6-1-1996

A Mathematician among the Molasses Barrels: Maclaurin's Unpublished Memoir on Volumes

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Proceedings of the Edinburgh Mathematical Society (1996) 39, 193–240 ©

A MATHEMATICIAN AMONG THE MOLASSES BARRELS: MACLAURIN'S UNPUBLISHED MEMOIR ON VOLUMES

INTRODUCTION: MACLAURIN'S MEMOIR AND ITS PLACE IN EIGHTEENTH-CENTURY SCOTLAND

by JUDITH V. GRABINER

Suppose we are given a solid of revolution generated by a conic section. Slice out a frustum of the solid [**14**, diagrams pp. 77, 80]. Then, construct a cylinder, with the same height as the frustum, whose diameter coincides with the diameter of the frustum at the midpoint of its height. What is the difference between the volume of the frustum and the volume of this cylinder? Does this difference depend on where in the solid the frustum is taken?

The beautiful theorems which answer these questions first appear in a 1735 manuscript by Colin Maclaurin (1698–1746). This manuscript [**14**], the only original mathematical work by Maclaurin not previously printed, is published here for the first time, with the permission of the Trustees of the National Library of Scotland. (An almost identical copy [**15**] exists in the Edinburgh University Library.) In this work, Maclaurin proved that the difference between the cylinder constructed as above and the frustum of the given solid depends only on the height of the frustum, not the position of the frustum in the solid. When the solid is a cone, Maclaurin showed that its frustum exceeds the corresponding cylinder by one fourth the volume of a similar cone with the same height. For a sphere, the cylinder exceeds the frustum by one half the volume of the sphere whose diameter is equal to the height of the frustum; this holds, he observed, for all spheres. He derived analogous results for the ellipsoid and hyperboloid of revolution. Finally, for the paraboloid of revolution, he proved that the cylinder is precisely equal to the frustum.

These results resemble some propositions from classical geometry, especially Archimedes' theorem that the difference between the volume of a cylinder with height and diameter equal and the volume of a sphere with the same diameter is a cone with that height and diameter. The volumes of the cone, sphere, and cylinder stand to one another in the ratio 1:2:3. Maclaurin's friend, the philosopher of aesthetics Francis Hutcheson, gave Archimedes' result as a key example of mathematical beauty [**10**, p. 49]. In fact Maclaurin later proved his theorems geometrically in his *Treatise of Fluxions* [**16**, pp. 24–27], where he extended them to frustums terminated by planes oblique to the axis of the solid. But these theorems originated in a far different setting: the present manuscript, a memoir addressed to the Commissioners of Excise for Scotland, that describes how to find the volume of molasses contained in barrels at Glasgow.

In this manuscript, after having explained how excise officers could use his results on frustums to measure the contents of barrels, Maclaurin proved the theorems by means of Newtonian calculus. Struck by the beauty and novelty of these results, he wrote, "It is a remarkable property of all the solids which can be generated by the Revolution of any Conick section upon either of its two Axes, that the Content of a frustum of a given Altitude in a Vessel of a given Species terminated by planes perpendicular to that Axis differs from the Contents of a Cylinder of the same Altitude upon a base equal to the Section of the frustum through the midle of its depth by a given or invariable quantity". [**14**, p. 81; spelling as in the original] Since these results were meant for measuring barrels, he included a formula going from the dimensions of the solids given in inches to the difference between the frustum and the cylinder expressed in English gallons.

As readers of Maclaurin's memoir will see, he did much more than prove the theorems we have stated. His work is an impressive piece of applied mathematics. Maclaurin concentrated not on mathematically-idealized barrels, nor on barrels of arbitrary geometrical shapes, but on the barrels used in 1735 to hold molasses in the Port of Glasgow. His purpose was to show how to set up a table for each barrel that would allow the excise officer to find the volume of molasses in that barrel simply by using a dipstick to measure the number of "dry inches" from a carefully-chosen place on the brim to the surface of the liquid. The twenty-eight propositions in the memoir are designed to achieve this goal.

He began by asserting that the barrels that hold molasses at Glasgow "are nearly frustums of Cones that stand on their lesser Base". [**14**, p. 2] The barrels were typically about five feet high and about six feet across, and could hold between 800 and 1,000 gallons of molasses. It would be easy to find the volume of molasses in any such barrel by means of a dipstick, if the barrels were actually frustums of cones and if they stood exactly level. But this was not the case. Maclaurin carefully analyzed and classified the irregularities which he observed "in taking the Dimensions and computing the contents" of many barrels at Glasgow, and gave strategies to compensate for them.

Maclaurin followed a practice, common among gaugers [**11**, p. 126], of finding the volume of a barrel by treating it as the sum of slices of some fixed height—ten inches was often chosen—and then approximating the volume of each slice by the volume of a cylinder with the same height and with diameter equal to the diameter of the slice halfway between its top and bottom.* Finding the difference between the slice and the approximating cylinder, which Maclaurin's theorems let him do, was thus crucial for his purpose.

But the slice method alone was not enough to deal with the ways these barrels departed from precise mathematical shapes. The irregularities were such that Maclaurin needed to decide which values to use for the diameters and heights of each barrel. One significant problem was dealing with "irregularities of the brim" resulting from staves in the same barrel having different heights. Others included interior bottoms which were not flat (some were concave down, more were concave up), individual staves which stuck out farther than others or which were unusually close to the axis of the barrel,

*The leftover part of the barrel had to be measured in some other way. See, for instance, [**14**, pp. 20–21].

and the fact that these barrels were made with a little hollow, called a chine, around the inside of the bottom [**14**, diagram, p. 22]. In addition, the barrels did not usually stand level, but were tilted in order to make them easier to clean, the downside being called the "drip". In his analysis of these and other irregularities, Maclaurin measured dozens of barrels, and also observed the way coopers made barrels and how the barrels' shapes changed after years of use.

Maclaurin used his mathematical knowledge in many ways to deal with the irregularities. For instance, when there was a drip, he said it was better to measure the depth of the liquid from the places where the opposite dipstick readings were equal than to average the greatest and least dip. This was because it is hard to identify the greatest and least values since "the dip does not vary sensibly" in their neighborhoods, but easy to identify the places where the opposite dipstick readings are equal because the depth of the liquid is changing fastest there [**14**, p. 52]. Finally, after explaining how to choose the appropriate measurements for the barrels, he showed how to apply his theorems to calculate the volumes.

Maclaurin emphasized both the novelty of his theorems on volumes and their usefulness, saying "I have insisted the more on these last four propositions, that these things have not been observed by the writers on this Subject, nor by the Geometers themselves; and because the knowledge of them may...be of use to those who have often occasion to measure solids". [**14**, p. 81]

Indeed, Maclaurin made the results useful for the excise officers. Often after doing the mathematics, he gave a simple formula or sample calculation that the officers could follow. And Maclaurin's work was put to use for its intended purpose. His biographer Patrick Murdoch wrote in 1748 that Maclaurin had applied himself "to terminate some disputes of consequence, that had arisen at Glasgow concerning the gauging of vessels; and for that purpose, presented to the commissioners of excise two elaborate memorials [presumably [**14**] and [**15**]], containing the rules by which the officers now act, with their demonstrations". [**19**, p. xix]

But why on earth, of all the possible problems in applied mathematics, did Maclaurin choose the volumes of molasses barrels in Glasgow? In 1735 he was in mid-career at Edinburgh. He taught six hours a day, was involved in founding the Edinburgh Philosophical Society and otherwise gracing the intellectual life of the capital, and had just begun writing the *Treatise of Fluxions* in response to Bishop Berkeley's attack on the calculus in 1734. One would think that Maclaurin had better things to do than to stroll along the quays of Glasgow and wonder about the volumes of the barrels he saw there to the extent of measuring their dimensions (it must have been a sticky job), observing their construction, and calculating dozens of tables of dry and wet inches for a handful of excise collectors.

But Maclaurin was not a head-in-the-clouds pure mathematician; he was attuned to practical mathematics and involved in it, and he associated with many people influential in the governing of Scotland and the raising of its revenue. As we shall see, the problems Maclaurin's memoir treated were central to his society.

By the eighteenth century, sugar products had become major components of the economy of Great Britain [**24**, p. 81; **25**, pp. 344, 347–8]. The Treaty of Union of

England and Scotland in 1707 opened all sorts of economic opportunities to the Scots, giving them freedom of trade and navigation to the entire United Kingdom and to the "dominions and Plantations thereunto belonging". [**12**, p. 58] It was the international trade in sugar and its byproducts, molasses and rum, that built the first great fortunes in Glasgow [**8**, p. 261; **6**, p. 53]. We must acknowledge also that, although the Glasgow merchants had no direct involvement in the slave trade, the growing market in sugar, molasses, and rum was based on slavery. As the demand in Britain and elsewhere made sugar more profitable and thus more intensively cultivated, the British West Indies saw a sugar revolution: one of monoculture, large estates, and the "industrial-type discipline of gang slavery". [**17**, p. 156; **25**, p. 242] Port Glasgow's trade with the West Indies began in 1732. By 1735, four ships from Glasgow were trading to Jamaica, one to Barbados, one to Antigua, and two to St. Christophe [**25**, p. 31]. By 1736 there were five sugar refineries in Glasgow [**25**, p. 31].

Why was Glasgow, in particular, the centre of success in the molasses trade? Glasgow is closer to the Americas than are the chief English ports, and insurance was cheaper because Glasgow was not subject to attacks from the Continent. The successful trade was also promoted by the availability of capital from the Royal Bank of Scotland, founded in 1727, and an enterprising and able merchant class [**20**, p. 9]. The economic value of the molasses trade was increased further by the passage of the Molasses Act of 1733. Since the Act prohibited French sugar, rum, and molasses from entering Ireland, it promoted an Irish move from the consumption of French spirits to British rum. For the trade with Ireland, Glasgow was ideally situated.

Such prosperous and successful trading was clearly a potential source of funds for the Crown, which depended heavily on customs and excise revenues throughout the eighteenth century. After 1700 customs and excise provided between two thirds and three fourths of government revenue, with excise, which brought in between 1.5 and 2 million pounds, accounting for more than the 1 to 1.5 million pounds produced by customs [**3**, p. 188]. Such revenue financed the founding of the Bank of England in 1694: the bankers lent 1.2 million at 8% to the government, with the interest guaranteed from customs and excise duties. This gave the government a reliable source of funds and thus "unprecedented power and flexibility in the conduct of foreign policy and war". [**9**, p. xix]

The excise revenues, then, were essential to the British government, and the Scots were required to pay their share. After the Union in 1707, the Scots fell under the customs and excise laws of the new Britain. The new burden was quite large, and the Scots responded with increasing levels of smuggling against which laboured "a despairing customs and excise service". [**12**, p. 61] There was also widespread fraud. In addition, serious political unrest arose out of the resistance to taxes and to crackdowns on smuggling. For instance, the Shawfield riots in Glasgow in 1725 protested Walpole's attempt to tax Scottish malt, and the Porteous riots in 1736 in Edinburgh, in which a soldier was hanged by a mob, began because of the execution of a smuggler.

Furthermore, even honest tax collectors had trouble gauging accurately, because of the inaccuracy of calibration of dipsticks, the unreliability of standard measures, and above all the imperfection of the approximations commonly used for volumes [**2**, pp.

24n, 110]. The problems of gauging provoked a good deal of literature before Maclaurin's day. Kepler wrote the *Nova Stereometria doliorum vinariorum* (1615), an important document in the prehistory of the integral calculus, to test some approximations used by wine merchants in Germany. In Britain writers on gauging in the seventeenth and eighteenth centuries included William Oughtred, Henry Phillipes, Thomas Everhard, and Charles Leadbetter, all of whom gave various formulas for approximating the volumes of solids. We know that these mathematical methods were used for tax collection purposes. A good witness is Crouch's *Complete View of the British Customs* (1724) [**5**], which used formulas published earlier by Oughtred and Everard for determining volumes and then told how to calculate the taxes from these volumes. Leadbetter, whose book [**11**] became a bible for the customs and excise services in London in the eighteenth century, gave many examples of finding the volumes of barrels by the method of slices we have already described.

In the 1680's, David Gregory wrote, though he did not publish, a *Treatise of Practical Geometry*, which included a discussion of finding volumes in ways useful for traders and tax officials. Maclaurin himself published Gregory's work in 1745, with some additions of his own which briefly describe how to find the volume of a "round vessel" by the use of approximating cylinders. The method is exact, Maclaurin noted, only when the barrel "is a portion of a parabolic conoid", but the error is "not considerable" in those vessels "as are in common use". [**7**, pp. 146–147] Maclaurin justified this by stating, though not proving, the theorems we have already described. Maclaurin's unpublished memoir of 1735, however, appears more sophisticated than all the others we have cited.

The gauging of barrels is not the only instance in which Maclaurin pursued the practical connections of mathematics. For example, he worked on mapping various islands north of Scotland to search for a northeast passage. He also did the actuarial calculations for the "Scheme for providing an Annuity to Minister's [*sic*] Widows and a Stock for their Children" in Scotland [**18**, pp. 105–108]. In discussing the history of mathematics he called attention to the empirical origins of geometry and arithmetic in the land-measurements of Egypt and the mercantile calculations of the Phoenicians [**13**, quoted in **21**, pp. 96–97]. And he had planned, before his untimely death, to write a mathematically sound work on practical mathematics [**19**, p. xix].

Maclaurin had many connections to Glasgow. He had been to university there, and had bought a nearby farm as a country retreat at Muirhouse, near Hamilton, in 1730. His cousin Neil Campbell was Principal of the University of Glasgow, and his brother John was a prominent minister there [**1**, p. 5].

He also had political connections with the Excise Commission. Among the five excise commissioners in 1735, who sat in Edinburgh where Maclaurin was Professor of Mathematics, were Richard Dowdeswell and Thomas Cochrane. (The others were Gilbert Burnet, Richard Somers, and Christopher Wyvil.) [**23**; **22**, pp. 83–86] The central power struggle in Scottish politics at this time pitted the followers of the Argyll family, the "Argathelians", against the group known as the "Squadrone". Control over the Excise Commission was an important battleground, and both Dowdeswell and Cochrane were supporters of the Argathelians [**22**, p. 83]. So was Maclaurin. The

Treatise of Fluxions, Maclaurin's major work, is dedicated to the second Duke of Argyll. (Compare Maclaurin's eulogistic remarks on the death of the Duke in 1743 [**18**, p. 111].) Maclaurin also sought and enjoyed the patronage of Archibald Campbell, Earl of Islay, the Duke's brother [**18**, pp. 32, 46, 60, 171], who inherited the title in 1743 and continued the Argathelian exercise of power. Helping the Excise Commission, then, was good politics for Maclaurin and for his friends.

