2017

Emergence and Complexity in Music

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May, 2017

Abstract

How can we apply mathematical notions of complexity and emergence to music, and how can these mathematical ideas then inspire new musical works? Using Steve Reich's *Clapping Music* as a starting point, we look for emergent patterns in music by considering cases where a piece's complexity is significantly different from the total complexity of each of the individual parts. Definitions of complexity inspired by information theory, data compression, and musical practice are considered. We also consider the number of distinct musical pieces that could be composed in the same manner as *Clapping Music*. Finally, we present a new musical compositions to demonstrate some of these ideas.

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Acknowledgments

Thank you to Michael Orrison for being a fantastic advisor, Bill Alves for being a great second reader, and my fellow senior mathematics students for your thoughts, ideas, and support.

Chapter 1

Introduction

> Emergence . . . refers to the arising of novel and coherent structures, patterns, and properties during the process of self-organization in complex systems.
>
> — Jeffrey Goldstein (Goldstein (1999))

Emergence is a concept that is hard to pin down, but can be seen in contexts throughout science, mathematics, and art. Many systems in nature have been described as emergent: flocking behaviors of birds, synchronization of fireflies flashing, and even the social behavior of humans. Music is another example of such a domain. Since many types of music are governed by simple rules that lead to complex output (for example, classical symphonies guided by rules of counterpoint), it seems natural that music could display emergent properties.

In this thesis, we wish to identify music with emergent properties and create new music with these properties. Thus, we need to consider some concrete definitions of emergence.

1.1 Definitions of Emergence

There is little agreement about what is precisely meant by "emergence". Different authors cite various qualitative aspects that characterize emergence such as "radical novelty", "dynamical" (Goldstein (1999)), or "recognizable and recurring" (Holland (1998)).

Some authors also distinguish between notions of *strong emergence* and *weak emergence.* The most common definitions of these two ideas is that strongly emergent phenomena are not deducible given the starting state of a system, while weakly emergent phenomena are simply unexpected (Thorén and Gerlee (2010)). By this definition, there are very few examples of strong emergence. Philosopher and cognitive scientist David Chalmers points to mathematical examples such as the behavior of cellular automata as examples of weak emergence, while the one observable example of strong emergence is the phenomenon of consciousness (Chalmers (2008)). Mark Bedau also puts forth the idea of "normal emergence", which is simply "a macro property that is the kind of property that cannot be a micro property" (Bedau (2002)).

Emergence can also be defined in terms of what is knowable only by simulation. For example, the behavior of certain cell patterns in Conway's Game of Life, such as the bounded growth of the R-pentomino, can only be determined by simulating the system, thus marking their behavior as emergent (Baker (2010)).

Following this idea of simulation, we can also consider the amount of computation needed to simulate a system as compared to other methods of finding the system's final state. Vince Darley defines two functions: $s(n)$, the amount of computation needed to simulate a system to the desired state, and $u(n)$, the amount of computation needed to arrive at the state of the system by another means—for example, by using known physical laws to determine the state of a physical system. Here, n is the number of elements of the system, such as the number of cells in a finite cellular automaton. According to Darley, if $u(n) < s(n)$, then the system is non-emergent, and if $u(n) \geq s(n)$, the system is emergent. That is, a system is emergent if the best way to understand it is to simulate it (Darley (1994)).

Finally, emergence can be understood in terms of complexity. According to Henrik Thorén and Philip Gerlee, whose work links complexity to notions of weak emergence, complex objects "lie somewhere in between complete order and randomness." A useful measure of complexity therefore doubles as a measure of how difficult a system is to forecast. However, there is often a disconnect between the complexity of the fundamental rules of a system and the complexity of its observed behavior: simple rules can cause complex behavior, and complicated rules can give rise to simple behavior. We can therefore consider both the "emergence of complexity" and the "emergence of simplicity" (Thorén and Gerlee (2010)). Given a way to measure complexity, we can use it to measure the emergent properties of a system.

With this in mind, we create the following definition: *emergence occurs when the resultant behavior of a system is significantly more or less complex than*

Figure 1.1 An excerpt from *The Clod and The Pebble*.

the sum of the complexities of system's components. This definition has a few advantages. First, given a definition of complexity, it provides a simple way to check for the presence of emergence. It is also particularly suited for music because we wish to create music of varying degrees of complexity. For example, we might want to compose a piece with a high complexity to create musical interest, or a piece with a low complexity to be performed by beginning musicians.

In order to use this definition of emergence, we must find a suitable way to define complexity. Notions of complexity based on randomness, compressibility, and cognitive difficulty are discussed in Chapter 3, and a new complexity metric for music is proposed in Chapter 4.

1.2 Composing Emergent Music: *The Clod and The Pebble*

Even without defining a particular metric, it is possible to consider complexity as a way to create emergence in music in a more general way. To give an example, consider *The Clod and The Pebble*, an original musical piece composed for this thesis. The score and performance notes are found in Appendix A, and Figure 1.1 contains an excerpt of the score.

This piece is written for two soprano soloists with accompaniment. The text is taken from William Blake's poem "The Clod and the Pebble", which discusses two contrasting ideas about love. The piece begins with each singer performing the text of one of the stanzas in its entirety. The singers then perform a "divided" version of the text of the first stanza, in which the words are broken into approximate syllables and divided between the performers. For example, the text written in IPA (the International Phonetic Alphabet, a way of representing phonetic sounds unambiguously) corresponds to the text "Love seeketh not itself to please, nor for itself ha. . . ", which is a portion of the first two lines of the original poem. However, the text has been divided between the two singers, so the line is essentially performed as "Lo**ve see**ke**th** **no**t i**tse**lf to **plea**se, no**r fo**r i**tse**lf ha. . . ", where the text in bold is performed by the first singer, and the remaining text is performed by the second. This technique is similar to the one found in Daniel Lentz's *Can't See the Forest . . . Music*, a piece for solo vocalist, wine glass, and electronics (Lentz (1984)). The performer says single syllables from well-known phrases, spaced apart in time and punctuated by taps on a wine glass that vary in pitch as the wine is consumed over the course of the performance. A recording device then plays back the performer's sounds, allowing the phrases to be slowly reconstructed. (The wine also adds some distortion to the performer's voice, and perhaps some additional cognitive difficulty.)

In both *The Clod and the Pebble* and *Can't See the Forest . . . Music*, each vocalist has a cognitively complex part, since it is more difficult to sing or speak gibberish than English text. However, the result heard by the audience is relatively simple—the text is familiar, and the words are still understandable when divided and then reassembled. This is an example of the idea of "the emergence of simplicity", or a situation where the complexity of a whole is less than the sum of its parts. In fact, one might consider that a hypothetical single performer with two voices might have an easier time with *The Clod and the Pebble* than would one of the two ordinary singers.

