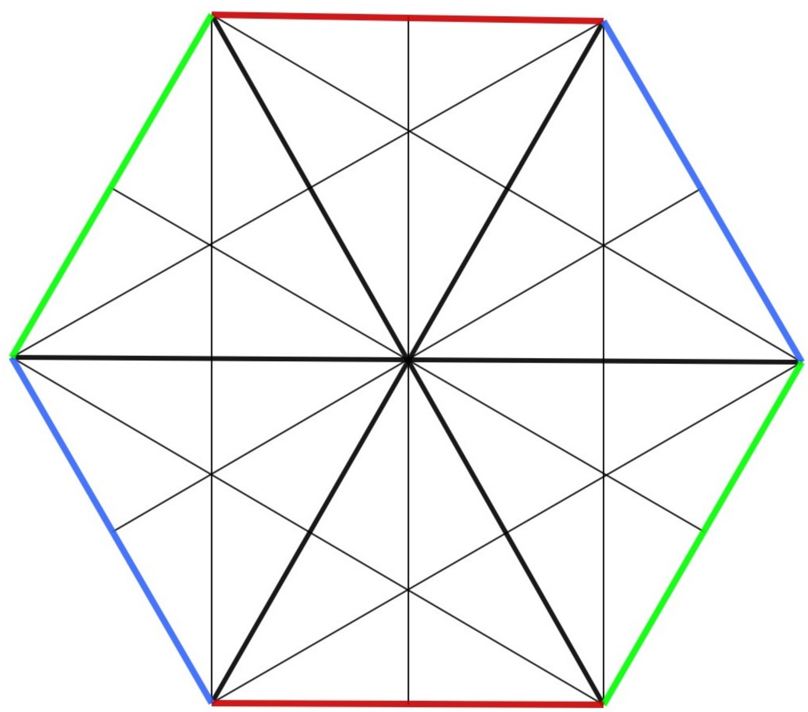


Figure 2.4 A Hexagonal Torus Made into a Sudoku-like Board

2.3 Terminology

To make sense of Figure 2.4, we must determine a set of words which will be used to refer to its various parts throughout this thesis. We do this as follows:

- **Board** will refer to the hexagon which is subdivided as seen above, and is blank.
- **Tiling** will refer to a board which has been completely filled according to the rules of the game.
- **Puzzle** will refer to a partially filled in board, which will produce a unique tiling.
- **Clue** will refer to a number or symbol which has been used to fill in a part of the board.
- **Big Triangle** will refer to one of the six triangles that form the hexagon.
- **Triangle** will refer to a triangle that helps form a big triangle.
- **Band** will refer to a set of twelve triangles which fall within a set of parallel lines.
- **Bar** will refer to one of the lines dividing the board up into a Band.
- **Vertical Band** will refer to the band whose bars are vertical to the bottom of the page.
- **Diagonal Band** will refer to the band whose bars start in the upper left of the hexagon.
- **Anti-Diagonal Band** will refer to the only other type of band there is.
- **Vertex** will refer to a point at which multiple big triangles meet.