More generally, Maclaurin's work on this problem embodies the ideal of "improvement", so prevalent in 18th-century Scotland, in which scientific and technical expertise could, and should, be applied to solve problems of economic and social importance, especially for this new and undeveloped North Britain [**4**, esp. pp. 127–134]. Finally, there is a key idea of the Enlightenment: that conflict and contention could be alleviated through applying reason and rationality, that disputes should yield to scientific knowledge. Maclaurin's "Memorial to the Honourable Commissioners of Excise" exemplifies this view. All of this suggests that a mathematician in Scotland, skilled in geometry, concerned with the economic success of Scotland and the political success of the Excise Commission, might want to turn his considerable skills to the problem of correctly determining the excise and customs duties on an economically important commodity.

Students of the history of the practice of customs and excise in the eighteenth century will find much of interest in this manuscript. It also sheds new light on the thought and practice of the leading mathematician of eighteenth-century Scotland. As we read the many pages analysing the problem of gauging the amount of molasses in a barrel, we may agree with what Maclaurin wrote in 1731 to his friend Martin Folkes: "In what I do I am apt to do it with keenness". [**18**, p. 32] This manuscript does more than exhibit Maclaurin's mathematical and analytical abilities and introduce and prove some beautiful theorems. It also shows Maclaurin learning from traders and tax collectors as well as from Archimedes and Newton, and applying his learning to solve a problem central, both politically and economically, for Scottish society.

Acknowledgements. I thank Alex Craik of St. Andrews University for his encouragement and suggestions. I also thank the Trustees of the National Library of Scotland, the Rare Book Department of the Edinburgh University Library, and the Flora Sanborn Pitzer Professorship for its generous support.

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COLIN MACLAURIN, *Memorial offered to the Honourable Commissioners of Excise concerning the Mensuration of Tuns or Backs that have some irregularity in the Figure and Situation of the Bottom, and in the Height and Position of the Staves; and of ascertaining a just place for taking the dry Inches in such a Vessel: To which is added a Method of correcting the common Tables, and some new Theorems concerning the Mensuration of regular Solids.* 1735.

Transcribed from the manuscript in the National Library of Scotland: Adv MS 23.1.13. Published with the permission of, and gratitude to, the Trustees of the National Library of Scotland.

A note on the transcription and the manuscript. The manuscript consists of 94 folio-sized pages, in a copper-plate hand which is not Maclaurin's. Bracketed numerals, e.g. [1], [2], ..., [94] indicate the page of the manuscript. The transcription reproduces the spelling, capitalization, and punctuation of the original.

Most of the manuscript is self-explanatory. The footnotes to the text are Maclaurin's. When I have added a comment, I have enclosed it in square brackets. In the interest of readability I have tried to use "sic" only when needed to clarify meaning, rather than remark on such variant (and inconsistent) spellings as "Vessell", "cheif", "ellypticall", or the mixing of "I" and "l" in numbering lists.

Another manuscript version of Maclaurin's *Memorial*... exists in the University of Edinburgh Library, call number DC.1.17. This version is in a different hand, but is otherwise virtually identical in content and wording to the one in the National Library of Scotland, though there are minor differences in spelling, capitalization, punctuation, and handwriting size (the copyist of the Edinburgh University version used only seventy-two folio pages). The two manuscripts appear to have been made from the same original (which we do not have), rather than one having been copied from the other.*

*The NLS copy reveals its dependence on the original because the original evidently came with numbered diagrams on two separate pages, with designations like "Plate 2, fig. 1" in the text to indicate where the diagrams should go; the NLS copyist has reproduced those designations along with the diagrams. The EUL copy does not include those designations. The EUL copy shows its dependence on the original by being correct on two occasions where the NLS copyist has introduced minor errors, in the drawing of a line in the last of the diagrams and in not completing the vinculum in an algebraic expression; these are pointed out in square brackets at the appropriate place in the transcription. But the occurrence of other, identical minor errors (also pointed out where they occur) in both versions argues for a common original containing them.

[Title page, unnumbered]

Memorial offered to the Honourable Commissioners of Excise concerning the Mensuration of Tuns or Backs that have some irregularity in the Figure and Situation of the Bottom, and in the Height and Position of the Staves; and of ascertaining a just place for taking the dry Inches in such a Vessel: To which is added a Method of correcting the common Tables, and some new Theorems concerning the Mensuration of regular Solids.

[1] When a Vessel is perfectly regular in its Figure and has its Bottom levell, it is easy to measure its content, and to take the dry Inches by the common Methods: Severall Deviations from a regular Figure are subjected to Computation by the usual practice of taking two Perpendicular Diameters at the middle of every ten inches of the Depth: and by taking the dry Inches at the Quarter from the Drip [*i.e.*, putting the dipstick, when the barrel is tilted, at a point 90° along the circumference from the lowest point or "drip"], some Errors are avoided which would arise from the Inclination of the Vessel to one side. But tho' these Rules are very good, yet they are not sufficient, without some further precautions to prevent Errors in measuring the Content of a Vessell that has a sensible irregularity in the Figure of the Bottom or in the Height and position of the Staves; The latter of which in some Degree is found in most Vessels, and both are generally found in such as are old and have been altered or mended, An equitable and skilfull officer may ascertain the Content of such Backs [Maclaurin uses the terms "Tuns" or "Backs" interchangeably for barrels shaped like the frustum of a cone] near the Truth by taking such precautions as his own Judgment may suggest to him; But it may be inconvenient to leave the Method of proceeding in such cases altogether in the Officer's choise: [2] A Difference of the Contents may arise from different Methods, which may be equally good in measuring a regular Vessel, while one or perhaps both of them may be improper for measuring a Vessel that has certain irregularitys in its Figure. The sum of what I observed in taking the Dimensions and computing the Contents of the Backs at Glasgow is contained in the following Propositions, where I have explained the Method which I followed, with the Reasons that induced me to chuse it, and some Cautions that may be necessary in applying it to Vessells more irregular than these are. Had I had Occasion to consider Vessels of other Kinds than these Backs perhaps I should have been able to offer something more complete on this Subject, As I shall have these and others of the same kind cheifly in my Veiw, I shall begin with a Description of them.

The Backs which serve for holding the Molasses at Glasgow are nearly Frustums of Cones that stand on their lesser Base. Their irregularities may be reduced to these Heads.

I°. The Staves are not equall in Height, some of them exceeding others by a very sensible Difference, and this produces what we shall call the irregu- [3] larity of the Brim or Lip. 2°. The bottom is not a regular plane, but is hollow, in most of them at or near the Center. In two or three of the Backs the Center rises a little, this we call the Irregularity of the bottom. 3°. The Staves in many of them are not in a regular position, some stand out considerably from the generall Contour of the Vessel, so as to fall without the ark formed by the adjoining Staves. Others stand in towards the Axis of the Vessel, being what the Coopers call proud Staves. This we shall call the Irregularity of the Staves, which I found to be always greatest near the Brim of the Vessell. 4°. In many of the Tuns the bottom is not levell, but inclines to one side and the Tun has what is called a Drip: Besides these, there are also some other irregularitys which I endeavoured to account for that are of much less effect in these Vessels than the four I have already mentioned. 5°. The opposite Staves do not always form equall angles with the bottom; but as this Error affects the Diameters much more than the Altitudes it is almost all accounted for by taking proper cross Diameters. However we had still further regard to it as will appear afterwards. 6°. The bottom was not only hollow in the middle, but its Circumference was not [4] all in one exact plane; a Consequence of which was that the rise and Drip were not exactly opposite. 7°. There might have been some small deviation from a perfect right line in those lines which we drew from the upper to the lower Quarters of the Back; but we prevented any error that might arise from this by comparing the depth of the Slice† that came out after the division with what it ought to have been and if they differed we brought them to agree by a new division. 8°. There is a Chine or little hollow that lyes alongst the Circumference of the Bottom on all the Vessels we have measured, only one expected where it was scarce perceptible. I need not take notice of the Ellipticall figure of the Sections of the Vessells, because the Effect of that figure is accounted for sufficiently in the common Method, by taking a mean betwixt the perpendicular Diameters, when their difference does not exceed four Inches; and we never found it so great but in one Vessel only.* How we endeavoured to gaurd against the Errors that might arise from any of those irregularities especially the first four which are the most considerable, will appear from the following Propositions.

[5] **Prop: I.**

In measuring a Tun or Back, the end we are to have in veiw is to obtain a Table so adapted to the Vessell that the Content shown by the Table at any Number of dry Inches may be equall to what is contained in the Vessell, when the same Number of dry inches are taken by the Gauging Officer.

This is too evident to need any Illustration, for by this means the Charges drawn from the Table are true. But as it is impossible to arrive at Mathematical preciseness in any Mensuration, so we are to take care that the Content in the Table be rather a little under than above the Content in the Vessell, that the turn of the Scale may be given to the Trader, according to the generall practice in the Excise.

†By the Slice I understand the frustum whose Depth is the Excess of the depth of the usual ["vessel" is meant] above 40, 20, 60, or any member ["number" (in the sense of "multiple") is meant] of ten Inches, for example if the Depth of the Vessell is 48 inches it is said to consist of 4 frustums of 10 inches depth and a Slice of 8 inches depth.

*Div: I. No. 2. [Maclaurin identifies individual barrels in the port by giving the Division number, and then the number of that barrel in the Division.]

Prop. II[d].

The totall Content of a Tun or Back ought to shew the Number of English Gallons which that Tun would hold if it was fill'd up to the dipping place, supposing the brim to be rais'd, if needfull, to the levell of that Place.

[6] For the totall Content is what the Table shews to be held in the Vessell when no Inches are dry, that is, when it is quite full to the place where the Gauging Officer takes his dip. If the brim of a Vessell is levell it is indifferent from what part of this Brim he take the dry inches: but if some parts of the brim rise above the levell of other parts, he must dip at that part which answers to the tabled Content of the back; that is, he must dip at a place of such an Altitude, that if the Back was fill'd with Liquor to that height, the quantity of Liquor filled in would be equal to the total Content of the Back, that is assign'd in the Table; suppose for Example, that the totall Content of the Back assigned in the Table is a 1000 Gallons; and that 1000 Gallons are fill'd into the Back, the Brim being supposed to be rais'd that it may hold this quantity, then the dipping place ought to be at the Surface of this Liquor.

Prop. III.

In a Vessell that has higher and lower Staves or an irregular Brim, a greater and lesser Content may be equally true, if the dipping places be higher or lower in a just proportion.

[7] This easily follows from the last; for each Content is true if it be equall to what the Vessell would hold when fill'd up to the levell of the dipping place ascertained for that Content, and when this place is higher, it is manifest that the Content ought to be greater. The difference of two such true Contents is equal to the quantity of Liquor which the Vessel would hold betwixt the levell of the two dipping places; and may be computed by multiplying the Area at or near the Brim by the Difference betwixt the Altitudes of the dipping places. If the difference of two Contents assigned for the same Back at different Altitudes be not nearly equall to this product, they cannot both be right.

Prop. IV.

In a Tun that is either fix'd [i.e., in position] *or is seldom altered, it is necessary to pitch upon a dipping place, which when the position of the Dip* [presumably "Drip" is intended] *is altered, must be again examined and altered if necessary.*

If the Vessell be so regular in its Figure and Situation as to have the Brim and bottom parallel to each other and levell, it is then of no Consequence where the Officer dips, but I have not met with any [8] such Tuns; and I doubt if any such are made for use: I find some parts of the Brim are always higher than others. There is but one particular height of the dipping place to which any Table can answer, and tho' the bottom of the Tun was level, if the Gauging Officer dip at a part of the Brim that has a greater altitude than this, the Charge from the Table will be below the truth; and if he dip at a lower part of the Brim he must have too few dry Inches, and will overcharge the Trader. In a Tun that is seldom moved, as well as in a fixed Tun, this ought not to be left to chance, or to the Officer's choice: It is not enough to direct him to dip at the quarters from the drip; for in the case we are now supposing, when the Bottom is levell, there is no drip, and yet the irregularity of the Brim may lead him into considerable Errors.

It is excuseable to have no determined place for dipping in such Tuns as are not only moveable, but actually are often moved, perhaps every time they are emptied: In these indeed it must be left to the Officer's Judgement where to dip, who, if he knows and minds his duty, will not keep too strictly to the common Rule of dipping at the quarters from the drip, when the Brim is very regular, for if these Quarters hit upon a lower Stave than that which agrees with the computed content of the Vessell he will wrong the Trader, and [9] if these quarters hit upon a higher Stave, he may charge him below the Truth. I need not observe that I am speaking of that case only when the Charge is deduced from the dry Inches, and when the Duty is so high that an Error which otherwise might be inconsiderable, becomes of some Moment.

I suppose some have been induced to leave the dipping places undetermined in the Backs at Glasgow, because they considered them as moveable; but I am well informed that these Backs are not moved for several years together, and that when they are moved, it is easy for the Officer to discover it: In which case he is to adapt his dipping place to its new Situation by an easy practicall Rule* we shall explain afterwards. This I am sure of, that if the Gauging Officer have no other Direction given him but to dip at the quarters from the drip, without regard to what Staves these quarters may hit upon, and does not supply this neglect by more than ordinary Caution and Skill, he may charge 10 or 15 Gallons wide from the true Content in some of these Backs, when the Vessell is fill'd within five or six inches of the Brim. Nay if this Table be computed from some mensurations [10] that have been actually made of those Backs, he may err by a much greater quantity.

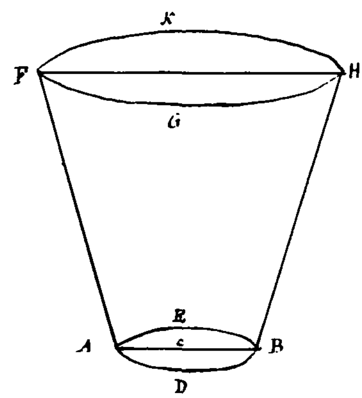
I did not however compute the totall Content of these Backs as if they were to remain always in their present Situation; but endeavoured to find the best mean Content that will answer nearly to the true Contents in the different Situations which the Trader may give them, which is described in the following proposition.