While the complexity metrics discussed in Chapter 3 and Chapter 4 are not capable of evaluating the complexity of a piece of music such as this, it is still an example of how the mathematics of complexity and emergence can inspire musical composition.

1.3 Musical Objects

We now move away from abstract notions of complexity and begin to fix a notion of a particular type of musical object that we will use to evaluate and develop complexity metrics. For our purposes, we wish to consider one very specific type of music, inspired by Steve Reich's *Clapping Music* (Reich (1980)).

Clapping Music is a work for two performers, both of which create sound

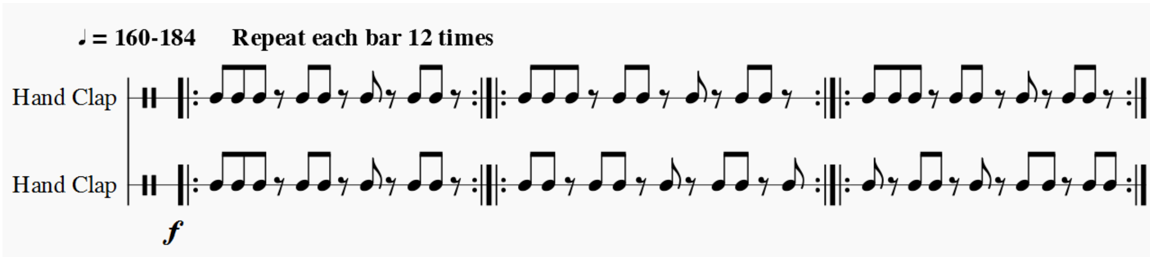

Figure 1.2 The first three measures of *Clapping Music* by Steve Reich Reich (1980).

only by clapping their hands. The piece begins with both players playing a specific pattern in unison. After a specified number of repeats, the second player shifts the pattern forward by one beat, effectively skipping a beat at the beginning of the next section. The first player's pattern remains constant. Since the original pattern is 12 beats long, the second player repeats the shifting process a total of 12 times, the last of which brings the players back into unison. The first three measures of *Clapping Music* are reproduced in Figure 1.2.

We consider a generalization of this piece. We want to have an arbitrary number of players who can either clap or not clap on any given beat, so each player's part can be rendered as a binary string, where a 1 represents a clap and a 0 represents a rest (a beat without a clap). We can then add m of these parts together to create an m-ary string, which would represent a piece being played by m musicians. For instance, if one player plays [1101] while another plays [0100], the piece will sound like [1201].

We assume that a listener without access to the musical score can tell how many players are clapping on a given beat, but not which players are clapping. For example, two players, one playing [1110] and one playing [0111] will together sound like [1221], which is indistinguishable from one player playing [1111] and one playing [0110]. Thus there are often multiple ways to create individual parts that sum to the same sounded piece.

We also wish to have players undergo a similar shifting process to that found in *Clapping Music*. We generalize this as well. All players are given identical starting measures consisting of n beats. At the end of each measure, each of the musicians shifts forward by skipping some number of beats. This number may differ between players, but a given player's shifting number may not change over the course of a piece. Finally, a piece is over after n

measures are played, so the total number of beats in a piece is n^2. Note that this is a different formulation from Steve Reich's original *Clapping Music*, since we do not play the first pattern again but instead end a measure before we would hear the new pattern. This distinction is mainly made for ease of computation, as used in Chapter 2. Our generalization also differs from *Clapping Music* in that there are no repeats—while each measure is repeated 12 times in the original Reich piece, we will play each measure only once.

In summary, a "clapping music" for m musicians with n measures can be represented in two ways: as a musical score or as a sounded piece. A *musical score* is an $m \times n^2$ matrix with binary entries, where a 1 in the ijth entry indicates that musician i is clapping on the jth beat, and a 0 means that musician i is resting on the jth beat. A *sounded piece* is the information that is recovered by a listener who can distinguish volume changes but not the identity of a particular performer, i.e. it is a list of n^2 entries, each of which contains an integer in $\{0, \ldots, m\}$ representing the number of musicians clapping on that beat. The sounded piece can be computed from the score by adding the entries in each column.

Finally, a clapping music is *realizable* if it has the following properties:

1. All musicians have identical parts for the first n-beat measure of the piece, and

2. For the ith musician, each subsequent measure is a shift by k_i beats of the first measure, where k_i remains constant throughout the piece.

If we consider realizable clapping musics as a subset of all possible pieces that could be written for m clapping performers, it seems that these realizable pieces make up a relatively small part of our musical universe. This idea is made more precise in Chapter 2.

Chapter 2

Counting Clapping Musics

Now that we have defined a general type of Clapping Music in Section 1.3, it is natural to consider the relationship between these objects and others in a more general musical universe. It seems that we have created a heavily restricted type of music, and that it might not be possible to compose very many distinct pieces that meet all of our requirements. To demonstrate that we have actually isolated a very small subset of the possible pieces of music that could be written for clapping musicians, we will consider the ratio of the number of possible realizable Clapping Musics to the number of more general pieces.

To do this, we will consider a piece of music for m performers as a list of n measures, each of which contains n beats. We wish to compare the number of such pieces that are *realizable* to the total number of such pieces. By realizable, we mean that these pieces begin with all players playing the same pattern, and that each player's part consists only of shifted versions of the first measure.

We will consider two separate cases: the *written* case, in which we have a complete description of which musician is playing on each beat, and the *sounded* case, in which we can only determine the number of musicians playing on each beat, but not the identities of the individual musicians.

2.1 Written Music

First, consider the case of written sheet music (i.e., we can tell which player is playing on a given beat). For the case of an arbitrary piece of music (potentially unrealizable), we can consider a piece of sheet music as an

$m \times n^2$ array with binary entries, where a 1 in the ijth entry indicates that musician i plays on beat j, and a 0 means the musician rests on that beat. Thus the number of unrestricted sheet musics is $2^{n^2 m}$.

In contrast, the number of realizable pieces is constrained by the first measure—given the first player's first measure in a realizable piece, we can reconstruct the entire piece if we know the way each player is shifting. Thus, the number of realizable sheet musics with a given shifting pattern and number of musicians is 2^n.