Chapter 3

First Variant: Sudodici

3.1 Finding a Solution

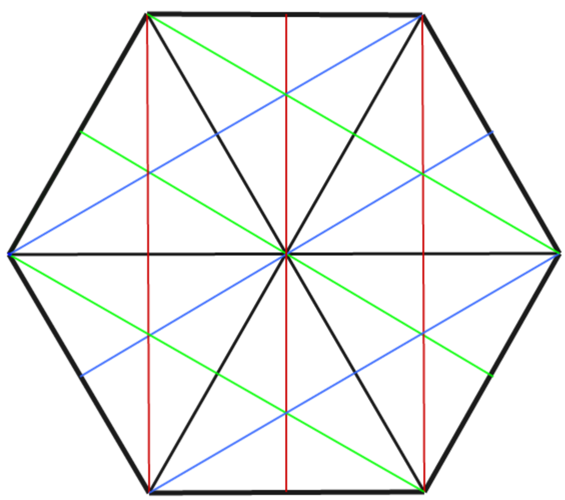


Figure 3.1 The Bands in Question

Now that we have a board, we should determine a set of rules with which this board is to be labeled. The most important analog seems to be use the bands, displayed in Figure 3.1, which mimic the columns and rows of a Sudoku or Latin Squares tiling. There seem to be 3 vertical bands of 12 triangles, each of which fit this requirement, where the external edges form a band together. Additionally, there are 3 diagonal bands in each direction, and so it seems sensible to require that each of these bands contain the numbers 1 . . . 12, and that the adjacent Big Triangles do the same. We will hence call this puzzle the **Sudodici**- keeping the Japanese 'su' for digit and using the Italian 'dodici' for twelve, as it sounds best. The overwhelming question, then, becomes whether or not there exists a valid tiling of this Sudodici board.

Luckily, the answer is yes, as displayed in Figure 3.2.

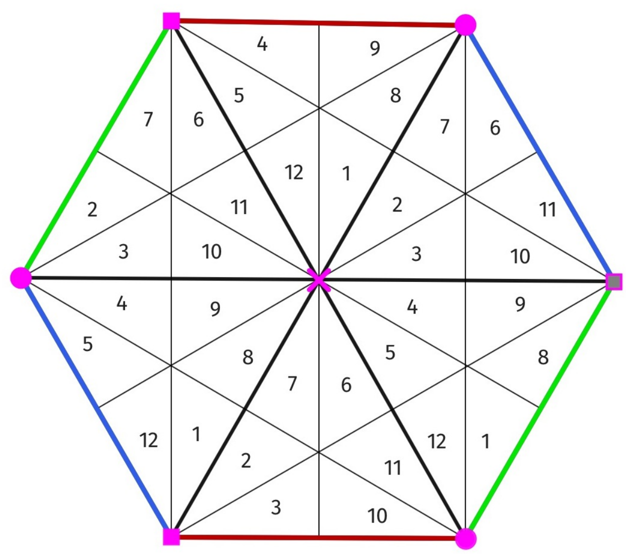


Figure 3.2 A Hexagonal Sudoku Tiling

3.2 Properties of the Sudodici

What is most interesting to me about this tiling was that in addition to the constraints I wanted, the tiling satisfies some additional constraints. The regions that have the numbers 1 . . . 12 appear once and only once are:

1. Each band, including the third band formed by the gluing
2. Each adjacent pair of big triangles
3. Each set of triangles adjacent to a vertex

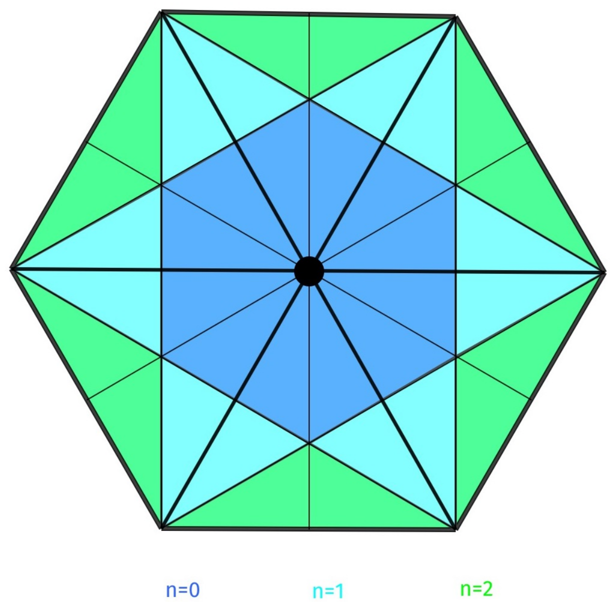


Figure 3.3 An Illustration for Constraint (4)

4. Each set of triangles at 'distance' n from a vertex (where $n = 0, 1, 2$), as seen in Figure 3.3

In addition, the solution is incredibly symmetric, with each 'diamond' being mirrored across the hexagon. This solution, then, seems improbable to be discovered by coincidence unless it reveals something about the overall set of solutions for this type of board.