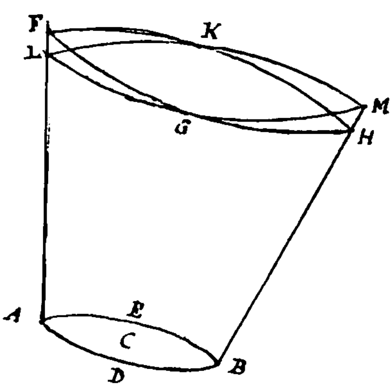
Prop. V.

The totall Content of a Back that is sometimes moved ought to be the Number of English Gallons which it would hold if the Circumference of its bottom was brought as near to a levell as possible, and it was fill'd up to the levell of the dipping place.

It is evident that this is the best mean Content that can be assigned for a tun that is sometimes moved; and any Error that might arise from the Tun's inclining to one side is prevented by adjusting the dipping place to its new Situation.

*See prop. VI.





[Along with each diagram, designations like "Tab. I. fig. 1" occur in the margin of the National Library of Scotland version of Maclaurin's memoir, apparently referring to the source of the diagrams; the diagrams themselves were then copied into the margins of the manuscript. As Maclaurin presumably intended, these designations do not occur in the Edinburgh University Library version, and are omitted from the present transcription.]

Let FACBH be a Section of a Tun by a plane thro' its Axis; AF, BH the two opposite sides, [11] ACB the common Section of that plane and the bottom of the Back. Suppose the Circumference of the bottom ADBE to be brought as near a levell as possible, and the Vessell to be fill'd up to the levell of the dipping place FGHK, Then the best Content that can be assigned for this back is the quantity of Liquor it now holds, and when the Back is in this Situation, the dry Inches may be taken from any part of the Circumference FGHK.

Suppose this same Back to be inclined to one side so as to have its drip at B, its rise at A, and the quarters from the Drip at D and E. Let K and G be the points directly over above E and D. then the same Content will serve for this Tun in this new Situation, if the dipping place be at G or K. For if you now Suppose this same quantity of Liquor to be fill'd in as before, it will rise to the quarters K and G in the new Situation of the Vessell nearly and there will be no dry Inches at those parts, The Liquor indeed will rise higher than H as far as to M upon the Stave BH and it will subside below F as far as L on the opposite Stave; Therefore the same Content will not serve if the Gauging Officer dip at F or H, but if he dip at K or G it will be right. [Apparently the dipstick is placed parallel to the side of the vessel, giving what he calls the "slope altitude", p. 14.] [12] By taking the dry inches at the quarters from the Drip, you will not be oblidged to alter the dipping place when the drip is made greater or less, the Vessell inclining the same way as before. It is to be observed that a certain lesser Content & a Table computed from it will serve as well, if the Officer dips at H over the drip, and a greater Content if he dips at F above the rise. But in these cases when either the quantity of the drip is altered, or the place of the Vessell where it lys, the dipping place must be altered to adjust it to the Table; whereas by taking the dipping place at the quarters from the drip, it is only to be altered in the latter of these cases.

The Surface FGHK to which the Liquor would rise, if the bottom was brought as near to a levell as possible may be called the plane of the dipping place, because this place ought always to be on that Surface at or near the Quarters from the drip. If we were to require great accuracy it ought to be a little from the quarters towards the drip; but in so small inclinations as are given to the Backs it is needless to be so precise: What is more materiall is, that the brim of the Vessell being irregular, we must not suppose it to coin- [13] cide with this plane FGHK which may in some places fall below the Brim, in others rise above it. If the Circumference of the bottom be in one plane (tho' its middle be hollow) then the Surface FGHK is parallel to the plane of this Circumference, and is always known by setting the same Altitude from the Circumference of the Bottom. But the best practicall Rule for finding the dipping place that agrees to a given Content and Table, seems to be that which follows.

Prop. VI.

A Content and Table being given to find just place for dipping for the dry inches; fill into the Vessell any quantity that is assigned near the lower end of the Table which is sufficient to cover the bottom of the Vessell; take the dipping place near the quarters from the drip at such an Altitude that the dry inches at that place may be equall to those that correspond in the Table to the quantity of Liquor filled in.

When this Altitude falls short of the brim, or, exceeds it at the quarters, a Slice may be taken off the Brim or nail'd upon it to bring them to an equality; or a Nail answering to that Altitude may be [14] fixed upon the brim to shew the gauging Officer from what point he ought to take his dry inches. In Vessells which I measured, the dipping places may also be found by setting off from the Bottom at the quarters of the drip the same slope Altitude which I took in my mensurations for determining the Content.

Prop. VIIth.

The Mensuration is much better regulated by the Circumference of the bottom (or a plane parallel to it passing through some point at the top) than by the brim of the Tun.

There are severall reasons which induced me to regulate the mensuration rather by the bottom than by the brim of the Tun. 1°. The bottom is always one of the Surfaces which bounds or terminates the quantity of Liquor and the Brim never is. Parts may be taken of ["off" is intended] the brim and no way affect the quantity of Liquor contained in the Vessell; but any alteration in the bottom affects it. If the bottom and sides be regular the mensuration may proceed regularly, however irregular the brim may be; and all the irregularity of the Brim is accounted for by fixing a proper dipping place, and that being done, it is of no Con- [15] sequence whether the Brim be in one plane, or whether some parts rise a foot or two higher than others. 2°. The Circumference of the bottom is generally the most regular Section of the Vessell. When the Vessels are new the Coopers generally make the bottom a regular circular plane; but they are not at so much pains to make the Staves of an equall height and the brim is somewhat irregular even in a new Vessell. In process of Time the bottom becomes hollow, or perhaps rises at the midle in some cases; but its Circumference even then varys little from one plane. In the Vessells I measured I found by the dips of the Water that it varied so little that the Error was easily accounted for, as will appear afterwards; But the Errors in the upper parts of the Vessell by stricking [refastening higher and tighter] up the hoops, and the irregular position of the Staves, become much more considerable; and it was evident to the Eye as well as from Experiment, that the Contour of the Vessell was much more regular below than above. The Bottom of Vessells becomes a little ellypticall by the swelling or shrinking of the bottom Staves; but it approaches generally much nearer to a Circle than the Brim of the upper Sections. This will appear by comparing the difference of the uppermost cross Diameters with the difference of the [16] lowest. I shall subjoin an Example or two.

Div: 1 N°. I.
The uppermost Diameters are
72.7 & 71.0. Diff: 1.7.
The lower Diameters are
65.7 & 65.5. Diff: 0.2.

Div: I. N°.2.
Uppermost Diameters
73.9—69.7. Diff: 4.2.
Lower Diameters
66.4—64.9. Diff: 1.5.

This was the Elliptick Back.

Div: II^{d}. N°.1.
Uppermost Diameters
85.6—83.1. Diff. 2.5.
Lower Diameters
75.9—75.1. Diff: 0.8.
Div. III^{d}.N°.1.
Uppermost Diameters
64.2—62.8. Diff. 1.4.
Lower Diameters
57.0 large—56.9. Diff. 0.1.
[17] Div: III^{d} N°.2.
Diff: of uppermost 1.2.
Diff. of the Lowest 0.3.

It is needless to give more Examples. I do not say that this always holds, but there are only two Vessells in which I have found it otherwise.

3°. If we should draw strings or Lines over the Brim, and descending by plum Lines, take the Diameters parallel to these Strings, it would be difficult to avoid some small Errors that would bring out the Content too great, and it would be hard to adjust a dipping place to this Content with as much exactness as in the other way. If one end of the String be fixed on the top of a higher Stave than the other, then the string will not be perpendicular to the axis of the Vessell, nor to the Plum line, except by chance the Vessell incline to one particular side. The Diameters being Oblique to the Axis, all of them will be taken too great, and the Altitude of every Frustum being estimated larger than the truth, the Content will also, on this Account, come out too great, No Rule in Mensuration is more fundamentall than that the Diameters and Altitudes ought to be taken [18] so as to be perpendicular to each other as accuratly as possible; but in the Method we are speaking of, that is obtained only when the String drawn over the Brim of the Vessell is levell which will rarely happen.

4°. It appears from the last proposition that the totall Content which is the best mean from which a Table may be calculated that will answer best the different inclinations of the Vessell, is what it will hold when the Circumference of its bottom is levell, and there is no drip; But by drawing lines over the brim, and regulating the Mensuration by these Lines, this mean is not obtained when the Brim is irregular, and the Content that comes out by this Method cannot well be said to answer exactly to any one Situation of the Vessell, or to any one dipping place.

For these and other Reasons which it would be tedious to describe at greater length, I chose rather to regulate my mensuration by the Circumference of the Bottom than That of the Brim: We therefore first quartered at the bottom, and then from these quarters sett off equall Altitudes to the Brim, and by this means obtained four points (some of which always were on the Brim, [19] some near to it) which lay in a plane nearly paralel to the Bottom; I considered this plane as terminating the Vessell at the top, according to the fifth proposition; for this is the plane to which the Liquor would rise if the Circumference of the Bottom was brought to a levell; And in this plane the dipping place ought always to be placed at the quarters from the Drip, to answer to the Content which I assign. By this Method I got rid of the irregularity of the Brim, and had no further regard to it but to mark by how much the Brim exceeded or fell under any of the four points above mentioned, for which I could make some estimation of the difference there ought to be betwixt my mensuration, and another taken from a higher or lower part of the Brim. After finding these four points, and taking the perpendicular Altitudes, we draw lines from the upper to the lower quarters, and then took the Diameters at the midle of every ten inches of perpendicular Altitude from the said four points downwards to the Bottom.

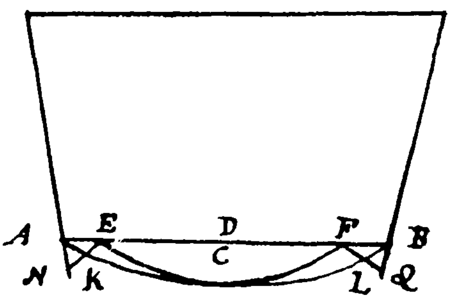
Prop. VIIIth.

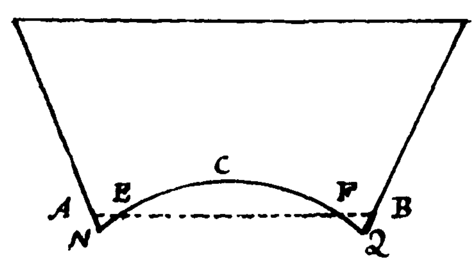
The Content of the irregular part of the Bottom is best determined by filling in [20] *Water by a just Measure till the Bottom be covered, and then computing the lower part of the Slice* [for Maclaurin's definition of this term, see above, p. 4n] *from this water. If we would compute it by a Rule, it ought to be such a Rule as will give the Content of the hollow less than the product of the Area at the Bottom multiply'd by half the depth of the hollow.*

Since the Days of Archimedes till this Time it has been allowed that the best Method to measure the Content of an irregular Vessell, is to fill it with water by some just measure. It is in use among the Officers for finding the Content of the lower part of Stills. It gives no favour to the Trader, and includes the Content of the Chine [the hollow in the bottom, illustrated on p. 21] with the effect of the various bendings of the Bottom Staves; so that it will not admitt of any dispute that this Method is the best for accounting for this irregularity if it be employed with due precaution.

But it would be of Service if we could find some Rule tolerably near the Truth by which the Content of the hollow of the bottom might be estimated by computation. With this Veiw I took the Altitude at the midle of the bottom, by [21] subducting from which the Mean perpendicular Altitude at the side, the remainder gives the depth of the hollow. From the decrease of the Diameters of the Vessell it is easy to find the Diameter at the bottom. Having computed the Area of this Diameter [i.e., the area of the circle with this diameter], I multiplyed it by half the depth of the hollow, I compared the product with the quantity of Water that was received by the hollow, and I find that this product exceeds always that quantity of Water; and this is accounted for from severall reasons.

I°. Let AB be the Diameter of the bottom, AE or BF be the breadth of the Chine (which sometimes exceeded an Inch and a half). Let ECF represent the hollow of the bottom, EF its Diameter, DC its depth; If ECF was a common parabola, then the Content of the hollow would be equall to the product of the area of the Diameter EF multiplied by half the depth DC. But the curve ECF may not be a parabola but may approach a little nearer to a right line, whereby the hollow ECF will be less than that product, and the irregular riseings of the Staves will contribute to this. If AKC was a parabola of a certain kind the Content of the Solid AKCLB [22] would be equall to the





product of the Area at the Bottom multiplyed by 2/5 of the depth of the hollow. [He empirically justifies this approximation for the barrels he is studying; see p. 24 below.]

2°. The Diameter at the Bottom AB is greater than EF sometimes by more than three inches; the product of the area of the Diameter AB multiplied by the half of DC would give the Content of the Figure AKCLB if AKC was a common parabola; But this figure manifestly exceeds the Content of the hollow, the parts CEK, CFL being Timber and receiving no Water. Nor can the Content of the Chine ANE or BQF balance this, except the points N and Q be much lower than the point C. I have computed the Content of a Chine an inch and a half over, and one inch deep, and find it does not come up to the space that would be generated by the Revolution of the Area KCE about the Axis of the Vessell, unless DC be very small; much less can ANK the part of the Chine that falls without the paraboloid AKC amount to that quantity; as it ought to do, to justify the multiplying the Area of the bottom by the 1/2 of DC the depth of the hollow.

In some Vessells where the midle of the bottom rises higher than its Circumference, if we were to compute the Content by a mean betwixt the Alti- [23] tude at the Center, and the Altitude at the side, there ought to be an Abatement from the Content on this

Account; but by the experiment of the Water this Abatement is ballanced by the Chine in a Vessell or two. In another Vessell* the Chine diminishes this Abatement, tho' it does not ballance it. When the Center rises the Chine lessens the Abatement for two Reasons, not only because it receives some Water itself, but also because it diminishes the Diameter of the Crown or riseing ECF.

After all the differences betwixt the Content of the hollow of the bottom as it is found by the Experiment of the Water, and as it is computed by multiplying the Area at the Bottom by half the depth of the hollow, is not considerable, and is below two English Gallons in fourteen of the sixteen Vessells which I measured. This difference amounts only

In Div: I.	N°. I to ——I. Gall.	9.	[*i.e.*, 1.9 gallons]
	N°. 2. to——0.	9.	[*i.e.*, 0.9 gallons, etc.]
Div. 4.	N°. 2.———0.	3.	
	N°. 3.———1.	0.	
	N°. 6.———1.	9.	
	N°. 7.———1.	4.	