Now we must determine how many possible shifting patterns there are for n musicians. Suppose musician i shifts by k_i beats at the end of every measure. We know that $0 \leq k_i < n$, since a shift of at least n beats is the same as some shift of less than n beats. We assign one k_i to each of the m musicians, and there are n^m ways to do this. Thus, the number of written clapping musics is bounded by $2^n n^m$. Note that this is certainly an overcount. For example, consider a clapping music for m musicians with the starting measure [0000]. No matter how the k_i are assigned, the resulting written music will be the same.

Finally, we consider the ratio of the number of realizable pieces to the number of unrestricted pieces. This ratio is given by

$$\frac{2^n n^m}{2^{n^2 m}}. \tag{2.1}$$

We now wish to find the limit of this ratio as m and n tend to infinity. First, we rewrite the ratio:

$$\frac{2^n n^m}{2^{n^2 m}} = \frac{2^n n^m}{2^n 2^{n^2 m - n}} = \frac{n^m}{2^{n^2 m - n}}. \tag{2.2}$$

By inspection, we can see that the limit as n tends to infinity of this ratio is zero, since the numerator grows as a polynomial and the denominator grows exponentially. Thus, we have

$$\lim_{n \to \infty} \frac{2^n n^m}{2^{n^2 m}} = 0. \tag{2.3}$$

Now consider the limit as m tends to infinity. We begin by rewriting the ratio further:

$$\frac{n^m}{2^{n^2 m - n}} = \frac{\left(2^{\log_2(n)}\right)^m}{2^{n^2 m - n}} = \frac{2^{\log_2(n) m}}{2^{n^2 m - n}} = \frac{2^{\log_2(n) m} 2^n}{2^{n^2 m}}. \tag{2.4}$$

The limit of this ratio as m tends to infinity will tend to zero if $n^2 > \log_2(n)$, which is true for all $n \geq 1$. Therefore, we have

$$\lim_{m\to\infty} \frac{2^n n^m}{2^{n^2 m}} = 0. \tag{2.5}$$

Since the limit of this ratio tends to zero as either m or n goes to infinity, we conclude that realizable written pieces constitute a small subset of the total number of written pieces.

2.2 Sounded Music

Now consider the case of sounded music (i.e., we can tell how many people are clapping, but not which ones). In the unrestricted case, each of the n^2 beats in the piece can contain a number in $\{0, 1, \ldots, m\}$, so there are $(m+1)^{n^2}$ distinct pieces.

Since we can consider a sounded music as an equivalence class of written music, we can use our result in Section 2.1: the number of realizable sounded pieces is bounded by $2^n n^m$. Note that this overcounts the number of sounded pieces, but this works for our purposes since we want to compare this value to the total number of sounded pieces.

Again we consider the ratio of realizable pieces to unrestricted pieces. This yields

$$\frac{2^n n^m}{(m+1)^{n^2}}. \tag{2.6}$$

By inspection, we see that the limit of this ratio tends to zero as n tends to infinity, since the numerator is exponential in n and the denominator is exponential in n^2. That is, we have

$$\lim_{m\to\infty} \frac{2^n n^m}{(m+1)^{n^2}} = 0. \tag{2.7}$$

Unfortunately, the limit as m tends to infinity does not exist. It would be useful to have a better way to count the number of realizable sounded pieces, since we wish to show that the ratio tends to zero as both m and n tend to infinity. Despite our inability to show this for m, our progress for the written case is promising.

Chapter 3

Complexity Measures

As discussed in Section 1.1, our definition of emergence relies on a notion of complexity. This chapter contains an exploration of several types of complexity metrics and a proposal for a new metric specifically constructed for the evaluation of clapping musics.

3.1 Shannon Entropy

Our first complexity metric is Shannon entropy, first described in 1948 by Claude E. Shannon (Shannon (1948)). Strictly speaking, Shannon entropy is not a measure of string complexity, but rather a measure of the information content of a discrete random variable. However, we can use it on strings by pretending each musical event (a note or rest) is drawn from some probability distribution, and calculating the probabilities by simply observing the relative frequency of each event's occurrence.

The Shannon entropy of a discrete random variable X with possible values $\{x_1, \ldots, x_n\}$ and probability mass function $P(x)$ is given by

$$H(X) = \sum_{i=1}^{n} P(x_i) \log_b \frac{1}{P(x_i)} = -\sum_{i=1}^{n} P(x_i) \log_b P(x_i). \tag{3.1}$$

Different values for b can be used—in general, $H(X)$ gives the minimal number of bits per symbol to encode the information in base b, so we might want to use $b = m$ where m is the number of musicians.

As a brief example, consider the first two measures of *Clapping Music*. If we take a rest, a single clap, and a double clap as three separate symbols, we can calculate the Shannon entropy of the first player's part as follows.

Sounded Pattern	Entropy
222022020220	0.918
221121111211	0.918
212112021121	1.325

Table 3.1 Calculation of the Shannon Entropy of the first three measures of *Clapping Music*.

First, we write the part as the string [111011010110] (claps are 1s, rests are 0s). Note that four of the twelve symbols are 0s while eight are 1s. Thus we say $P(0) = 1/3$ and $P(1) = 2/3$. Using the above formula, we have $H(X) \approx 0.918$.

The first player's part does not change throughout the piece, so its entropy does not change between measures. Since entropy is independent of order, the second player's entropy is also constant and equal to the first player's. The entropy of first three measures of the sounded piece is shown in Table 3.1.

This complexity metric does make intuitive sense in some ways—the more different types of notes there are, and the more randomly distributed these notes seem, the more difficult and possibly complex the music will be. We can also think about the "flatness" of a probability distribution (which Shannon entropy measures): the flatter a distribution, the more interesting it is to choose from (Toussaint (2013)).

However, there are some clear drawbacks to using Shannon entropy for musical purposes. The sequences evaluated as most complex are entirely random, which is contrary to both musical intuition (random music is just white noise) and notions of emergence as a sort of order. Shannon entropy's independence of the order in which events occur is also particularly problematic from a musical standpoint—re-ordering the notes of a musical piece can clearly change its complexity. For example, imagine ordering all the notes in a symphony by pitch.

Despite these issues, we can take some useful information from this idea. We want to consider sequences that display some amount of "sameness" or "repetitiveness" less complex than sequences with more varied content. This concept is explored more in Section 4.3, where we define a repetition parameter that reduces the complexity of a sequence that repeats a pattern.

3.2 LZ-Complexity

Another possible way to think about complexity is to consider the compressibility of a sequence, i.e., the amount of information needed to reconstruct a given string. For instance, we might think that [1010101010] is less complex than [0110101111], since the former can be described as "[10] five times", while the latter does not have an obvious description.