3.3 Answering Sudodici Questions

In order to answer Sudoku-type questions for this board, we begin by looking at what was implied by the presence of one clue. That is, how does the positioning of one triangle affect the others, in order to determine how many distinct completed tilings there could be. We start by placing one such clue, x, and seeing how it can exclude other positionings for x, first by the bands and then by the triangle constraints, as these are sufficient to imply all further constraints. We will use this to prove our first theorem.

Theorem 3.1. *There exists a unique tiling for the Sudodici board with constraints* (1)-(4).

Proof. We begin by placing one x, and looking at where we can place the other xs. We do this by a process of elimination. First, we notice which triangles are ruled out from the band constraint, as seen in Figure 3.4.

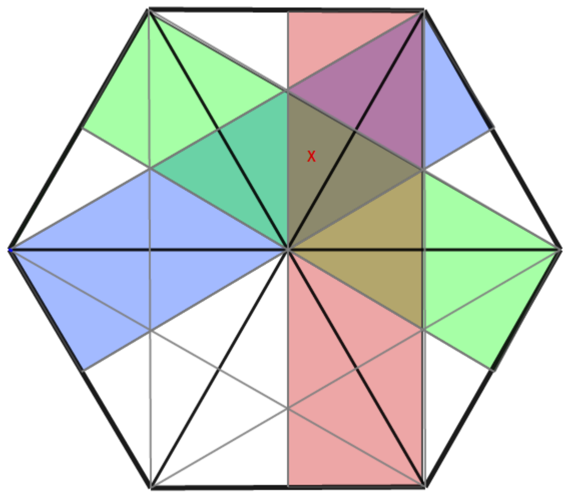


Figure 3.4 An Illustration of Which Triangles are Ruled out by the Band Constraint

We next look at which triangles are ruled out by the neighboring big triangles constraint, as in Figure 3.5.

We now use the fact that each band needs x at least once, and use that to find where the other numbers must go. By placing an x where necessary for each diagonal band, we then use the vertical bands to complete our proof. We now have found where all xs must be placed based solely on one clue.

By placing one x, then, we followed a series of logic steps to determine that the other two xs must only appear in the positions determined in Figure 3.6. Trying to place an x anywhere else will lead to an inability to complete the puzzle, as you are encouraged to try. We can further see that by symmetry, this will carry over to all numbers.

We have now proved that there exists precisely one tiling for this board with this constraints, and can now back-build puzzles to find the total number of puzzles.