[24] How these differences are computed will appear from the Examples in the next proposition.

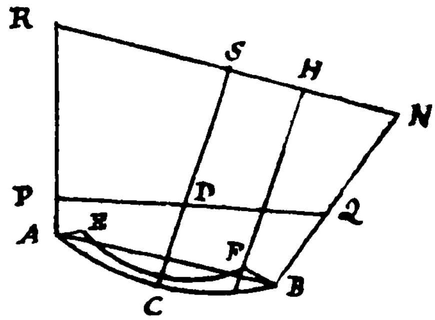
After comparing the different Experiments by the Water, I find the Content of the hollow is nearly equall to the product of the Area at the bottom multiplyed by four tenths of the depth of the hollow; and perhaps this Rule might serve tolerably well for computing the hollow at the bottom when its depth does not exceed two inches. In N°. 1. Div: 1. the Content of the hollow by this Rule is 7.55 [gallons], by the Experiment of the Water it is 7.39. In N°. 6. Div. 4. the Content of the hollow computed by this Rule is 7.56. By the Experiment of the Water it is 7.5.

Prop. IXth.

In computing the totall Content I suppose the Slice at the bottom to be divided into two parts, the upper or dry part, and the lower or wet part, the latter of which is measured by the Water filled in. I find the mean depth of the dry part by two Methods which check each other, either by subducting the Mean of the dips of the Water at the four quarters from the whole depth of the Slice; or by subducting the dip at the Center from [25] *the Altitude at the Center, and throwing away from the Remainder the Altitudes of the upper frustums. Having found this depth I multiply the lowest Area by it, then by adding the whole Water filled in; I obtain the Content of the whole Slice.*

This appears to be the best Method of computing the Content of the Slice, the rather that there are two checks upon it; Let BACBN [RACBN is intended] be a Section of the Slice thro' the Axis of the Vessell and the drip; Let AB be the Diameter of the

*N°. 5. Div. III.



Bottom B the drip, ACB the hollow of the Bottom, AE or FB the breadth of the Chine. Let PQ be the Surface of the Water filled in; which distinguishes the Slice into the upper or dry part RPQN, and the lower or wet part PAECFBQ. The Content of the lower or wet part is equall to the Water filled in. To find the Content of the dry part RPQN substract DC the dip at the Center from SC the depth of the Slice at the Center, and the Remainder SD will give the mean depth of the dry Slice or its depth at the Axis of the Vessell. This depth may also be found by substracting from FH (which is the depth of the Slice at the side) the mean of the [26] dips at the quarters of the bottom; and these two ways may serve as Checks upon one another. The depth DS being found, multiply the lowest area [*i.e.*, the area of the circular cross-section at the bottom of the dry part] by it, and the product shall give the Content of RPQN the dry part of the Slice; which being added to the Water that was filled in, you obtain the Content of the whole Slice RACBN; and this Method is equally good when the midle of the bottom rises as when it is hollow. We suppose the Diameters RN and AB to be perpendicular to the Axis of the Vessell, and not paralel to the horizon except when there is no dip ["drip" is intended; EUL version has the same error]; So that when the Vessell is inclined to one side, the dip at the Center will not be precisely equall to DC the part of the Axis that is under Water; but the difference is altogether inconsiderable in any of the Backs which I measured; for the inclination of the Back must be very great to make it sensible.

It is evident that in this method it is of no Consequence how much Water you fill in, providing it be sufficient to cover the bottom, for when you fill in more Water the dry part of the Slice is diminished by as much as the wet [27] part is increased. Nor is it of any Consequence what Situation the Back has when you measure it in this Method if the inclination be not very great: However if the Vessell can easily be moved it is best to bring the bottom as near a level as you can.

This proposition needs no further proof, but we shall illustrate it by an Example or two. In No. 1. Div. 1. from the Altitude at the Center or 51, I substract the Dip at the Center 1.85 there remains 49.15. which gives 9.15. for the mean Depth of the dry Slice.

To check this I take the Mean of the Dips of the Water at the quarters of the Vessell and find it to be 47 [.47 is intended]. I substract this from the depth of the whole Slice at the side, which is 9.7. (because the mean perpendicular Altitude is 49.7.) and there remains 9.23 for the depth of the dry part of the Slice which exceeds the former computation of it only by 8/100 of an inch. The mean betwixt Computations is 9.19; Therefore I take 9.2. for the depth of the dry slice and multiply it by the lowest Area 14.587. [Here and elsewhere, Maclaurin gives the area in square inches divided by 231, the number of cubic inches in a gallon, so his final result will be in gallons] and by adding to the product the Water that was filled in, which is [28] the wet part of the Slice I find the whole Slice to be 149.08. The Calculation is as follows.

Lowest Area	14.587
Depth of the dry slice.	9.2.
	29174
	131283.
Content of the dry Slice	1342004
Wet Slice or Water filled in	14.88
Content of the whole Slice	149,080,4

This Slice being added to the frustums above it which are computed in the usuall manner, the whole Content of the Back or the quantity which it holds when filled to 49.7 Inches of Altitude from the Circumference of the bottom is 801.19 Gallons.

In order to compute how much water was received into the hollow part of the bottom in this Vessell I multiplied the lowest area

	14.587
by the whole depth of the slice or	9.7
	102109
	131283
And the product is the Content of the Slice without the hollow	141,4929
Subduct this from the Content of the whole slice or	149.08
	141.49
And there remains the Content	7.59

of the hollow or what is allowed for it by my mensuration.

[29] Having thus found the Content of the hollow by the experiment of water, I then tryed how this Content might be computed by a Rule. By the decrease of the Diameters of the Vessell I found that the Diameter at the bottom was 64.75. the Area of which is 14.252 [again, the area $\pi d^2/4$ is divided by 231 in^3/gal.]. [F]rom ["from" in manuscript] the Altitude at the Center 51. I substracted the mean perpendicular Altitude at the side 49.72. And the remainder 1.28 is the depth of the hollow part of the bottom. I computed

this depth another way be [sic] substracting the mean depth at the quarters, 47 from the depth at the Center which was 1.85. and the remainder being 1.38. I take the mean betwixt this and 1.28 viz.' 1.33. for the depth of the hollow, the half of which is 0.66. Having multiplied the Area at the Bottom 14.252. by the half depth of the hollow or 0.66. I found the product to be 9.40. which would have been the Content of the hollow if it had been an exact Paraboloid of that Area and depth. But I observed in the preceeding proposition that the true Content of the hollow is less than the Content of such a paraboloid for severall Reasons. To see how much it is less [30] I substracted 7.59. which is allowed in my mensuration for the hollow from 9.40. the Content of the Paraboloid, and there remains of difference 1.81. gallons.

I then multiplied the area at the bottom by 2/5 or 4/10 of the depth of the hollow (viz.' of 1.33.) which is 53, and I found the product (as in the margin) to be 7.55. which differs from what is allowed by my mensuration for the Content of the hollow by less than .04 of a Gallon.

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14.252
    53
------
  42750
 71260
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 7,55356.
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In like manner I checked my Computations of all the other Vessells, especially as to this irregular part of the bottom by different Methods till I was satisfied that the Contents I had computed were very near the Truth.

An easy and plain Rule for practice is to substract the dip of the Water at the Center from the Altitude at the Center and throwing away the Altitudes of the upper frustums, Consider the Remainder as the depth of the dry part of the Slice; multiply the lowest Area by this remainder, add to the product the Water filled in, and so you obtain the whole Slice. This Method will [31] serve whether the middle of the bottom rise or sink into a hollow whether the Circumference of the bottom be exactly in one plane or not.

Another Example.

In No. 6. Divis: 4 I found the depth of the dry part of the Slice,	
by the Methods described above to be 8.1.	16.094
The lowest Area is 16.094. By multiplying these	8.1
	16094
	128752
I find the Content of the dry part of the Slice to be	130,3614
The wet part of the Slice or the Water filled in	13.90
The Content of the whole Slice is	144.2614.

This I added to the upper frustums computed in the usuall manner, to find the totall Content of the Back.

To find how much Water in this Vessell was received into the hollow of the bottom, I multiplied the lowest Area 16.094 by 8.5. the depth of the Whole Slice at the side, and substracted the product 136.76. from the whole content of the Slice or 144.26. and the remainder 7.5 is what is received by the hollow of the bottom. [32] To see how this would agree with the Rule for Computation above mentioned; from the Altitude of the Center 49.7. I substracted the mean perpendicular Altitude 48.55. (In these Computations it was necessary to have some regard to the hundreths of an Inch) the Remainder 1.15. is the depth of the hollow. By the other Method this depth comes out something greater; so I consider the depth of the hollow as 1.2. The Diameter of the bottom is about 68.1. the Area 15.76. This multiplied by 2/5 or 4/10 of the depth of the hollow, that is by 48 the product is 7.56 as in the Margin [box below] which agrees with the Content found by the Experiment of Water.

Area at the bottom - - -	15.76
2/5 of the depth of the hollow	48
	12608
	6304
Cont^{t}. by this Computation - -	7,5648

I do not however propose this of multiplying the Area at the bottom by 2/5 of the depth of the hollow for finding its Content, as a Rule that may be depended on, especially if the depth exceed two Inches, or be under half an Inch, I only mentioned it as what agrees best [33] with the Content of most of the Vessells I measured as they are discovered by the experiment of the Water. It does not however agree with all of them, nor can it be expected that any exact Rule can be found that will serve for determining what is so irregular as the hollow of these bottoms is. But in all the Vessells which I have measured the Content of the hollow comes out greater than the product of the Area at the bottom multiplied by 1/3 of the depth of the hollow, which, with what was observed in the last proposition of the Diameter of the hollow (which is some inches less than the Diameter of the bottom) gives a presumption that my Measures are not sensibly below the truth.

Prop. X.

When the Circumference of the bottom differs considerably from a plane, the Vessell is then best measured by bringing first the bottom as near a levell as you can, then filling in Water by measure till the bottom be covered or more, and setting of [*f*] *equall Altitudes, not* [34] *from the Circumference of the bottom, but from the Surface of the Water at the four quarters towards the brim, and then proceeding as in the other Method.*

When there is not only a hollow in the midle of the bottom, but a considerable variation from one plane in the Circumference of the bottom; it will then be better to sett off equall Altitudes from the Surface of the Water than from the Circumference of the bottom especially if the Axis of the Vessell can be brought near to a plum line or the bottom to the most levell situation it will admitt of. If great nicety is required, or the Staves stand so as to make Angles sensibly unequall with the bottom; then it will be more accurate if you sett of[f] equall perpendicular Altitudes from the Surface of the Water rather than equall slope Altitudes. But these errors were so inconsiderable in the Backs at Glasgow, that by setting off equall slope Altitudes and taking a mean betwixt the perpendicular Altitudes att the quarters, no sensible errors could arise. And we obtained the Content of the Back [35] when fill'd up, so as to have that mean Altitude wet. By taking a mean betwixt the dips at the quarters and by comparing the mean depth of the dry Slice which it gave with the depth of the same Slice which was got by subducting the dip at the midle of the bottom from the depth of the Slice there; we avoided any errors that could arise from the irregularity of the plane of the Circumference of the bottom.

We have now considered the two first Irregularitys of the Backs; and have shewn how we endeavoured to avoid the Errors they might have led us into. The third we mentioned at the beginning was that of the position of the Staves which stand sometimes out, and sometimes stand in more than the humour or Contour of the Vessell requires. The irregularity may have a great effect if it is not guarded against. In one Vessell* in avoiding unfair Staves, while we were dividing the Section at the brim into six equall parts, one of the Diameters hit upon a Stave that stood out sensibly. To correct it I took two Diameters, one on every side of this Dia- [36] meter upon unexceptionable Staves, and found that each of them fell short of it by a whole Inch. From this and many other instances it appeared that it requires a good dale [sic] of Caution and that a great regard ought to be had to equity in pitching upon fair Staves for quartering these old and irregular Vessells; 'twas this oblidged me often to take four Diameters, and in two instances to take six; But in taking four or more Diameters, care must be taken that they do not hit upon seams, splinters or unjust Staves, equally at the upper part of the Vessell. Some farther precautions relating to this matter are explained in the following propositions.

Prop. XIth.

It is better to quarter at the top from the lower than from the higher parts of the brim.

The Vessells are never filled to the brim, or within five Inches of it; it is therefore of no use in quartering at the Top to pitch upon a high part of the brim. It is even better to quarter at the top from a lower part of the brim, [37] because the upper part of the Vessell is the most irregular, and you may avoid some Errors in the uppermost Diameter by quartering from a lower Stave, and descending upon it. Besides if you

*Div: I. No. 4.

regulate the mensuration by the Altitude of one of the higher Staves, the quarters from the drip at the top will often fall upon lower Staves, and you will wrong the Trader if you do not nail on something to raise the Brim at that place; which does not seem to be so convenient and sure a Method. When we pitch upon one of the lower Staves, it may happen that the dipping place may fall under the brim; but it is easy to cut a Slice off the brim in that Case. When we measure from a higher Stave, it indeed raises the Content, but it is in name only. If Justice is done the Trader by adapting the dipping place to that Mensuration, the charge will be the same as if we had measured from a lower Stave, and it seems to be very needless to alarm him with the name of a great Content, when it is no more: but when a Content is determined by a Mensuration upon [38] a higher Stave; and the Officer takes the dry inches from the top of a lower Stave, 'tis then that a manifest injury is done to the Trader.

It appears to be proper to mark in the Books the Altitude from the Circumference of the Bottom within the Chine which corresponds to the compleat totall Content; for by this means different mensurations may be compared together easily; and it will be of use for finding the dipping place which ought to be at a perpendicular distance from the bottom equall to that Altitude. But when a certain part of the depth of the hollow is added to this Altitude, and the sum only is marked in the books which is considered as a mean Altitude of the whole Vessell, these advantages are lost.

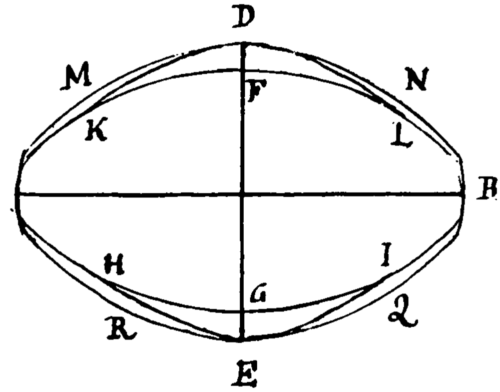
Prop XIIth.