This idea is made more precise with the idea of Kolmogorov complexity, or algorithmic complexity, first proposed by Andre Kolmogorov (Kolmogorov (1968))). Under this metric, the complexity of a string is the length of the shortest computer program that outputs the string. This metric seems to capture many of the intuitive ideas that we want in a complexity metric. Unfortunately, Kolmogorov complexity is not computable in general. Thus, we need to find a more practical way of measuring compressibility.

One such measure is LZ78-complexity (Ziv and Lempel (1978)). This metric can be considered as a measure of how repetitive a sequence is. The measure is computed by writing down a string from left to right and noting the number of new substrings that occur in this process.

There are several variants of this process, with slightly different emphases and applications. We choose to use the method used by Godfried Toussaint specifically to evaluate music (Toussaint (2013)). The process works by passing over a string from left to right and noting the start of a new substring if the current pattern does not occur to the left of our current position.

This can best be shown with an example. Consider the string 212112021121 (the third sounded measure of *Clapping Music*). Passing over the string from left to right, we first encounter a 2, which forms the first substring in our decomposition, and then a 1, which becomes the second. The next symbol is a 2, which we have already seen, so we proceed to the next character, yielding the string 21. However, 21 has already occurred to the left of our marker (even though it is not in our dictionary), so we continue on to the next symbol, giving 211. This string has not yet occurred, so we write it down. We continue in this manner until we have passed over the entire string. This process is outlined in Table 3.2. Since there are five substrings in our decomposition, we conclude that the LZ-complexity of this string is 5.

We calculate the LZ-complexity of the first three measures of *Clapping Music* in Table 3.3.

This measure makes intuitive sense for music: music that is highly repetitive is easier to play, and thus somehow less complex. However, there is not much variation in this metric—in particular, short strings are likely to

String seen so far	Current decomposition
2	2
21	$2 \cdot 1$
21211	$2 \cdot 1 \cdot 211$
2121120	$2 \cdot 1 \cdot 211 \cdot 20$
212112021121	$2 \cdot 1 \cdot 211 \cdot 20 \cdot 21121$

Table 3.2 Example calculation of LZ-complexity.

Sounded Pattern	LZ-complexity
222022020220	4
221121111211	4
212112021121	5

Table 3.3 Calculation of the LZ-complexity of the first three measures of *Clapping Music*.

be given approximately the same complexity.

To illustrate this problem, consider the highly repetitious string 121212121212. This has a LZ-complexity of 4, the same value as assigned to the seemingly more complex strings in the first two rows of Table 3.3. This poses a problem if we wish to compare measures of music, which contain a relatively small number of beats.

3.3 Pressing Complexity

A third notion of complexity comes from the field of psychology and deals with the cognitive difficulty of performing various rhythms. Created by Jeffrey Pressing, Pressing complexity is a measure of the "cognitive cost" of rhythmic patterns (Pressing (1999)).

The basic idea of Pressing complexity is to define five fundamental types of rhythm and to locate these patterns at different musical hierarchical levels. We begin by defining these hierarchical levels. In Pressing's work, this is done by considering the prime factorization of the number of beats in our selection. For example, since 8 can be factored as $2 \cdot 2 \cdot 2$, we have levels consisting of 2 beats, $2 \cdot 2 = 4$ beats, and $2 \cdot 2 \cdot 2 = 8$ beats. These can be thought of as corresponding to considering a pattern on a quarter note, eighth note, and sixteenth note level. Figure 3.1 demonstrates a decomposition of a particular 8-beat pattern.

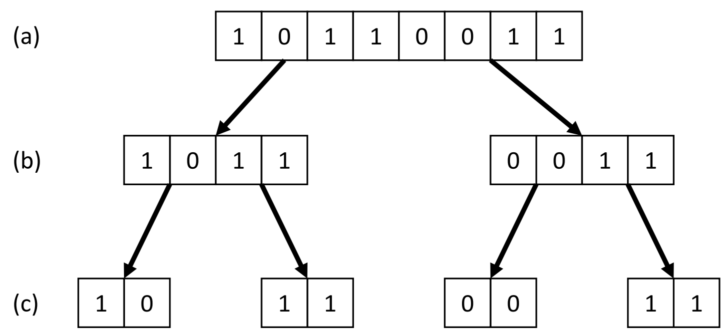

Figure 3.1 An example of computing the Pressing metric on a 8-beat pattern. The complexity is computed for each sub-rhythm, and these complexities are averaged across a level of subdivision. Then these averages are added to give the sum of the total rhythm.

Once the complexity is determined for each sub-rhythm, we use an averaging process to determine the overall complexity of the rhythm. This is done by averaging the complexity scores at each level and then adding these together. To illustrate this with an example, consider Figure 3.1. To evaluate this rhythm using the Pressing metric, we would need to compute the complexity of each of the seven sub-rhythms shown, including the top level "sub-rhythm" that includes all 8 beats. Once we have these complexities, we average them across each level. For instance, we average the complexity of each of the 2-beat sub-rhythms in level (c) of the figure. Once we have the average complexity of each level of the decomposition tree, we add these averages together to yield the complexity of the rhythm. To put it another way, if the complexities of each sub-rhythm in Figure 3.1 are given by $a_1, b_1, b_2, c_1, c_2, c_3, c_4$, according to the corresponding level of the decomposition tree, the complexity of the rhythm is given by

$$a_1 + \frac{b_1 + b_2}{2} + \frac{c_1 + c_2 + c_3 + c_4}{4}.$$

Note that Pressing's work deals only with patterns whose length is a power of two, so there is no unique way to resolve the question of order – for instance, should a pattern of length 12 be decomposed as $2 \cdot 2 \cdot 3$ or $3 \cdot 2 \cdot 2$? For our purpose of evaluating the complexity of *Clapping Music*, which is

Pattern Name	Description	Pattern Form	Difficulty
Null	A pattern containing only rests or a single note on the first beat	$[0^n]$ or $[1\ 0^{n-1}]$	0
Filled	A note occurs on every beat	$[1^n]$	1
Run	A note occurs on the first beat and a series of subsequent beats	$[1^k\ 0^{n-k}]$	2
Upbeat	A note occurs on the first beat, and there are one or more pick-ups into the next downbeat	$[1^k\ 0^\ell\ 1^{n-(k+\ell)}]$	3
Syncopated	A pattern that both starts and ends on off beats	$[0^k\ 1^\ell\ 0^{n-(k+\ell)}]$	5

Table 3.4 Difficulty types in the Pressing Metric.

composed of 12-beat measures, we choose to make the smallest subdivisions as close to powers of two as possible. Thus we will consider three sets of two quarter notes, each of which contains a set of four eighth notes.