□

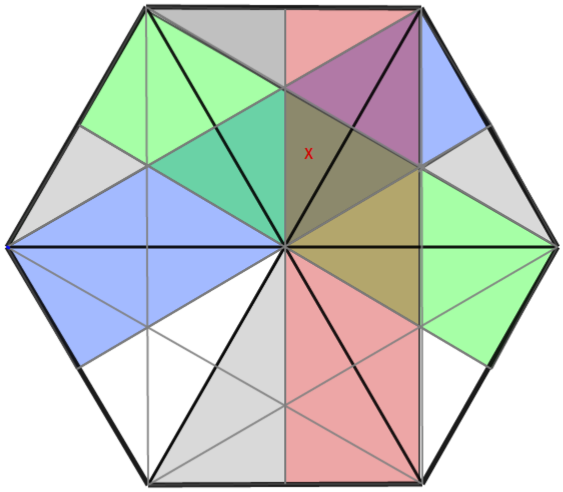


Figure 3.5 Available Positionings for x

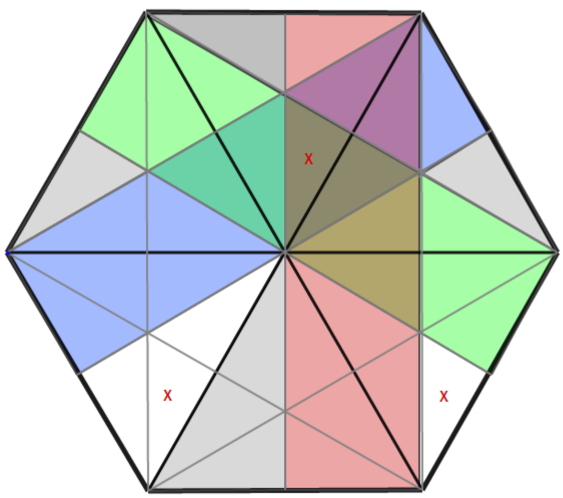


Figure 3.6 All xs Have Been Determined by One Placement

In order to have a meaningful puzzle, however, it should have a *minimal* number of clues, where minimal means that each clue gives information that cannot be gained otherwise. We thus look for the minimum and maximum number of clues possible for a puzzle.

Theorem 3.2. *The minimal and maximal number of clues necessary for a Sudodici*

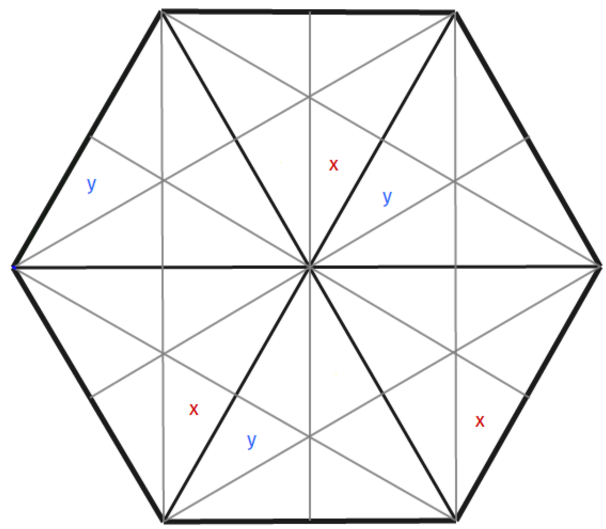


Figure 3.7 The Positioning of x Does Not Affect Positioning of y

puzzle is 11.

Proof. We begin by noting that a puzzle with 11 clues, each a distinct number, is indeed solvable, as seen in Figure 3.8. As we have shown in the previous section, these clues determine where the other iterations of 1-11 appear, and leave a remaining 3 places for 12. We further note that as the positioning of one clue determines the other two iterations of it, these 11 clues can be positioned wherever in the puzzle, and not just around the center, as pictured here.

We next note that if we had more than 11 clues, we would either have an extraneous instance of the 12th number, which we argued above is unnecessary, or we would have the same number twice, which is unnecessary as a clue need only appear once in order to know where the other two iterations are. Thus, 11 is the maximum number of clues needed.

Further, we argue that if less than 11 clues appeared, we would have at least two numbers that didn't appear at all on the starting puzzle, but when the rest of the board was filled in at least 6 triangles would be left empty, and we showed with our symmetry argument earlier that the positioning of one clue does not affect another, so these two numbers that didn't appear would be effectively interchangeable. Hence, 11 is also the minimum number of clues needed.