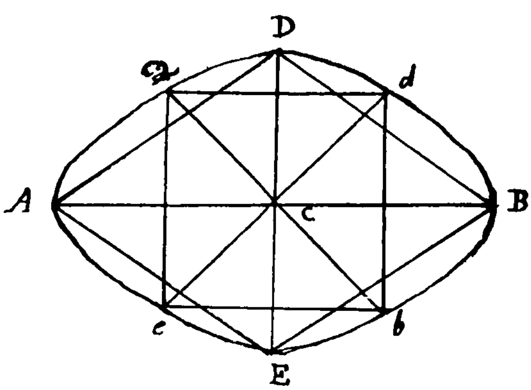
In an Elliptick Vessell that is regular the Content is best determined from a Geometrical mean betwixt the greatest and the least Diameters which are the two Axis [the manuscript often uses "Axis" for "Axes"] *of the Ellipse. But if the Vessell is irregular and the Diameter which is perpendicular to the greatest be not itself* [39] *the least in that Section of the Vessell, it is unjust to compute the Content from the mean of these two Diameters.*

The former part of this proposition is demonstrated in the Doctrine of the conick Sections. If AB and FG are the two Axis of the Ellipse ABGF, the Area is accuratly computed from the Geometricall Mean betwixt AB and FG; but if the figure of the Vessell be AKDLBIEH so that DE be not the least Diameter of the figure, it would be unjust to compute the Area from the Mean betwixt AB and DE, for thus you would obtain the Area of the Ellypse AMNBQR which has no Diameter less than DE, and exceeds the figure AKDLBIEH.

Prop. XIIIth.

The same Chord will not quarter an Ellipse from different points; the greatest Chord which quarters it is that which terminates upon the Axis, and the least is perpendicular to the Axis, wherever you quarter you obtain two Diameters that are perpendicular to each other, the sum of which is greatest when they are the [40] *Axis, and is least when the Chords that quarter the Ellipse are perpendicular to the Axis.*





All these follow from the properties of the Ellypse. Suppose AB and DE to be the Axis, then AD is the greatest Chord that can quarter the Ellypse. Suppose the Ellypse to be quartered from any other point as Q [as is clear from the diagram, the point Q was originally called a; the copyist changed some, but not all, occurrences of a in the text] so that the Chords Qd, db, be, eQ, may all be equall to each other, then are ab, de, Diameters of the Ellypse or lines passing through the Center and perpendicular always to each other: Their sum is greatest when they coincide with the Axis, and is least when the Chords are perpendicular to the Axis; in which case only the figure adbe, inscribed in the Ellypse is a square, the Diameters, ab, and, de, are equall, and ad is the least Chord that quarters the figure. These Diameters are not what the Geometers call Conjugate Diameters.

Prop. XIV.

It must be observed that when the Ellypse is quartered a Diameter ab *is not the greatest line that can be drawn through* [41] a *within the Ellypse* and that if the greatest line which can be drawn through* a *be taken in place of the Diameter* ab *and the Area be computed from it, the Content will be computed from it,* [so that] *the Content will be computed too great to the Trader's prejudice.*

This caution I thought necessary, for if it be not observed very consideable Errors will arise. By computing the Area from the Diameters ab, de, when the Chords are perpendicular to the Axes the Area is a litle below the Truth; but the Error is very small when the difference betwixt the Axis is but an Inch or two as will be shewn in the next proposition, whereas that which would arise by taking the greatest line which can be inscribed in the Ellypse passing through a may be very sensible.

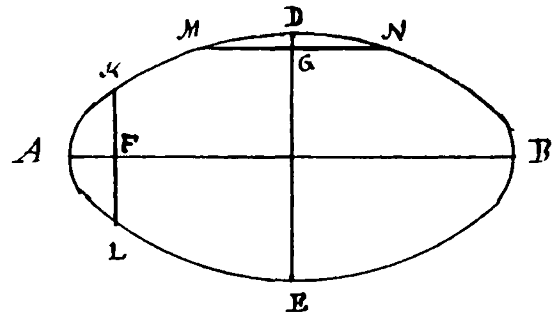
Prop. XV.

In the ordinary Vessells the Error that may arise from the Ellypticall figure is inconsiderable compared with that which may arise from taking the Diameters on unfair Staves.

[42] If the two Diameters that are taken in an Ellypticall Vessell be the Axes, then the Area may be computed from the Geometricall mean betwixt them without any Error. When the Diameters are those that quarter the Vessell, and are equall to each other (which is the case when the Chords that quarter the Ellypse are perpendicular to the Axes) the Area computed from these Diameters will be a little below the Truth: But this Error is inconsiderable in the ordinary Vessells. The proportion which it bears to the whole Area is that which the square of the difference of the two Axes bears to the sum of their squares. For Example, if the greater Axis be 72 the lesser 70, this proportion is that of 4 to 10084 so that it is only the 1/2521 part of the whole Area and amounts only to .006 of a Gallon. If the greater Axis be 74 the lesser 70, it is less than the 1/548 part of the whole, and amounts only to about 26/1000 of a Gallon; yet the equall Diameters are these from which this Error, when greatest, arises.

But the Error from the irregular position of the Staves may be very considerable, and this is one of the cheif causes of the difference [43] betwixt the Measures of the same Vessell taken by different officers. A Diameter in N°. 4. Div. 1. above mentioned that hit upon a Stave that stood out exceeded each of the Diameters on either side of it at 18 inches distance by a whole inch, which would have produced a difference in the Content of the uppermost frustum alone of more than two Gallons. The remedy is to be cautious in choosing such as are fair and agree best with the Generall Cantour [sic] of the Vessell; or to correct a diameter that you find unjust by taking one on every side of it upon fair Staves and substituting a mean betwixt them in its place, allowing (if it seems

*excepting only when a is the extremity of the longer or shorter Axis.



necessary) in taking this mean sometimes a tenth or half a tenth on account of that irregular Stave, which stands out and receives a little more Liquor by its position than regularly it ought to do, tho' not enough to justify the computing the whole Area from it.

When a Vessell is very irregular, it may be right to take four or perhaps more diameters if those you first take be found injurious to the Trader or revenue. In a Vessell whose [44] Sections are regular Ellypses the Content may be found with the greatest preciseness by taking two Diameters, providing these Diameters be the Axes of the Section. But when the Staves stand irregularly and the figure cannot be reduced to a regular Ellypse, it is better to take more diameters than two if the figure does not differ much from a Circle. Whenever any Diameters were complained of by either party I took two others that formed half right angles with the first. But an easy Method of determining the best Diameters for the Mensuration is explained in the next proposition.

Prop. XVI.

By applying a line or Ruler to the brim of the Vessell and comparing the versed Sines [in a unit circle, 1-sin (ϕ); in the ellipse below, the lines AF and DG] *or the distances betwixt the midle of this Line and the Brim of the Vessell, you may readily judge if the Staves are fair before you take the Diameters.*

If you apply a line or ruler of a given length as KL within an Ellypse perpendicular to the longer Axis cutting it in F and a line MN (equall to KL) perpendicular to the shorter Axis [45] meeting it in G, it is evident that AF will be greater than DG. Tho' the difference betwixt the two Axes AB, DE be but small, yet AF will be greater than DG in a very sensible proportion if KL be very little compared with the Axis; For when KL is very little, AF is to DG as the cube of AB to the cube of DE nearly; so that tho' the greater Axis was 72, the lesser 68, and therefore their proportion that of 18 to 17, the proportion of AF to DG will be nearly that of 7 to 6.

We call the Lines AF, DG the versed Sines; these are found equall in a Circle wherever you take them. In the Ellypse the versed Sine is greatest at the end of the longer Axis and least at the end of the shorter Axis: Therefore if you find the versed Sine greater at the end of one of your Diameters than in most other parts of the Vessell, and also greater at the end of the other Diameter than in severall parts of the Vessell, you have reason to conclude that these are not diameters from which you can fairly compute the Areas and Content: In like manner the Diameters are to be rejected [46] when both of them give versed Sines less than what are to be found round the Vessell in most parts. In making these trials you must choose such parts of the Brim as are tolerably regular; at least the ends and midle of the line KL must answer to parts that have a regular Situation.

Prop. XVII.

It is more just you should compute the Area from a Diameter that is half a tenth below the mean of the two perpendicular Diameters than half a tenth above it.

The Officers give their Diameters in inches and tenths without marking the hundredths, and in stating the Mean betwixt two perpendicular Diameters, they must often State it either half a tenth below or above the true mean. For example, if the two Diameters are 70.1. and 73.2. the true Arithmeticall mean betwixt these is 71.65 which they must either call 71.6 or 71.7. It is more just to call it 71.6 not only because it is the practice to take care [47] that the Trader be not wronged, but because the Area of 71.6 will be nearer the true Area of the figure than the Area of 71.7. For if this figure be an Ellypse, and these Diameters be its Axes, the Area is equall [to] that of a Circle whose diameter is the Geometricall mean betwixt the two Axes: But the Geometricall mean is always less than the Arithmeticall Mean. In this Example it is 71.63. which is nearer to 71.6. than to 71.7. For this Reason therefore it is just to take the diameter under the mean rather than that above it, especially when it is considered that when the difference of the two diameters is an even Number, and consequently the Arithmeticall mean is assigned precisely in inches and tenths, then the Area is computed a little above the truth. For example, let us suppose that the two diameters are 70.1 and 73.3. in which case the Arithmeticall Mean is precisely 71.7. and is assigned therefore as the diameter of the Vessell from which they compute the Areas and Contents: But the Diameter of [48] a Circle equall to the Ellypse is the geometricall mean betwixt 70.1. and 73.3. which is 71.66. and is almost .04 below that which the Officers compute the Area; so that in this Computation the Trader is overcharged. From this therefore it appears how just it is in the Commissioners to order the Officers in taking Diameters to mark the tenth below when the higher tenth does not plainly appear in taking the dimensions by the cane, if they did not give this order, and did not also throw away the hundredths in taking a mean betwixt the perpendicular Diameters, the Trader upon the whole would be injured by the Mensuration. The difference betwixt the Arithmeticall and Geometricall mean becomes so considerable when the difference between the two diameters exceeds four inches as to require to be accounted for, and it is actually considered by skillfull Officers in that case.

Prop. XVIII.

The perpendicular Altitudes of the Ves- [49] *sell ought to be taken with the greatest care, and the same favour ought to be shown to the Trader in taking them as in taking the Diameters.*

The perpendicular Altitude is the nearest distance betwixt the Bottom and the String that is stretched at the Top; therefore it may be taken too great if the End of the rod rest not on that precise point which is nearest to the string: And if it touch the bottom at all it cannot be taken too little. For this reason it ought not to be taken very sharp, and as it is the most important dimension it ought to be taken with particular care. There was another reason that induced me not to take it very sharp. In the Method of mensuration which I followed I ought to have had the Altitude from the highest part of the Bottom, when it was hollow at the inner side of the Chine, And if it fell within this towards the hollow it must have been taken too great. But after all by the Method of finding the dipping place described in the sixth proposition if it be used any prejudice or favour that can be done the Trader [50] or Revenue in taking the Altitudes at the quarters, or in taking the mean betwixt these is rectified; and therefore when the Altitude of the dipping place is taken by the rule in the sixth proposition, it ought not to be taken too hard, but rather large; otherwise the Trader has not the turn of the scale in this particular: For the taking it large in the Mensuration is of no service to him, if it be not taken large in fixing the dipping place.

Prop: XIX.

The dips of the Water ought to be taken as soon as possible after it is filled in, or if they be taken at the end of the work, they will be found to be less than they ought to be.

The Water will decrease a little from the Evaporation which becomes sensible by the continual dashing of it against the sides of the Tun (by the motion of the person who quarters the Vessell and takes the Diameter) which soon drys and becomes wet again and again in the progress of the work. I found the effect of this in two Vessells which took a great deal [51] of time and pains, having taken six Diameters in them at every depth. In these the dips were sensibly less at the end than at the beginning of the work, in one of them it amounted to a full tenth of an Inch there having been some boards putt in the Water for the Convenience of one of those who took the Diameters which drunk up some quantity of it, and there being a good deal of heat in the house which contributed with the continuall dashing of the water against the sides, to make the Evaporation or loss sensible. I have mentioned this that it may be a Caution to others who may measure in the same method. The Remedy is to take these dips as soon as you can determine the quarters of the Vessell.

Prop. XX.

The best way of finding the quarters from the drip, is to find the two places where two opposite dips of the Water are equall.

The drip is not easily discovered in the Backs by pouring in a little Water, because the Chine or hollow in the middle swallows [52] it up. The drip cannot very accurately be determined by finding where the greatest dip at the side is; for the Ark formed by the Water at the drip being nearly parallel to the Ark formed by the bottom (or to speak more accurately their tangents being parallel) the dip does not vary sensibly for a considerable space there. The same is to be said of the part opposite to the drip. But the dip varies most sensibly in a little space at the quarters, and these places are to be considered as quarters where the opposite dips are equal, and consequently the Diameter over the bottom is level, and the string drawn parallel to it at the top is also level, I generally endeavoured to find such a Diameter, and if the Staves were right, I chose to quarter from it, but when the Staves were unjust I took others that appeared fair near to them.

Prop. XXI.

The true Content of a Vessel with the Altitude of the dipping place that is adapted to it being given, and any other Content, with the Altitude of* [53] *the dipping place which was used while it obtained, being also given, to find the overcharge or undercharge at any Number of dry inches while this latter Content obtained.*

If the dipping place that was in use while the latter Content obtained be higher than that which is adapted to the true Content, substract the difference of their Altitudes from the dry inches at the false dipping place, and the remainder will shew the dry inches at the true dipping place. If the false dipping place is lower than the true, then add the difference of their Altitudes to the dry inches at the false dipping place, And the Sum will give the dry inches at the true dipping place. From these in the Table computed for the true Content you will find the true quantity in the Vessel, and thereby discover the overcharge or Undercharge. Lastly if the Altitude of the two dipping places is the same, then the difference of the Contents as they are shewn in the two Tables at the same Number of dry inches, is the Error of the Charge.