We will call the pattern of length n at the level we are working at a *sub-rhythm*, and the pattern at the next level down a *sub-sub-rhythm*, as in the work of Eric Thul (Thul (2008)). With this decomposition, we can fix a pattern of length n and look for it at quarter note, eighth note, or sixteenth note levels by considering which pattern type is shown in each sub-rhythm.

The five types of rhythm and their difficulty scores are summarized in Table 3.4. Note that the sub-rhythm patterns as described by Pressing are not sufficient to cover all possible rhythmic patterns—for example, consider the string [0101], which does not have the structure of any of the named patterns. To account for this, we will consider all patterns that do not fit into one of these categories as syncopated and having difficulty 5. In addition, another difficulty level, *subbeat*, which corresponds to a score of 4, is mentioned by Pressing but not defined, so we omit it here.

With this information, we can compute the Pressing complexity of sections *Clapping Music*. Since the metric is defined only on binary sequences, we will consider the first musician's part in the first measure: [111011010110]. We write this as a set of three four-beat sub-rhythms: [1110], [1101], and [0110]. At the quarter note level, the first string is a run (difficulty 2), the second string is an upbeat pattern (difficulty 3), and the third string is a

First Player's Part	Pressing Complexity	Second Player's Part	Pressing Complexity
111011010110	$31/3 \approx 10.3$	111011010110	$31/3 \approx 10.3$
111011010110	$31/3 \approx 10.3$	110110101101	$25/3 \approx 8.3$
111011010110	$31/3 \approx 10.3$	101101011011	$32/3 \approx 10.7$

Table 3.5 The Pressing complexity of each musician's part of the first three measures of *Clapping Music*.

syncopated pattern (difficulty 5). We then repeat this process and find the difficulties of each of the six patterns of two eighth notes. We then average the scores of the eighth note patterns (yielding 2) and add this to the average of the scores for the quarter note patterns (yielding 16/3). Finally, we need the score of the total pattern – since the measure does not have any of the structures suggested by Pressing, we say it is syncopated and has difficulty 5. Adding these together, we arrive at $31/3 \approx 10.3$ as the total complexity of the first measure of the first musician's part of *Clapping Music*.

The complexity for each musician's part of the first three measures of *Clapping Music* are displayed in in Table 3.5. Note that we are unable to compute the Pressing complexity of the sounded measures, since this metric does not have a way of distinguishing beats containing a 1 from those containing a 2.

Despite the difficulties with this metric, it shows considerable promise. It is grounded in mathematics, music, and psychology, and will serve as a starting point for the new metric developed in Section 4.

Chapter 4

Toward a New Musical Complexity Metric

Based on the ideas of complexity discussed in Chapter 3, we wish to create a new metric for rhythmic complexity that is easily computable, mathematically interesting, and musically meaningful. We begin by taking inspiration from a common musical situation.

Suppose there is a long rhythm that a musician wishes to play. To make the task easier, they might choose to think of the rhythm as a series of very short sub-rhythms that they might already know how to play. If the partitioning into sub-rhythms is done in a clever manner, the piece might be revealed to be simply a list of very easy rhythms or a highly repetitious pattern, for example. This idea is already used in musical pedagogy: teachers will often instruct students to think of a rhythm as a series of smaller, easier rhythms. James Morgan Thurmond writes that modifying "note groupings" can be advantageous for both musical expression and technical performance of rhythms by students. For example, he argues that the common dotted eighth-sixteenth note pattern can be accurately mastered by practicing the rhythm both with the dotted eighth note as a downbeat and with the sixteenth note as a downbeat (Thurmond (1982)).

This idea allows us to create a new sort of complexity metric. If we can decompose a rhythm into short sub-rhythms, and we can assign a complexity value to each possible sub-rhythm, then we can assign a complexity to any rhythm by considering the sum of these complexities.

This is the basic idea of this new metric: we will consider the various ways of breaking a rhythm into two- and three-beat sub-rhythms, where each sub-rhythm has an associated complexity score. We will then find the decomposition that results in the smallest total complexity, and that complexity is the complexity for the rhythm. In addition, we will introduce a repetition parameter ρ which expresses a decrease in difficulty for repeated patterns, and difficulty parameters d_1 and d_2 which express the differences in difficulty when playing accented and unaccented beats.

4.1 Complexity of Sub-Rhythms

To begin, we will assign a difficulty to each possible sub-rhythm that we might encounter. For a variety of reasons, we will only consider sub-rhythms of length 2 or 3. Sub-rhythms of length 1 are too short to be musically meaningful, while sub-rhythms of length 4 are difficult to deal with mathematically because they could also be viewed as two sub-rhythms of length 2, and viewing the sub-rhythms as a set of 4 is almost always unnecessary in the metric developed in this chapter, since it results in a non-optimal decomposition. In addition, a huge amount of real-world music is based around rhythms in groups of 2 or 3 (Toussaint (2013)), so these numbers make sense from a musical perspective as well.

The complexities of each possible sub-rhythm are largely based on the Pressing complexity metric. However, there are some necessary changes. The Pressing metric does not work with strings whose length is not a power of two, but since we are considering only sub-rhythms of length 2 and 3, we do not need to worry about dividing our sub-rhythms and can use the characterizations presented in Table 3.4. Due to the absence of examples for the patterns of difficulty 4 in Pressing's original work, we have assigned a score of 4 instead of 5 to all patterns classified as syncopated. We continue classifying rhythms that do not fit one of the designated patterns as syncopated and also assign them a score of 4.

Another problem with the original Pressing metric is that it only considered binary strings: a rhythmic pulse could either contain a sound or not. For our purposes we wish to consider at least two types of sound in addition to the empty beat – this corresponds to zero, one, or two people clapping in a piece like *Clapping Music*, or a single player who can play both accented and unaccented beats. Thus we must make another modification.

We consider a 1 to be an ordinary sounded beat, and a 2 to be an accented beat. We begin by computing the difficulty of a sub-rhythm with our modified Pressing metric, considering both 1s and 2s as 1s. We then use our two *difficulty parameters* d_1 and d_2: we add d_1 to the sum if the first beat of the sub-rhythm is accented, and we add d_2 for every beat other than the first that is accented. With $d_1 < d_2$, this represents the relative difficulty of accenting a downbeat (the first beat of a sub-rhythm) and an upbeat (other beats in a sub-rhythm), since it is generally easier to accent a downbeat than an upbeat. For example, if the string [222] received a base score of s, its total score would be $s + d_1 + 2d_2$, since there are three accented beats: one on the downbeat and two on the upbeats.