□

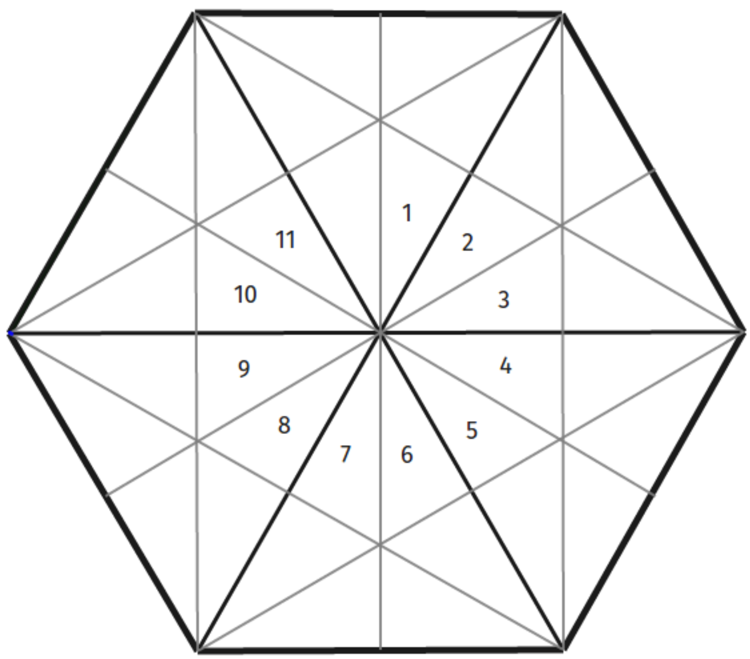


Figure 3.8 A Solvable Puzzle

We can thus find the number of minimal puzzles using simple combinatorics. If we have 11 clues, selected from 12 numbers, which can each be put in one of 3 positions, we will end up with $3^{11}x12!$ puzzles. However, as they all have the same solution, they may not be the most enjoyable game to play.

3.4 Conclusions on the Sudodici

The Sudodici is an interesting Sudoku-like problem because it allows us so much information with just the placement of a single clue. For a traditional Sudoku, placing one symbol will rule out 20 (out of 81) other positions for another iteration of that symbol, for a Sudodici, we end up ruling out 33 (out of 36). Traditionally, a Sudoku's difficulty is ranked in part by how much information one clue gives the player. This should should imply that the Sudodici is much 'easier', as the placement of one clue gives us a lot of information. However, the complicated form of the board means that this doesn't actually hold true. Indeed, while there is only one unique tiling for the Sudodici board, it is remarkable that this randomly constructed board could both have a solution, but also such symmetric properties. Further, the topological nature of the board and the varying shapes mean that you could almost certainly stump someone with this puzzle multiple times, especially if you varied your labels.

Chapter 4

Second Variant: Suroku

4.1 Constructing a Non Trivial Variant

Seeing that the first variant had a unique tiling, there were only so many questions we could ask about it, and they all had relatively easy to compute solutions. Therefore, it made sense to try and create a more complex variant, whose questions and answers could be more easily related to a traditional Sudoku.

To create this variant, more flexibility had to be built into the puzzle. This was achieved by changing it so that each band has the numbers 1-6 appear twice, and each Big Triangle having the numbers appear once. We call this new puzzle the **Suroku**, where 'roku' is Japanese for six, indicating that we will now have the natural numbers up to six appear.

4.2 Features of the Suroku

We first note that this board with these rules does have multiple tilings which obey the rules. We do this by observing Figures 4.1 and 4.2.

By looking at the upper left corner of these boards, it is observable that these tilings are not just relabelings of each other. It is further noticeable that these tilings are very similar to each other. This is because there are a series of moves through which we can translate one tiling to another, and still have them be valid labelings.