In this Rule however we suppose that the [54] dipping places are both at the quarters from the drip according to the usual practice: But if the false dipping place be towards the drip, and the false Content exceed the true, the Error of the Charge will be greater than it comes out by this Rule, and if the false Content be less than the true, the Error of the Charge will be less than this Rule gives it. It is the contrary when the false dipping place is towards the rise or part opposite to the drip.

For example, suppose that the true Content of N°. 1. Div. 1. when it is filled to an Altitude of 49.7 inches from the bottom is 801.2. Suppose that an Officer computes the Content of the same Vessel to be 812.3. and that the Gauging Officer dips, while he charges from this last Content, at the same place which is adapted to the former, which we suppose to be the true Content; it is evident that the overcharge, if the Vessel was

*By the perpendicular Altitude in these propositions we understand the Altitude above the Circumference of the bottom.

full, would be 11.1 gallons. If the overcharge is desired when there were five inches dry in this Vessel, it is easily found from the Tables computed from the two Contents; or thus; From the difference of the Contents substract [55] five times the difference betwixt the uppermost Areas, when the mensuration which assigns the greater Content gives the greatest uppermost area; But to the difference of the Contents add five times the difference of these Areas when the mensuration which assigns the greater Content has the lesser uppermost Area: In the Example abovementioned the false Content was 812.3. and the uppermost area according to this mensuration 17.53, the true Content 801.2, and the uppermost area agreeable to it 17.38 the difference of these areas being .15 I multiply it by 5 and substract the product .75 from 11.1. the difference of the Contents, and the remainder 10.35 is the Overcharge when five inches are dry.

But if the uppermost Area from which the true Content is computed be greater than the uppermost Area from which the false Content is computed, then their difference multiplied by 5 is to be added to the difference of the Contents, in order to find the overcharge when five inches are dry in the Vessel.

Suppose now that the dipping place, when [56] the false mensuration obtained, was higher than that which is adapted to the true mensuration; in this case the difference betwixt the Altitudes of the dipping places is to be multiplied by the uppermost Area that is assigned by the wrong mensuration; and this product being substracted from the difference of the Contents, the remainder shows the Error of the Charge if the Vessel was full; from this remainder substract the product of the difference of the uppermost Areas multiplied by 5, or add this product according as the uppermost Area is greater in the false or true Content, and the remainder, or sum, will shew the Error of the charges when five inches are dry in the Vessel. For Example, the true Content of a Vessel, when filled to the 48.4 inches from the bottom being 802.14. Suppose that another Content assigned for the same Vessel is 825.7 but that while this Content obtained the Gauging Officer dipt at a place 8/10 of an inch higher than that which is adapted to the true Content. Let the uppermost area according to the wrong mensuration be 17.82, multiply this by 8, and [57] substract the product 14.256 from the difference of the Contents, and the remainder 9.3. would be the overcharge if the Vessel was full to the true dipping place. Suppose the uppermost Area, by the true mensuration, to be 17.63. the difference betwixt this and the former Area is .19 which multiplied by 5 gives .95, and this product being substracted from 9.3 there remains 8.35 Gallons of Overcharge when five inches are dry.

If the dipping place that obtained while the false mensuration took place, be lower than that which is adapted to a lesser Content which is supposed to be true, then you are to add to the difference of the Contents the product of the difference of the Altitudes of the dipping places multiplyed by the uppermost Area that is assigned in the false mensuration: But if the true Content was greater, you are to substract that product from the difference of the Contents, and the sum or remainder will shew the Error of the charge, when the Vessel is full.

In the preceeding cases we supposed the dipping places to have been always at the [58] quarters from the drip; but if the dipping place that had been in use while the false Content obtained was not at these quarters, but either towards the drip or rise, then the computation is different. In Div. II^{d}.N^{o}. I. my Content exceeds that which was computed for the same Vessel from the mensuration taken in July last by about five Gallons: But the dipping place that was then fixed being towards the drip of the Vessel (whereas all my Contents are computed so as to answer to the quarters from the drip) by this means the difference of these Contents, as it will affect the Revenue or Trader amounts only to about 1 Gallon; so that either of these Contents, with their proper dipping places, will be almost equally just, while the Vessell continues in the present situation. But if the drip should increase or decrease (the quarters remaining as they are) the dipping place which answers to the mensuration in July must be altered. If the Content which comes out by my mensuration be admitted, then the dipping place is to be taken at the quarters from the drip according to Prop. VI.th and it will not need to be altered when the drip becomes [59] greater or less, but only when it is shifted to another part of the Vessel.

Prop. XXII.

The table computed in the usual method for a Tun or Back that stands upon its lesser Base gives the Content a little above the truth at all dips except at 10, 20, 30, 40, 50 *inches from the top. The Error is greatest at* 5, 15, 25, 35, 45 *inches from the Top, and at these depths sometimes exceeds a Gallon.*

When a Conical Vessel stands upon its greater Base, the Error of the Tables is to the Traders advantage; but it is the contrary when the Vessel stands upon its lesser base. This will appear by considering the Method by which these Tables are computed. After finding the whole Content of the Vessel they substract from it the uppermost area, and mark the remainder as its Content when one inch is dry. But there is realy more wanting in the Tun if it stands on its lesser base when one inch is dry at the Top than is expressed by the uppermost Area; because the Area at the midle of the uppermost inch is greater than the [60] Area at five inches below the Brim. This Error however would be inconsiderable if it was not almost doubled at the second operation, the Area at five inches below the Top being still less than the Area at the second inch from the top. The error continues growing till they come down 5 inches, and then it is at the height. After this the Error abates indeed, but the Content in the Table still exceeds the Content in the Tun till they come to the depth of ten inches. If they should continue after this to substract still the same Area, the Error would now pass to the other side, which might be some Compensation, but they do not allow the Content in the Table ever to become less than the real Content of the Vessel: For after they find the Content when ten inches are dry, they lay aside the uppermost Area, and substract in place of it the area at 15 inches depth, which being less than the Area at the midle of the 11 inch, the Error therefor appears again, and grows till they come to the depth of 15 inches in computing the Table from which term it abates till they come to 20 inches, when it vanishes again, immediately after which it appears [61] again and proceeds as before. By this way of computing the Table the Contents are assigned true at 10, 20, 30, 40, 50 inches from the Brim, only, and the Error is greatest at 5, 15, 25, 35, 45 inches from the Brim; and when there is a Slice the Table is true at the upper and lower end of the Slice, but a little above the truth at every intermediate depth of it. We suppose the perpendicular depth of the Vessel to be 50 inches or more.

There is a like Error when the Vessel stands on its greatest base; but in that case it favours the Trader and is to his advantage; From which it appears reasonable that in the present case where the Error is to his disadvantage, care should be taken that it be corrected so as that it may do him no prejudice at least. I shall here propose a very easy method by which it may be corrected so that the Table shall not err 1/40 of a Gallon and shall illustrate my Rule by an Example.

From the Content of the uppermost frustum substract the Content of the second frustum from the Brim, divide the difference by 8, and the quotient is the Error of the Table at the midle [62] of the depth of each frustum; That is at 5, 15, 25, 35, 45 inches from the Top: call this error P, then shall, $96/100 \times P$, $84/100 \times P$, $64/100 \times P$, $36/100 \times P$, respectively shew the Errors at 1, 2, 3, 4, inches above and below these midle depths. Thus [.]$96 \times P$ is the Error at 4, 6, 14, 16, 24, 26, 34, 36, 44, 46 inches from the Top. And, $84 \times P$ [in this paragraph the copyist repeatedly uses a comma and space where a decimal point is intended; *e.g.*, $.84 \times P$ is intended here] is the Error at 3, 7, 13, 17, 23, 27, 33, 37, 43, 47 inches from the top. And, $64 \times P$ [$.64 \times P$] is the error at 2, 8, 12, 18, 22, 28, 32, [38], 42, 48 inches from the Top. Lastly, $36 \times P$ [$.36 \times P$] is the Error at 1, 9, 11, 19, 21, 29, 31, 41, 49 inches from the top. Thus by substracting these five quantities P, $96 \times P$, $84 \times P$, $64 \times P$, $36 \times P$. [$.96 \times P$, $.84 \times P$, $.64 \times P$, $.36 \times P$] from the Contents assigned at the said depths, the Table will be corrected so as not to err 1/40 of a Gallon from the truth.

Example.

Suppose that the Diameter and Areas of a Conicall Back that stands on its lesser base are found by mensuration to be such as are shown in the following Table [see p. 35 below] where we suppose the Back to be gauged from the top downwards, and also from the bottom upwards. The use of the 4.th, 5.th, 9.th, and 10.th Columns will appear [63] from the next proposition.

I shall next subjoin the Table [see p. 36 below] computed for this Back in the usual Method, for the uppermost ten inches. I have continued the decimals farther than is usual in this as well as in the former Table that the Example might agree with the Theory and prove its accuracy. In the first Column of the following Table You have the dry inches of the Back; In the second Column are the [64] Contents that correspond to these dry inches computed in the common Method; In the third Column I have given the true Contents of the Back at the same dry inches, computed in a way that is described afterwards. In the fourth Column are the differences betwixt the Numbers in the second and third Columns, and these shew the Errors of the Table when computed by the common Method. In the fifth Column I have given the Errors of that Table computed by the Rule which I have proposed. In the last Column are the differences betwixt these errors which shew how near the Rule I have given is to the truth. In this Example the difference of the Errors does not exceed 23/1000 or 1/43 of an English

A Tun of a Conicall figure gauged downwards The Diameter at the top being 71 inches The Diameter at the bottom 62 The whole depth 45					The same Tun gauged upwards				
Depths	Diameter	Contents of each frustum computed from the area of its mean Diameter	True content of each frustum	Error	Depths	Diameters	Contents of each frustum computed from the Area of its mean Diameter	True Content of each frustum	Error
10	70	166.6	166.611 1/3*	.011 1/3	05	70.5	084.49425	084.495 2/3	.011 5/12
10	68	157.216	157.227 1/3	.011 1/3	10	69	161.874	161.885 1/3	.011 1/3
10	66	148.104	148.115 1/3	.011 1/3	10	67	152.626	152.637 1/3	.011 1/3
10	64	139.264	139.275 1/3	.011 1/3	10	65	143.65	143.661 1/3	.011 1/3
05	62.5	066.40625	066.407 2/3	.011 5/12	10	63	134.946	134.957 1/3	.011 1/3
45		677.59025	677.637	.046 3/4	45		677.59025	677.637	.046 3/4

*The fraction 1/3 in this and in the other parts of this Table signifies 1/3000 of a Gallon or 1/3 of what unit signifies in the preceeding place.

[Table from p. 63 of Maclaurin's memoir]

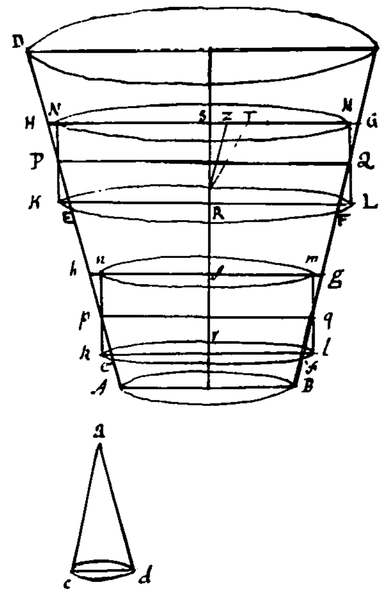
Gallon; and this difference would be still less if it was not for a reason which shall be explained in the next proposition. [65]

Dry Inches	Table computed in the usual way for the said Back.	Table shewing the true Contents of the Back.	Differences shewing the Errors of the common Method.	Errors by our Rule	Differences of the real Errors and those computed by our Rule
0	677.59	677.590	0.000	0.000	0.000
1	660.93	660.499	0.431	0.422	0.009
2	644.27	643.504	0.766	0.751	0.015
3	627.61	626.605	1.005	0.985	0.020
4	610.95	609.802	1.148	1.126	0.022
5	594.29	593.094	1.196	1.173	0.023
6	577.63	576.482	1.148	1.126	0.022
7	560.97	559.964	1.005	0.985	0.020
8	544.31	543.542	0.768	0.751	0.017
9	527.65	527.213	0.437	0.422	0.015
10	510.99	510.979	0.011	0.000	0.011

It appears from the fourth Column that the Error of the Table computed in the usual way, sometime exceeds a Gallon by a fifth part; & that it is to the Traders disadvantage. The errors in the fifth Column are computed according to the Rule above mentioned in the following manner. From the Content of the uppermost frustum 166.6 we substract the Content of the second frustum 157.216. The remainder is 9.384; This we divide by 8, and the quotient, 1.173. [66] is the Error of the Table at the midle of the depth of each frustum, that is at 5, 15, 25, 35, inches from the Brim: This error we call P then we compute the other Errors by the Rule delivered above and find

$P = 1.2$ or more accurately to 1.173
$.96 \times P = 1.1$ or ____________ 1.126
$.84 \times P = 1.0$ or ____________ 0.985
$.64 \times P = 0.75$ or____________ 0.751
$.36 \times P = 0.4$ or ____________ 0.422

These Numbers will serve for correcting not only the uppermost part of the Table, but all the Numbers in it till you come to the Slice; and if so much exactness is required, the Contents corresponding to the inches of the Slice may also be corrected by a similar operation. But because charges are never made from the Backs at Glasgow when there is so little Liquor left in the Back as to fill a part of the Slice only, we need not insist upon this. The Error computed from this Rule is a little under the true error at the upper end of the Table, and is a little above it at the lower end; but I have chosen this of several Methods as the most easy for computation, and at the same time sufficiently accurate.



[67] **Prop. XXIII.**

In a Conicall Vessell the Content that is computed from the Area at the midle of every frustum multiplied by the depth of that frustum, is a little below the truth. In a frustum of a given depth the Error is always of the same Magnitude in the same Vessel. In frustums of different Altitudes in the same Vessel it is as the Cube of the Altitude: And in frustums of equal Altitudes in different Vessels this Error is proportional to the square of the difference betwixt the upper and lower Diameters of the Frustums.

It may be demonstrated strictly from Geometry that the value of a frustum of a Cone which arises by multiplying the Area of the Circle that passes thro' the midle of its Altitude by that Altitude is not precisely true, and that the Error is equall to the Content of a Cone of the same Altitude with the frustum that stands upon a Base whose Diameter is half the difference of the upper and lower Diameters of the frustum.