Values for d_1 and d_2 can vary based on musical context, proficiency of performers, and personal preference. This is one of the strengths of this metric: it can be adapted for many different contexts.

The complexity scores for all 2-beat rhythms are given in Table 4.1, and the scores for 3-beat rhythms consisting of 0s and 1s are given in Table 4.2. 3-beat rhythms containing 2s have been omitted in the interest of space, since their scores can be easily calculated.

While the ordering of these rhythms by complexity would depend on the particular values chosen for d_1 and d_2, it is worth checking to see if the complexity scores make sense. For instance, the rhythm [12] has complexity $1 + d_2$, so it is more complex than [21], which has complexity $1 + d_1$ since $d_1 < d_2$. This makes sense because of the difficulty of accenting an upbeat. In addition, note that the rhythm [21] has complexity $1 + d_1$, while [22] has complexity $1 + d_1 + d_2$, so [22] is more complex. This can be understood by considering the physical and cognitive difficulty of performing an accented beat—even if all of the beats in a particular rhythm are accented, it is still more difficult to perform accented beats.

With these in place, we can now move on to consider the decomposition of a particular rhythm.

4.2 Decomposition into Sub-Rhythms

The other unique feature of this metric is that it deals explicitly with how a rhythm is parsed into sub-rhythms. We wish to find the partition into sub-rhythms that yields the lowest total complexity.

In general, there are many ways to do this. For example, an 8-beat pattern could be written as four 2-beat patterns or two 3-beat patterns and one two-beat pattern. This is shown in Figure 4.1, which shows that there are a total of four ways to decompose an 8-beat rhythm.

We can count the number of ways to partition a number in this fashion

Rhythm	Complexity
00	0
01	4
02	$4 + d_2$
10	0
11	1
12	$1 + d_2$
20	d_1
21	$1 + d_1$
22	$1 + d_1 + d_2$

Table 4.1 The complexity values of all possible 2-beat rhythms using our new metric.

Rhythm	Complexity
000	0
001	4
010	4
011	4
100	0
101	3
110	2
111	1

Table 4.2 The complexity values of all possible 3-beat rhythms without accents using our new metric.

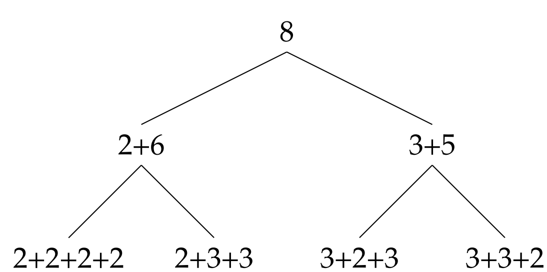

Figure 4.1 A tree showing the ways an 8-beat rhythm could be decomposed into 2- and 3-beat sub-rhythms.

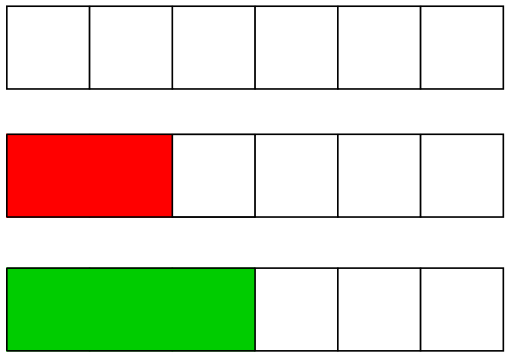

Figure 4.2 An example of how partitioning a rhythm into sub-rhythms can be seen as a tiling problem.

by considering a recurrence relation. Consider the representation in Figure 4.2. For each beat in our rhythm, we draw a single box – in this case, we have six beats, so we draw six boxes. We then consider the number of ways we can tile this with tiles of size 2 (the red tiles shown in (b)) or 3 (the green tiles in (c)).

Call the number of such tilings of a n-beat sequence $f(n)$. Because we can start by placing a tile of length 2 or 3 at the beginning of the sequence, we can obtain $f(n)$ by adding $f(n-2)$ and $f(n-3)$. Thus, we have the recurrence relation

$$f(n) = f(n-2) + f(n-3) \tag{4.1}$$

with initial conditions $f(1) = 0$, $f(2) = 1$, and $f(3) = 1$. This sequence is a slightly modified version of A000931 from the On-Line Encyclopedia of Integer Sequences (Sloane (2010)), which contains much information about this sequence. For example, we have the identity

$$(n) = \frac{r^n}{2r+3} + \frac{s^n}{2s+3} + \frac{t^n}{2t+3}, \tag{4.2}$$

where r, s, and t are the three roots of $x^3 - x - 1$. Because the magnitudes of the two complex roots s and t are less than one, the latter two terms tend to zero as n increases. Thus the formula is asymptotic to $\frac{r^n}{2r+3}$, where $r \approx 1.325$ is the real root of $x^3 - x - 1$.

Since we currently have no algorithm to determine what decomposition of a rhythm will be the least complex, it is important to know how many decompositions we will have to try. For example, since $f(12) = 12$, there are 12 partitions of a measure of *Clapping Music* that we will need to check. This is still fairly small, but the amount of computation does increase quickly—if we wanted to compute the complexity of the first three measures of *Clapping Music*, we would need to check over 10,000 partitions of the 36 beats.

It is possible that musical and mathematical knowledge and intuition could be used to greatly reduce the amount of computation needed, as certain decompositions could be judged as non-optimal before even computing their complexity. For example, the string [101021] is likely to be better parsed as [10]+[10]+[21] than [101]+[021], since the former partition results in every sub-rhythm beginning with a sounded beat, while the latter partition has one sub-rhythm that begins on an upbeat and incurs an additional penalty for having an accented note on an upbeat.

4.3 The Repetition Parameter

The last component of this metric is the repetition parameter ρ. By setting $0 < \rho < 1$, we can multiply the difficulty of a component sub-rhythm by ρ to represent a decrease in difficulty to the performer – and therefore a decrease in complexity – if a sub-rhythm is repeated.

For example, suppose we have the rhythm [011011011], and we have chosen to decompose this as [011] + [011] + [011]. The complexity of the sub-rhythm [011] is 4, as given in Table 4.2. However, since it is repeated three times, the complexity of this sequence is $4+4\rho+4\rho^2$, as we accumulate a factor of ρ for each time the sequence is repeated.

A consequence of this property is that the complexity of a sufficiently repetitious infinite rhythm is bounded. For example, consider the pattern [11111111 . . .], which we will parse as [11] + [11] + $\cdots$. Each of these sub-rhythms has complexity 1, so the complexity of this infinite rhythm is given by the geometric series

$$1 + \rho + \rho^2 + \cdots = \frac{1}{1-\rho}, \tag{4.3}$$

since $0 < \rho < 1$.