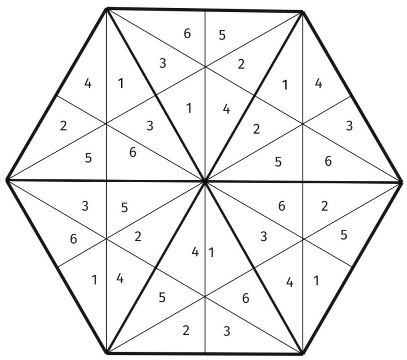


Figure 4.1 A Valid Tiling

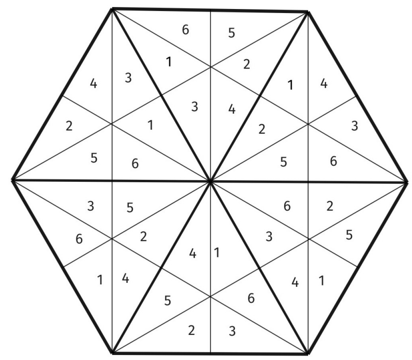
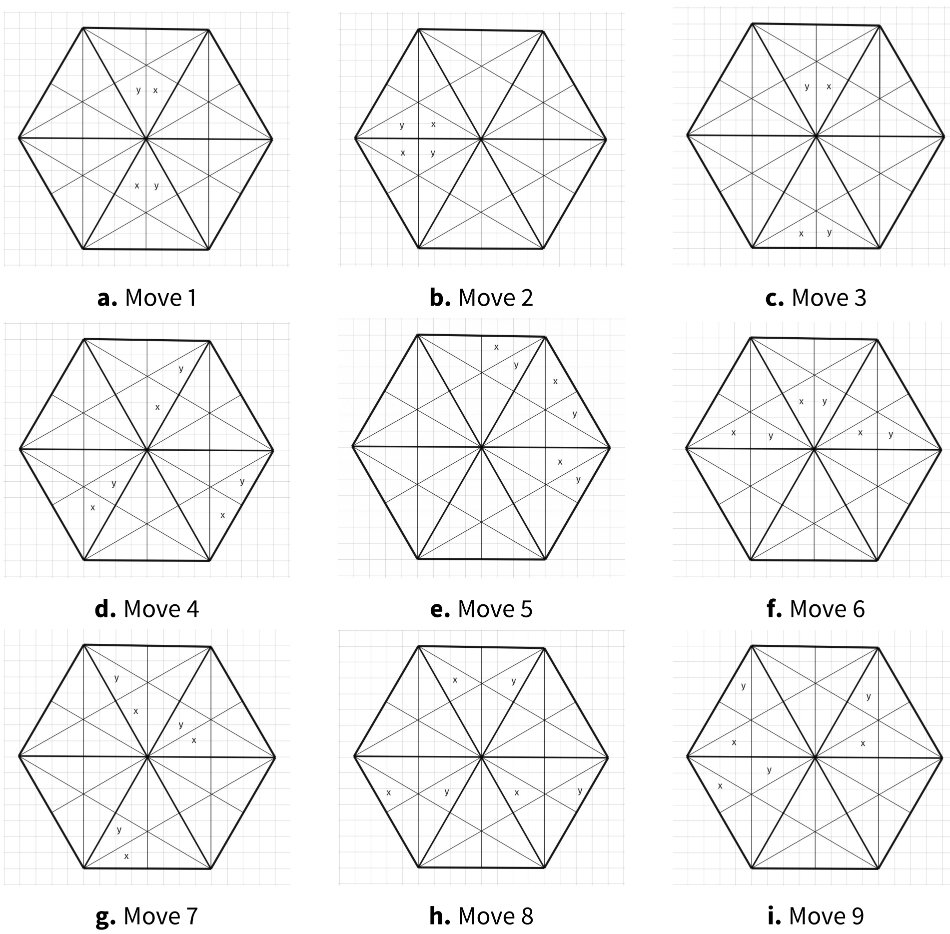


Figure 4.2 Another Valid Tiling

4.2.1 Moves

It seems like a useful task to catalog the ways in which we can move from one valid board to another, then. These moves will be seen as ways of switching labels for arbitrary symbols x and y, which aren't just relabel the tiling. It thus becomes reasonable to assume that these moves can only be formed by pairs or triplets of xs and ys, as moving just one clue will obviously mean that the initial tiling was wrong, and two and three are the only factors of six. That is, because we have up to six symbols, if we tried to swap out four or five iterations of xs and ys, we would end up with an invalid permutation as it would disrupt either too many or not enough things, and we would have to continue making swaps until we ended up back with something that was a multiple of 6 swaps. Else, the four or five iterations must have actually just been a combination of a move upon a pair or triplet.

Hence, we have managed to look at all possible ways of arranging pairs or triplets of xs and ys so that switching them does not change the validity of the board. These moves were found by looking at the distinct ways that we could position an x and y in relation to each other, and then seeing what swaps preserved the band and triangle constraints. These are displayed in Figures 4.3a through 4.3i.