In the conicall Vessel ABCD, [68] let EFGH efgh be any two frustums that have their depths RS, rs equal. Let PQ and pq be the mean Diameters that pass thro' the midle of RS and rs. The frustum EFGH exceeds the Cylinder KLMN and the frustum efgh exceeds the cylinder, klmn, by the same quantity precisely. Wherever the Altitude RS is taken this excess is always the same; and it is equal to the Content of a Cone acd that has dc the Diameter of its base equal to the difference of the Diameters HG and PQ (or to the half of the difference of HG and EF) and its height equal to RS the height of the frustum; draw RT parallel to BC the side of the Cone meeting GH in T, bisect ST in Z, join RZ, and the Cone that would be generated by the revolution of the right angled triangle RSZ about the side RS will be precisely equal to the foresaid error; or if you suppose the right angled Triangles PNH, PKE to revolve about the sides PN, PK the sum of the two Cones that will be generated will be equal to that error. From this it is evident that when the height RS is given, the Triangle RSZ is always of the [69] same magnitude in the same Cone and that the Content of the Solid generated by the Revolution of this triangle about the side RS is invariable and consequently the error in computing a frustum of a given depth from its mean Area is always of the same quantity in the same Vessel. 2.° The errors in computing different frustums in the same conicall Vessel are as the Cubes of the Altitudes of the frustums. 3.° The Errors in computing frustums of equal Altitudes in different conical Vessels are in the same proportion as the squares of the differences betwixt the upper and lower diameters of the frustums.

For example in the Table of page 63 we gave the Contents of each frustum of the Back computed according to the usual Method from its mean Area multiplied by its depth in the third Column the true content in the fourth Column; and the Error or difference betwixt the true and computed Content in the fifth Column. This Error we found to be always .011 1/3 of a Gallon when the depth of the frustum was ten inches; so that this error is the same whether the frustum be taken higher or lower [70] in the conical Vessel. From the 9.th and 10.th Columns it appears that this error is the same whether the Back be gauged upwards or downwards [;] according to the Rule given above this error ought to be equall to the Content of a Cone of ten inches height on a base of a Diameter equall to the increase of the diameter of the back in five inches of Altitude which in this example was one inch, and this will be found precisely true by computing the Content of such a Cone. For multiplying the square of this Diameter or 1 by 1/3 of the height of the frustum, or by 10/3 and then by .0034 [to convert from inches to gallons. $.0034=(\pi/4)\div 231$; *cf.* Maclaurin's footnote on his page 71] the product comes out precisely .011 1/3. By the Table the error in computing the Content of the slice of five inches Altitude whether it be at the top or bottom, or indeed wherever it be is .001 5/18. And this will be found precisely equal to the content of the Cone of 5 inches height upon a base of a Diameter equal to the increase of the Diameter of the Back in the half of that Altitude or 2.5 inches which increase in this Example is half an inch. The error in this case is the eight[h] part of the former, as a Cone is the eight[h] part of a similar Cone of a double height. The whole error in computing this example is .046 3/4 of a Gallon.

[71] The true contents of the frustums are found by the common Rule viz.' by squaring the upper and lower Diameters of the frustum adding to the Sum of these squares the product of the same Diameters, multiplying this sum by 1/3 of the Altitude of the frustum, and by .0034.* The following Rule is generally speaking easier. Find the product of the upper and lower Diameters of the frustum, add to it the third part of the square of the difference of these Diameters, multiply the sum by the Altitude of the

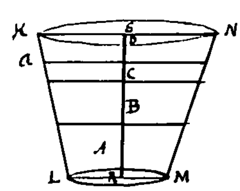
*We here suppose with all the late Writers on Gauging that the Circumference of the Circle is to its Diameter as 31416 is to a 1000[0] which indeed is a very little matter above the truth; but so little, that the Error is of no Consequence.

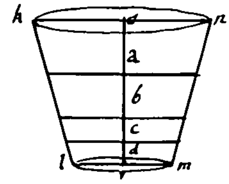
frustum; and this product multiplied by .0034 gives the true Content of the frustum in English Gallons. Thus the true Content of a frustum of ten inches Altitude at the top of the Vessel in the Table page 63 is had by multiplying 71 by 69 adding 1 1/3 (the third part of the square of 2 the difference of the upper and lower diameters) to the product, then multiplying by ten the height of the frustum and by .0034, the product 166.6113 1/3 is the true Content of that frustum. When you have a Table of Areas already computed, the true Content of a frustum may be found by this third Method; To the Area that corresponds to the half sum of [72] the Diameters of the two Bases, add 1/12 of the Area that corresponds to their difference, and the sum multiplied by the depth of the frustum gives its Content.

Prop. XXIV.

In measuring a conical Vessel the Content will be found the same whether you take the Diameters at the midle of every ten inches from the top downwards, and leave the Slice at the bottom, or take the Diameters at the midle of every ten inches from the bottom upwards leaving the Slice at the top, the Content of the Slice being computed in both cases from the Area that passes through the midle of its depth.

It follows from the last Proposition that if the depth of the Vessel be divided into any Number of parts equal or unequal and the Areas be computed from the Diameters at the midle of each part, the Content of the Vessel will be found the same in whatever order these parts are taken; whether the same parts be taken from the top downwards, from the bottom upwards, or in any other order that can be imagined, the parts being of the same magnitudes (however [73] unequal they may be amongst themselves) in every case.





Let KLMN, klmn be any two equal and similar Vessels. Let their equall Altitudes RS and rs be divided into the parts ABCD and abcd, so that A=a, B=b, C=c, D=d. Let the order of the parts a, b, c, d be varied at pleasure. Compute the Contents of these vessels by the Areas at the midle of each part and the whole Content will be found to come out always the same, the varieties of the order of the parts shall no way affect this content which will always differ from the true Content by the same quantity: For wherever the part a be placed the Error of the computed Value of the frustum a from the truth is equal to the error of the computed Value of the frustum A from the truth the Altitudes of these frustums being equal by the supposition. In like manner the Errors of the computed Values of the frustums b and B from the truth are equall, and so of the rest. Therefor the whole computed Values of the two Vessels are equally deficient from the truth, and the Vessels being equall, it follows that their computed Values must be equall, so that when the depth of such a [74] Vessel is divided into a certain Number of parts or given Values, it is of no consequence in what order these parts are taken upon the depth in calculating the content from the Areas at the midle of each part.

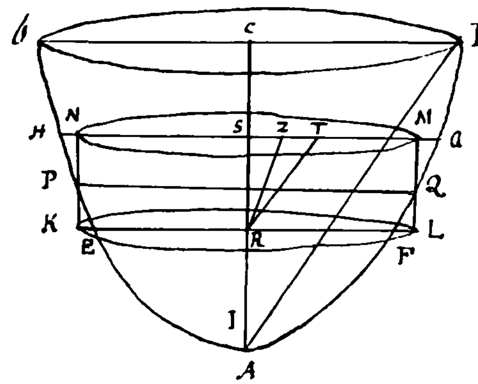
An Example of this we have in the 63 page where the errors of the frustums appear to be the same whether the Back is gauged upwards or downwards; and the totall Content is found to be the same in either way.

If the Brim of the Vessel is irregular and the Circumference of the bottom ly nearly in one plane, then you must either Gauge from the bottom upwards or sett off equall Altitudes from the bottom to four points on or near the brim, and gauge downwards from these points.

The Table is computed in the same manner in either method, only if the depth of the Slice terminate in the fractional part of an inch, a caution must be observed in computing the Table in this case, which will be best understood from an Example. Suppose the depth of the Slice is 5.4 of an inch; in computing the Table you are to substract the uppermost Area 5 times in order to find the Content when 1, 2, 3, 4 or 5 inches are dry. But in order to find the content when 6 inches are dry, [75] you are to multiply the uppermost area by, 4/10 and the second area by, 6/10 and to substract the sum of these products from the Content at 5 inches dry, for the remainder is the Content of the Back when six inches are dry. The rest of the Table (one Number in each frustum excepted) is computed in the usuall way; However to avoid this last operation, and that the Table may be the more easily corrected according to the Rule in the 22^{d}. proposition, I would recommend rather after finding four points at the brim in a plane parallel to the Circumference of the bottom, to gauge downwards from these four points. At the same time it must be owned that the uppermost parts of the Tables that are computed in the usuall Method, without the foresaid correction, are a little more accurate when the Slice is left at the Top than when it is left at the bottom of the Vessell.

Prop. XXV.

In a sphericall or spheroidical Vessell the Content of a frustum computed from its mean



area multiplyed by its depth, is above the truth. When the depth of the [76] *frustum is given the Error is always the same in the same Vessell or in similar Vessels; and the totall Content will be found the same whether the Vessel be gauged upwards or downwards.*

In a Vessel that is of a spherical Figure the Error that arises by computing the Content of a frustum from the Area that passes through the midle of its depth is precisely equall to half the Content of a Sphere that has its Diameter equall to that depth. It is a remarkable property of such Vessels that the Error in computing the Content of a frustum of a given depth is not only the same in all parts of the same sphere, but is the same in all spheres whatsoever. If the depth of the frustum be ten inches the Error is always one Gallon and two fifteenths of a Gallon, and in measuring a segment 40 inches deep the usual way there would be an overcharge of more than four gallons and a half. If the frustum be six inches deep the Error is almost 1/4 of a Gallon; so that in computing the Content of a segment 36 inches deep from the areas at the midle of every six inches there would arise an overcharge of almost a Gallon [77] and a half. The Errors in computing frustums of different Altitudes are as the Cubes of these Altitudes.

In a Spheroid [the solid produced by rotating an ellipse about one of its axes] the error always produces an overcharge and is in proportion to the Error in computing the Content of a frustum of the same height in a Sphere as the square of the horizontal Axis of the spheroid, to the square of its perpendicular Axis. In such a Vessel the overcharge may become very considerable: Suppose for Example the Horizonal Axis to be the double of the vertical Axis, and the Error in computing the Content of every frustum 6 inches deep will amount to almost one Gallon, and if the depth of the frustum be 10 inches, the error that arises in computing its Content from the Area at the midle of these 10 inches would amount to more than four Gallons and one half; and in computing a segment of 40 inches deep from the Areas at the midle of every 10 inches there would arise an Overcharge of 18 Gallons. When the proportion of the horizontal Axis to the vertical is not known it may be easily found in the following manner.

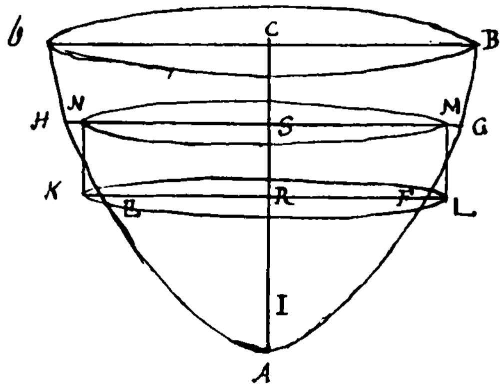
Suppose e f & g to be [78] three Diameters of the Spheroid taken upon its depth at equal distances from each other [*i.e.*, the diameters of three circular cross-sections of the solid, equally spaced on the vertical axis], from the square of f substract the square of g, and from the square of e substact the square of f; then take the difference of these two remainders and as this difference is to 8 times the square of the distance [between] the diameters e and f so is the square of the horizontal Axis to the square of the vertical Axis of the Spheroid; and the Error of computing a frustum in the Spheroid is to the Error in computing a frustum of the same height in the sphere in the same proportion. Let the horizontal Axis of the spheroid be Bb, C its Center, CA half its vertical Axis; Join AB and let RT parallel to AB meet GH in T; bisect ST in Z. Then the error in computing the Content of the frustum EFGH from the Area at the midle of its depth (or the excess of the Cylinder KLMN above this frustum) is equal to the Content of a Cone that has the diameter of its Base equal to ST and its height equal to RS the height of the frustum which is the Cone that would be generated by the revolu- [79] tion of the Triangle RSZ about its side RS. From this proportion it appears I.° That when the height of the frustum is given the Error by which the computed Content exceeds the true Content of the frustum is always the same not only in the same Spheroid, but in all similar spheroids. 2.° In the same or in similar spheroids the Errors are as the Cubes of the heights of the frustum. 3.° In Spheroids of different kinds the Errors in computing frustums of the same height are as the squares of the Numbers that express the proportion of CB to CA. 4.° The error that arises by computing any frustum of the Spheroid BAb from the Area that passes through the midle of its height is equal to the Error that arises by computing by the same Method a frustum of the same height in the Cone that would be generated by the Revolution of the Triangle BAC about its side AC.

Prop. XXVI.

In a parabolic Conoid [the volume] *of a frustum computed from its mean Area multiplied by its depth is precisely true; In* [80] *a Hyperbolic Conoid it is below the truth. In both it is the same thing whether they are gauged upwards or downwards.*

The only kind of round Vessel in which there is no error in the common Method of measuring or no difference betwixt the frustum EFGH and the Cylinder KLMN is the parabolick Conoid which is generated by the Revolution of the Area ACB, terminated by the parabola AFB, about its Axis AC. In such a Vessel the true Content of the frustum is obtained precisely by multiplying the Area that passes thro' the midle of its depth by the height of the frustum. Some Casks are commonly thought to be of this Figure, and an easy way for measuring their Contents follows from this property.

As for Hyperbolick Conoids the Error in computing a frustum of a given height is always of the same magnitude in the same or in similar Vessels; The Errors in computing frustums of different heights are as the Cubes of these heights. In these Vessels the error produces an Undercharge which is deter- [81] mined by a Construction similar to that which we gave for finding the Error in a Spheroid [solid formed by



rotating an ellipse] and is equal to the Error that would arise in computing a frustum of that same Altitude in the Cone that would be generated by the Revolution of the Asymptotes of the Hyperbola about its Axis.

In general it is a remarkable property of all the solids that can be generated by the Revolution of any Conick Section upon either of its two Axes, that the Content of a frustum of a given Altitude in a Vessel of a given Species terminated by planes perpendicular to that Axis differs from the Contents of a Cylinder of the same Altitude upon a base equal to the Section of the frustum through the midle of its depth by a given or invariable quantity. From which it follows that the Content will be found the same whether such a Vessel be gauged upwards or downwards. I have insisted the more on these last four propositions, that these things have not been observed by the writers on this subject nor by the Geometers themselves and because the knowledge of them may in some cases be of use to those who have [82] often occasion to measure Solids.