Having ρ as a parameter that can vary among performers and musical contexts, like the parameters d_1 and d_2 which represent the difficulty of accenting downbeats and upbeats, makes this model more flexible and more widely applicable. For example, beginning musicians might gain a great benefit from having a single rhythm to play over and over again, while more experienced performers might be equally comfortable with a constantly changing pattern, so ρ might be smaller for beginners.

4.4 Example: *Clapping Music*

Finally, we consider a concrete example of computing the complexity of a piece of music: Steve Reich's *Clapping Music*. We will walk through the computation of the score of one particular decomposition of the first sounded measure (i.e., the sum of both player's parts in the first measure), and then present the overall scores of the first three measures in a table.

The sounded first measure of *Clapping Music* is [222022020220]. Suppose we decompose this as [22]+[20]+[220]+[20]+[220]. Consulting Table 4.1 and Table 4.2, we find the scores for each sub-rhythm and add them to find the complexity score for this decomposition of this measure:

$$(1+d_1+d_2)+(d_1)+(2+d_1+d_2)+\rho(d_1)+\rho(2+d_1+d_2) = 3+2\rho+(3+2\rho)d_1+(2+\rho)d_2 \quad (4.4)$$

This particular expression isn't very meaningful without values for ρ, d_1, and d_2. In fact, it is impossible to compare the complexity scores of various decompositions of a rhythm without fixing these parameters. Thus, we will choose the following arbitrary values: $\rho = 0.75$, $d_1 = 0.5$, $d_2 = 1$. Note that these values are not chosen to align with any real-world data. The need to find reasonable parameters is one example of the potential future work discussed in Chapter 5.

Using these values, our complexity score for this decomposition of this measure is 9.5. To find the actual complexity of this measure, we would need to compute the scores for each of the 12 possible decompositions of this measure and take the lowest. We happen to have chosen the best decomposition in this case, however.

The complexities of the first and second players' parts are given for each measure of *Clapping Music* in Table 4.3, and the complexities of each sounded measure are given in Table 4.4. In addition, we give the decomposition that results in the minimal complexity score. These results are also given in Figure 4.3.

Part	Rhythm	Complexity	Decomposition
1st player, all measures	111011010110	4.5	[11]+[10]+[110]+[10]+[110]
2nd player, 2nd measure	110110101101	8.5	[110]+[110]+[101]+[101]
2nd player, 3rd measure	101101011011	4.5	[10]+[110]+[10]+[110]+[11]
2nd player, 4th measure	011010110111	7	[01]+[10]+[10]+[110]+[111]
2nd player, 5th measure	110101101110	4.5	[110]+[10]+[110]+[11]+[10]
2nd player, 6th measure	101011011101	6	[10]+[10]+[110]+[11]+[101]
2nd player, 7th measure	010110111011	7.75	[010]+[110]+[11]+[10]+[11]
2nd player, 8th measure	101101110110	4.5	[10]+[110]+[11]+[10]+[110]
2nd player, 9th measure	011011101101	5.75	[01]+[10]+[11]+[10]+[11]+[01]
2nd player, 10th measure	110111011010	4.5	[110]+[11]+[10]+[110]+[10]
2nd player, 11th measure	101110110101	4	[10]+[11]+[10]+[10]+[101]
2nd player, 12th measure	011101101011	8.5	[01]+[110]+[110]+[10]+[11]

Table 4.3 The complexity of the individual parts of each measure of *Clapping Music* using our new metric, with $\rho = 0.75$, $d_1 = 0.5$, and $d_2 = 1$.

Part	Rhythm	Complexity	Decomposition
Sounded, 1st measure	222022020220	9.5	[22]+[20]+[220]+[20]+[220]
Sounded, 2nd measure	221121111211	6.875	[22]+[11]+[211]+[11]+[211]
Sounded, 3rd measure	212112021121	5.75	[21]+[211]+[20]+[211]+[21]
Sounded, 4th measure	122021120220	8	[12]+[20]+[211]+[20]+[220]
Sounded, 5th measure	221112111220	9.5	[22]+[111]+[21]+[11]+[220]
Sounded, 6th measure	212022021211	8.125	[21]+[20]+[220]+[21]+[211]
Sounded, 7th measure	121121221121	8.875	[12]+[11]+[21]+[22]+[11]+[21]
Sounded, 8th measure	212112120220	8.125	[21]+[211]+[21]+[20]+[220]
Sounded, 9th measure	122022111211	7.5	[12]+[20]+[22]+[111]+[211]
Sounded, 10th measure	221122021120	9	[22]+[11]+[220]+[211]+[20]
Sounded, 11th measure	212121120211	5.75	[21]+[21]+[211]+[20]+[211]
Sounded, 12th measure	122112111121	7.125	[12]+[211]+[211]+[11]+[21]

Table 4.4 The complexity of each sounded measure of *Clapping Music* using our new metric, with $\rho = 0.75$, $d_1 = 0.5$, and $d_2 = 1$.

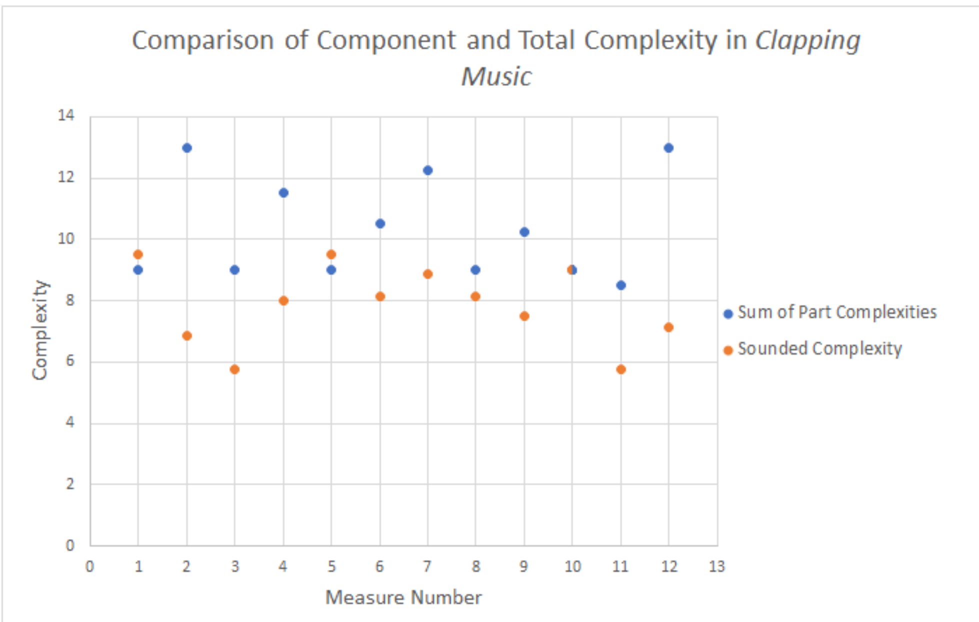

Figure 4.3 A graph of the complexity of the sound and component parts of each measure of *Clapping Music*.