a. Move 1 **b.** Move 2 **c.** Move 3

d. Move 4 **e.** Move 5 **f.** Move 6

g. Move 7 **h.** Move 8 **i.** Move 9

Figure 4.3 The Moves

4.2.2 Tiles

We gained intuition for the Sudodici by looking at how one clue constrained the other iterations of that symbol. In order to understand how Suroku tilings operate, we can do a similar procedure by placing two clues. As there are 1260 ways of doing this, we instead look for different "equivalence classes" of these pairs. We now bring in the topological intuition to note that all the vertices can, and should, be viewed as equivalent. As there are only twelve triangles around each vertex, this means if we are to place both clues around one vertex, we have brought down the number of cases to only 132. We can further note that due to the fact that each clue will knock out its entire big triangle, we're actually down to 120 cases. This is still a large number, so we next look at how these clues can share bands. Each pair of clues can be in either 0, 1, 2, or 3 bands together. This means we only have four cases, if we rely upon symmetry, for having both of the pair be around the same vertex. It further turns out that while it is possible for any vertex to have 0, 1, 2, or 3 of the same symbol around it, at least one vertex will have 2, meaning that by looking at all boards formed by having a pair around an arbitrary vertex, we have captured all possible ways of laying out the clues.

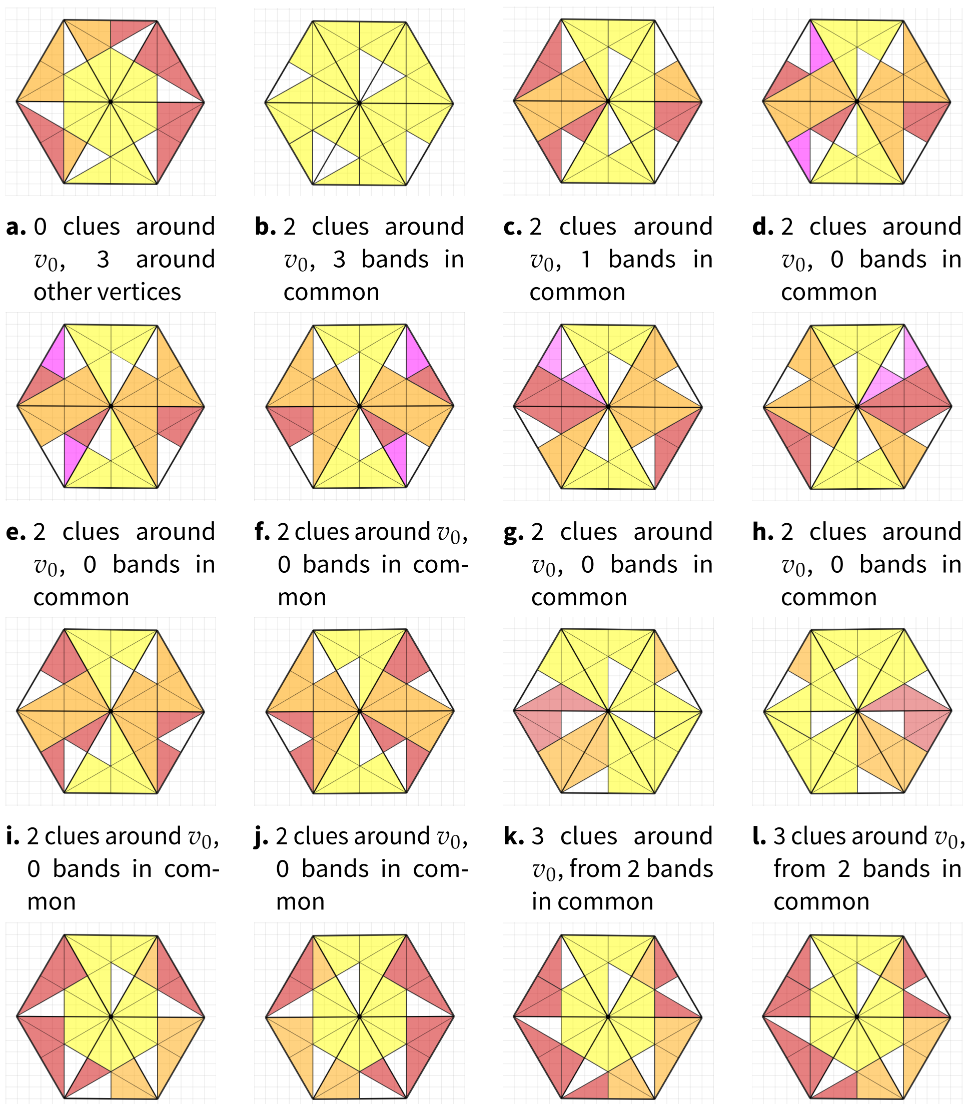
The options are below, where v_0 denotes the center vertex:

An interesting and important note is that the number of iterations of a symbol around each vertex is not constant across vertices. Indeed, we have found $(0,3,3), (2,2,2)$, and $(1,2,3)$ pairs. These are indeed the only options, as placing any more iterations of a symbol around one vertex would interfere with the bands constraint. Further, the fact that the number of clues around a vertex is not constant can be seen in Figure 4.3g, as Move 7 changes this value. It would thus be insensible to not be able to have different numbers around different vertices in different boards.

4.3 Relations of Boards

We now have a sequence of moves as well as a grasp on how different boards can be tiled. We can view the possible tilings for each symbol as a sort of puzzle of their own, where the number of possible tilings is the complete set of ways in which we 'fit' the tilings together.

The major question which arises out of these sets of moves and tiles is whether or not all completed tilings are 'reachable' from one another by the moves in question. A proof strategy is to observe how each of the tile-options only have a subset of moves even possible for the individual symbol, and



a. 0 clues around v_0, 3 around other vertices

b. 2 clues around v_0, 3 bands in common

c. 2 clues around v_0, 1 bands in common

d. 2 clues around v_0, 0 bands in common

e. 2 clues around v_0, 0 bands in common

f. 2 clues around v_0, 0 bands in common

g. 2 clues around v_0, 0 bands in common

h. 2 clues around v_0, 0 bands in common

i. 2 clues around v_0, 0 bands in common

j. 2 clues around v_0, 0 bands in common

k. 3 clues around v_0, from 2 bands in common

l. 3 clues around v_0, from 2 bands in common

m. 1 clue around v_0

n. 1 clue around v_0

o. 1 clue around v_0

p. 1 clue around v_0

Figure 4.4 Ways to Tile One Symbol for Suroku that these boards will become even more constrained once the other symbols are placed. A loose proof is to show that each of the individual symbol tilings is reachable via a series of moves, showing that the different board types are indeed related. However, while this does give us a connected graph of some sort of tilings, it doesn't actually give us a connected graph of all tilings. We would have to further include at least one more variable in order to show that just because one board of type a is reachable from one board of type b, all boards of type a are reachable from type b.

4.4 Conclusions

This board type will be more similar to a Sudoku than a Sudodici in a few interesting ways, despite the fact that it looks more like a Sudodici. First, the number of minimal clues in a puzzle should vary, as we can see in the 'tiling' section that we can have multiple ways of placing a couple of symbols that can cause different final tilings. The results from this puzzle should allow us to begin answering questions in a way which is built upon our work with the Sudodici, but can be compared to the Sudoku and give us intuitions and insights into its combinatorics as well.

Bibliography

Danburg-Wyld, Mark. 2005. Sudoku place.

Hayes, Brian. 2006. Unwed numbers the mathematics of sudoku, a puzzle that boasts" no math required! *American Scientist* 94(1):12–15.

Shapiro, Henry D. 1978. Generalized latin squars on the torus. *Discrete Mathematics* 24:63–77. doi:10.1016/0012-365X(78)90173-5.

Su, Michael Starbird & Francis Edward. 2016 Draft. *Topology Through Inquiry*.

Taalman, Jason Rosenhouse & Laura. 2011. *Taking Sudoku Seriously: The Math Behind the World's Most Popular Pencil Puzzle*. USA: Oxford University Press.

Taalman, Laura. 2007. Taking sudoku seriously. *Math Horizons* 5–9.

Weeks, Jeffrey R. 1985. *The Shape of Space*. New York, New York: Marcel Dekker, Inc.