Prop. XXVII.

The Backs or Tuns ought to be examined upon any Alteration of the position of the Hoops, and to be regauged every third or fourth year and in some cases more frequently.

It is evidently the Traders interest that the Diameters be examined when the hoops are struck up higher than they were before; for by this means the Diameters are contracted, and the Content of the Vessel diminished. In N°. 6 Div. 4. it appears from some peices of Timber that were nailed under the hoops in their former position that they have been raised almost 5 inches from their first place, which is undoubtedly the cheif reason why the Content of this Vessel has been found so much less by the late mensurations than that which is assigned for it in the Old table books. In some other Vessels the Alteration of the Hoops appears also to be very considerable.

When the Vessel is new, the Bottom is plain and Circular, as we observed above. Afterwards the Bottom Staves shrink and are contracted [83] a little in their breadth;

The figure of the bottom becomes Ellypticall and this also contributes to the lessening the Contents of the Vessel. The figure of the Bottom from the pressure of the liquor, the shrinking of the Staves, the stricking up the Hoops, or other causes, commonly in the Backs at Glasgow becomes hollow, which increases the Content, but not so much, generally speaking, as to balance the decrease from the other causes already mentioned. However it is plain that it is not sufficient to measure the Content of a Back when it is new, but that it ought to be regauged every three or four years.

Prop. XXVIII.

The English Wine Gallon contains about two Scots pints and a fifth part of a pint of the usual measure of the Pewtherers Jugs.

By a Scots pint I do not here mean the Content of the Standard measures; such as are kept by the Dean of Guild of Edinburgh and in other parts of the Kingdom; but that which is usually the measure of the Pewtherer's [84] pint Jug. These are generally found to hold about 105 Cubick inches, the double of this is 210, the fifth part of 105 in [should be "is"] 21, and the sum of these is 231, which by Act of Parliament is the Measure of the English Wine Gallon. The proportion of the Scots pint (according to the Pewtherer's Jug) to the English Wine Gallon is that of 5 to 11. By some Experiments made with a good deal of Care upon the Standard Juggs kept by the Dean of Guild of Edinburgh (by several ingenious Gentlemen) the Scots pint was found not to exceed 102.3 Cubick Inches; but in this proposition we speak only of the Scots pint according to the ordinary Pewterer's Juggs.

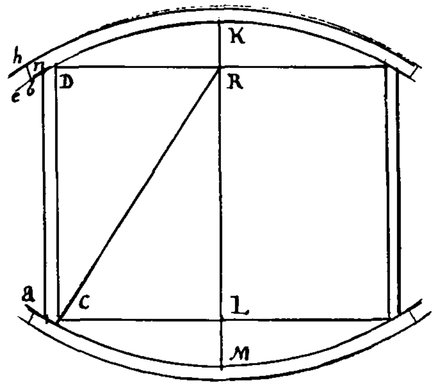
Appendix I

Of taking the Dimensions of Casks [barrels shaped as in the diagram on Maclaurin's page 85].

I had occasion to observe at Glasgow that the Officers use different methods for taking the head Diameters [CD or ab in the diagram, as opposed to the bung diameter KM] of Casks, and that the difference may be of considerable importance in the mensuration of so large Casks as those are which belong to the Distilleries. An Error in a method of mensuration is of greater Con- [85] sequence than a Mistake in a particular cask, and therefore ought to be more carefully examined.

[The line CR is drawn, instead of the required line KC, in the diagram in the NLS version; KC is, correctly, drawn in the diagram on p. 65 of the EUL version.]

Suppose ab to be the Diameter of the head of a Cask on the outside; CD the diameter on the inside from which the Content is computed. Let eb be the length of the Stave without [*i.e.*, outside of] the head, eh its thickness at its termination. It is usuall in measuring smaller Casks to take this distance ah, and to suppose CD equal to it. This Method may be tolerably exact in measuring lesser Casks. But it may be reasonably doubted if it is not erroneous for measuring the Diameters CD in large Casks, and therefore some Officers take an in place of ah in measuring these, the point n being near the midle of the thickness eh. This latter practice seems preferable, because the



bending or Curvature of the Staves in great Casks is less in proportion to their length and other dimensions than in small Casks. However this deserves to be farther examined. In order to compare these methods, in a Cask belonging to Div: 2. Mr. Chapman took the Diameter at the bung KM and the Diagonal KC[,] from which and CL [86] half the length of the Cask it is easy to compute KL by the 47.1 of Euclid [i.e. by the Pythagorean theorem]; By subducting LM or KR from KL there remains RL equal to CD. By this Computation CD was found to exceed an by 1/10 of an Inch only. Till this matter is examined by more trials it does not seem reasonable to measure the Diameter CD by the distance ah; for this distance depends upon the length of be, the thickness of the Staves he, and the magnitude of the angle abe all which may be different in Casks that may have the Diameters CD of the same magnitude.

In order to be able to judge better of these two Methods of taking the Diameters of Casks at the head, I went to a Coopers in the Grass Market of Edinburgh where [there] was a Cask that had one of its heads lately taken out, so that I had access to the Diameter of the other head in the inside. Having taken a streight stick in length equal to the distance ah we endeavoured to apply it to the head of the Cask on the inside; but found that the Stick was too long. When one end of the stick was applyed to the Circumference of the head at C on the in- [87] side the other end rested upon a point that was about two inches and a half above the point D[;] we were oblidged to cut off above 3/10 of an inch from the Stick before we could apply it to CD the Diameter of the head. This trial confirms me in my opinion that these officers who measure the Diameter CD by the distance ah take it too large and overcharge the Trader.

In the same Cask I observed that there was a thin Slice taken off the Staves round the head on the inside; This by the Coopers is called Howelling [*i.e.* planing], and is done for smoothing the part where the bottom joins the Staves. By this means the Diameter that is immediately within the head is made a little larger than it ought to be in proportion to the Dimensions of the Cask, so that a Computation of the Content from it would produce an overcharge. This observation made it still more evident that the distance ah exceeds the Diameter at the head from which the Content of the Cask ought to be computed, in the larger sort. In some instances perhaps CD may be found equal to ah, but to me [88] it appears very probable that, generally speaking, in the large Casks it will be found less than ah. Upon the whole it would seem to be better to take the Diameter ab somewhat large for the Diameter at the head; which as I am informed, is the practice of the Officers in some other branches of the Revenue.

II.

The Demonstration of Prop. XXIII.

See the Figure of Prop. 23. [Maclaurin's page 68, above]

Suppose HG the upper Diameter of the frustum EFGH equal to A, EF the lower Diameter equal to B, RS the depth of the frustum equal to D then the Content of the frustum EFGH by the ordinary Rules of mensuration expressed in English Gallons is $\overline{AA+AB+BB}\times .0034\times D/3$. The mean Diameter of the frustum (viz.PQ) is 1/2 A + 1/2 B the Area is $1/4\times\overline{AA+2AB+BB}\times .0034$. And the Content of the Cylinder KLMN is $\overline{AA+2AB+BB}\times D/4\times .0034$. Therefore the Excess of the frustum EFGH above the Cylinder KLMN is $\overline{AA-2AB+BB}/12\times D\times .0034$ which is equal to $\overline{A-B}^2\times D/12\times .0034$. But in a given Conical Vessel [89] when D the depth of a frustum is given then the increase of the Diameters in that depth or $A-B$ is also given, and the foresaid quantity which is equal to the Excess of the frustum EFGH above the Cylinder KLMN is of a given or invariable Magnitude, so that the Error of the common Method of computing the Content of a frustum of a given depth being equal to the foresaid excess, it is therefore always of the same magnitude in the same Vessel.

If d be the depth of a lesser frustum in the same Vessel, a and b the upper and lower Diameters of this frustum, then the Error in computing it in the usual way by multiplying the area that passes through the midle of its depth by that depth itself is $\overline{a-b}^2\times d/12\times .0034$ and the Errors in computing these two frustums are to each other as $\overline{A-B}^2\times D$ is to $\overline{a-b}^2\times d$ that is (because $A-B{:}a-b{::}D{:}d$) as D^3 is to d^3. These Errors therefore are as the Cubes of the depths of the frustums in a given Vessel. For Example the Error in computing a frustum of ten inches depth is a thousand times greater than [90] the Error in computing a frustum of one inch deep; but because there are ten frustums of the latter sort in one of the former, therefore the Error in computing a frustum of ten inches depth from its mean Area is only a hundred times greater than the Error that arises by computing its Content from the Area at the midle of every inch. It is evident that in general this Error is equal to the Content of a Cone that has the Diameter of its Base equal to half the difference of A & B and its Altitude equal to D, for the Content of such a Cone by the common Rules of Mensuration is $\overline{A-B}^2\times D/12\times .0034$ which we found to be equal to the said Error.

The practical Rule for computing this Error is, multiply the square of the difference betwixt the upper and lower Diameters of the frustum by its Altitude, and by .000.85 [$.00085=1/12\times .0034\times 3$], the third part of the product is always equal to the Error that arises by computing the Content of the frustum from the Area that passes through the midle of its Altitude in the usual Method. If the height of the frustum be ten inches, the Error will be easily computed by multiplying [91] the Square of the Difference betwixt any two Diameters that are at five inches distance from each other by .011 1/3. [By similar triangles the difference between the measured diameters squared must be multiplied by 4 to get $(A-B)^2$. Since $D=10$ he obtains 1/3 of $10 \times .0034 = .011$ 1/3.] This Error in the Backs that are commonly made is so inconsiderable that it may very well be neglected. In the Example of the XXII[d] Prop. it does not amount in computing the whole Content of the Back to the 20.[th] part of a Gallon.

Because this Error is equal to $\overline{A-B}^2 \times D/12 \times .0034$ it follows that when D the depth of the frustum is given it is as the Square of $A-B$; That is in different Vessels the Errors in computing frustums of equal depths are as the Squares of the differences of their upper and lower Diameters.

III.

The Demonstration of the 25 and 26 propositions.

See the figures of Prop. 25 and 26. [Maclaurin's pp. 77, 80, above]

These Propositions may be demonstrated from some of Archimedes Theorems; but more breifly by the Method of fluxions thus. Let AR be equal to x, EF equal to y then the Equation expressing the Relation of EF and AR when AEB is any conick Section [92] that has AC one of its axes is $y^2 = Ax^2 + Bx$. From this it follows by the inverse Method of fluxions [*i.e.*, integration from $x=0$ to $x=x$, and letting m/n stand for π] that the Solid AEF (the proportion of the Area of the Circle to the Square of its Diameter being that of m to n) is equall to $m/3n\ Ax^3 + mBx^2/2n$. In like manner if RS be equal to d the Solid AHG [integrating from $x=0$ to $x=x+d$] is equal to $m/3n\ A \times \overline{x+d}^3 + m/2n\ B \times \overline{x+d}^2$. The frustum EFGH is equal to the difference of the Solids AHG and AEF that is to $m/3n \times \overline{Ax^3 + 3Ax^2d + 3Axd^2 + Ad^3} + m/2n \times \overline{Bx^2 + 2Bdx + Bd^2} - m/3nAx^3 - mBx^2/2n$ [the manuscript omits both the Ad in the term Ad^3 and the vinculum over the three terms multiplied by m/2n] $= m/n \times \overline{Ax^2d + Axd^2 + Ad^3/3} + m/n \times \overline{Bdx + Bd^2/2} = m/n \times \overline{Adx^2 + Ad^2x[+]Bdx + Ad^3/3 + Bd^2/2}$. The Cylinder KLMN is equal to the Area of the Diameter PQ multiplied by RS. The square of the Diameter PQ is equal to $A \times \overline{x+d/2}^2 + Bx + Bd/2$ [evaluating $y^2 = Ax^2 + Bx$ when $x = x + d/2$] and therefore the Cylinder KLMN is equal to $m/n \times \overline{Adx^2 + Ad^2x + Bdx + Ad^3/4 + Bd^2/2}$ [the part of the vinculum over the last two terms is omitted in the manuscript]. The difference betwixt the frustum EFGH and the Cylinder KLMN is $m/n \times Ad^3/12$ which when d the depth of the frustum is given is of an invariable magnitude in a given Vessel or in similar Vessels: from which the 25 and 26 propositions are easily demonstrated. For since the Error in computing a frustum of a [93] given depth in any of these Vessels is always the same in whatever part of the Vessell this depth be taken it follows that any such Vessel is gauged with the same exactness from the bottom upwards as from the top downwards, and this is true not only of Conical Vessels but of all these which can be generated by the revolution of a Conick Section on its Axis.

The quantity of the Error in all such Vessels being $m/12n\ Ad^3$ or in English Gallons $Ad^3/6 \times .0017$ it produces an Overcharge when A is negative but an Undercharge when

A is positive; for in the first case the Cylinder KLMN exceeds the frustum EFGH, in the latter case the frustum is less than the Cylinder. In the sphere A is equal to -1 [both manuscripts, the NLS and EUL (p. 71), have -4, an obvious error, which is repeated twice more in this paragraph; I have changed it to -1, and noted the change, in each case] and the overcharge in computing a frustum of the depth d is equal to $m/n \times d^3/3$ or in English Gallons to $d^3/3 \times .0034$ which is half the Content of a Sphere of the Diameter d. This error is evidently the same not only in all parts of the same Sphere but also in all Spheres whatsoever when d is given. In a Spheroid A is also negative and is to -1 [corrected, as noted above, from -4] as the square [94] of the Horizontal Axis to the square of the Vertical Axis of the generating Ellypse. In the parabolick Conoid A vanishes, there is no Error, and the frustum EFGH is precisely equal to the Cylinder KLMN. In the Hyperbolick Conoid A is positive the Error is an Undercharge and the frustum EFGH exceeds the Cylinder KLMN; the quantity A in this case is to 1 [corrected, as noted above, from 4] as the square of the Axis of the generating Hyperbola that is parallel to the Horizon is to the square of the Axis that is vertical to it. But a compleat Demonstration of all these propositions would lead us too far from our chief design.

The preceeding Propositions and Appendix on 94 pages, as an Essay towards a more just mensuration of Vessels that are somewhat irregular, is offered to the Consideration of the Hon^ble^ Commissioners of Excise by

their most obedient most humble

Servant signed

Colin MacLaurin m.p.e. [presumably "mathematics professor, Edinburgh"]

Edinburgh Nov^r^. 6. 1735.