Part	Rhythm	Complexity	Decomposition
Sounded, 1st measure	222022020220	26	[22]+[20]+[220]+[20]+[220]
Sounded, 2nd measure	221121111211	15.5	[22]+[11]+[211]+[11]+[211]
Sounded, 3rd measure	212112021121	15	[21]+[211]+[20]+[211]+[21]
Sounded, 4th measure	122021120220	23.5	[12]+[20]+[211]+[202]+[20]

Table 4.5 The complexity of the first four sounded measures of *Clapping Music* using our new metric, with $\rho = 0.5$, $d_1 = 0.5$, and $d_2 = 2$.

As shown in Figure 4.3, the complexity of a given sounded measure is almost always less than the sum of the component complexities. In only three cases does the sounded complexity match or exceed the sum of the components. This is suggestive of the notion of the "emergence of simplicity" discussed in Section 1, as we here have a piece that is more complex for two players than it would be for one player playing both parts.

It is also important to note that the numeric complexities assigned to each measure, and thus the complexities of each measure relative to one another, are highly dependent on the chosen parameter values. To illustrate this, consider Table 4.5, which contains the complexity results for the first four sounded measures of *Clapping Music* with a different set of parameter values: $d_1 = 2$, $d_2 = 3$, and $\rho = 0.5$.

By comparing these results to those in Table 4.4 we can see the impact of changing the parameters. While we have maintained our ordering of the first four measures by complexity (i.e. measure 1 is the most complex, followed by measure 4, then 2, then 3), the difference between the complexities of each measure is greater with the new parameters. In addition, the optimal decomposition of the 4th measure into sub-rhythms has changed from [12]+[20]+[211]+[20]+[220] to [12]+[20]+[211]+[202]+[20].

We have now created a new complexity metric for rhythmic patterns that incorporates many of the advantages of earlier metrics. However, due to our metric's reliance on parameters, there is much work that remains to be done to ensure that this metric aligns with real musical experiences.

Chapter 5

Next Steps

By considering a very specific type of music and some ways complexity has been defined in different context, we have created a new complexity metric that could be used to look for emergent properties in some kinds of music.

This research leads naturally to a wide variety of topics for future research in both mathematics and music.

5.1 Mathematical Work

While the idea of decomposing rhythms into sub-rhythms of length 2 and 3 is certainly useful, the ideas could still be extended. First, there is no real reason to stop at 3 beats—a 4-beat sub-rhythm seems natural from a musical perspective, and skilled musicians may be comfortable using sub-rhythms of length 5 or more. The main difficulty with this approach is that it is tempting to give higher complexity to longer sub-rhythms in general, but this approach basically ensures that the lowest-complexity decomposition will only contain the shortest possible sub-rhythms, making the longer sub-rhythms worthless. This could be fixed with careful refinements of the basic Pressing metric idea and perhaps the use of more parameters.

Because of this metric's reliance on the repetition parameter ρ and the accent difficulty parameters d_1 and d_2, there is a wide variety in the potential complexity scores for various pieces of music. Without concrete values for these parameters, it is difficult to find meaningful complexity scores for musical pieces. Therefore, it would be worthwhile to collect data about these parameters, perhaps by surveying musicians about the relative difficulty of various rhythmic passages. It would also be interesting to see if these values changed based on the musicians' preferred genres or instruments. This work could also reveal the need for new parameters or adjustments in the base sub-rhythm complexity scores if there is no reasonable way to align our results with collected data.

In addition to refining our metric, it would be useful to have an efficient way to calculate it. We would like to have an algorithm to find optimal or near-optimal decompositions of various rhythms. There are some useful heuristics: in general, it is better to choose a decomposition that results in the largest accents falling on downbeats most often. If this idea is formalized, it could greatly speed up the computation time needed for scoring the complexity of a particular piece. We could also examine this topic with empirical data, as we could compare the decompositions created by musicians to those this metric designates as "optimal".

While we have considered the existence of some emergent phenomena in *Clapping Music*, it remains to be seen if emergence is more common in generalized clapping musics than in arbitrary rhythmic patterns. This problem could be approached at least partially using computational methods, since the space of clapping musics is relatively small, as shown in Chapter 2.

5.2 Musical Work

While there is one piece of music based on these ideas (see Chapter 1 and Appendix A), there is much room for new musical compositions. For example, a piece based on the most simple or most complex possible rhythm of a given length could be very interesting. Alternatively, a composer could attempt to create maximally emergent music by composing pieces where the individual musicians' parts are much more or much less complex than the sounded piece.

The metric developed in Chapter 4 is obviously limited in the music it can analyze—only pieces consisting of a maximum of two types of sound in addition to silence. To extend this to more types of music, more parameters could be used, perhaps denoting the difficulty of moving between two types of sounds instead of the difficulty of performing the sounds themselves. For example, it might be easier for a vocalist to sing a very fast passage that stays in one part of their range than a slower piece that crosses through registers. Again, these parameters could be adjusted for all types of instruments, performers, and musical contexts.

There is also a potential for this metric to be used as a tool by musicians learning new pieces of music. Sometimes, the easiest or best way to think about a piece of music is not readily apparent, and knowing the complexity of a piece and a good decomposition into sub-rhythms could enable a musician to learn more quickly and perform more comfortably.

Appendix A

The Clod and the Pebble

A.1 Background and Performance Notes

This piece is based on the idea of the emergence of simplicity from complexity. The text is taken from William Blake's poem *The Clod and the Pebble*, which presents two different views about love (Blake (1967)). The piece is written for two soprano soloists, who each first sing through a stanza of the poem in its entirety. Then, the words of the first stanza are broken up into syllables and distributed among the two singers. The end effect is that each singer's text is gibberish—a very cognitively difficult part—but the audience's perception is of one voice broken into two parts with a simple melodic/harmonic structure.

Singers should strive to match their tone as much as possible to contribute to the illusion of a single voice. The accompaniment here is not specified, but can be performed on any instrument capable of playing chords – for instance, a guitar or piano, and should be kept light.

A.2 Vocal Score

The Clod and the Pebble

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