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Random Tropical Curves

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Abstract

In the setting of tropical mathematics, geometric objects are rich with inherent combinatorial structure. For example, each polynomial $p(x, y)$ in the tropical setting corresponds to a tropical curve; these tropical curves correspond to unbounded graphs embedded in $\mathbb{R}^2$. Each of these graphs is dual to a particular subdivision of its Newton polytope; we classify tropical curves by combinatorial type based on these corresponding subdivisions. In this thesis, we aim to gain an understanding of the likeliness of the combinatorial type of a randomly chosen tropical curve by using methods from polytope geometry. We focus on tropical curves corresponding to quadratics, but we hope to expand our exploration to higher degree polynomials.

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Acknowledgments

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Chapter 1

Introduction

Tropical mathematics is a relatively new field of mathematics in which we explore various questions over the tropical semiring. The tropical semiring uses the min and + operations and is defined in more detail in Chapter 2. By asking familiar questions over the tropical semiring, we can often use the combinatorics inherent to the tropical setting to gain valuable insights, and then apply these combinatorial insights back to the classical setting.

For example, in tropical geometry, we explore questions similar to the ones we ask in algebraic geometry; namely, what can say about the geometric objects determined by roots of polynomials? In tropical geometry, we examine geometric objects that arise when we define polynomials over the tropical semiring. When we consider polynomials over two variables, the corresponding objects, called tropical curves, take the form of graphs embedded in $\mathbb{R}^2$ that satisfy a few additional requirements. In this thesis, we classify these curves based on their combinatorial type, which records the structure of the curve up to graph isomorphism.

In this thesis, we base our explorations on the following questions:

Question 1.0.1. *What does a randomly generated tropical curve look like? Are some combinatorial types more likely than others?*

In this thesis report, we discuss the background information necessary to explore this topic and some of the prior research that has been done in this area. In Chapter 2, we introduce some concepts from tropical geometry in more detail and define the main objects of study in this thesis: tropical curves. In Chapter 3, we introduce Newton polytopes. We discuss how we can use these objects to distinguish tropical polynomials by combinatorial type. In Chapter 4, we will discuss some previous approaches to this problem. In

Chapter 5, we will introduce the concept of secondary fans, and outline how we use these objects to answer our questions. In Chapter 6 and 7, we will use the method outlined in the previous chapter to explore the cases of quadratics in one variable and two variables respectively. In Chapter 8, we discuss lingering questions and directions for researchers.

Chapter 2

Tropical Curves

In this section we provide some background regarding tropical polynomials and their hypersurfaces, which are the main objects of study in this thesis.

2.1 Tropical polynomials

In this section, the treatment of this material follows from **?**.

Definition 2.1.1. The *tropical semiring* is the set $\mathcal{R} = \mathbb{R} \cup \infty$ together with the following operations on $x, y \in \mathcal{R}$:

1. $x \oplus y = \min(x, y)$
2. $x \otimes y = x + y$.

We now list some of the properties of these operations. We first note that $\oplus$ and $\otimes$ are commutative and associative. Furthermore we can see that $\otimes$ and $\oplus$ both have an identity element:

For all $x \in \mathcal{R}$,

$$x \oplus \infty = \min(x, \infty) = x$$

$$x \otimes 0 = x + 0 = x.$$

Thus, the identity element of $\oplus$ is ∞; $\otimes$ inherits the identity of 0 and the existence of inverse elements from classical arithmetic over $\mathbb{R}$. One can also show that $\oplus$ and $\otimes$ satisfy the distributive property. Thus, we can see that the tropical semiring satisfies all of the conditions of a ring (and in fact, a field) except for the existence of an additive inverse. We established that the additive identity of the tropical semiring is ∞, however for $a \neq \infty$, there is no $b \in \mathcal{R}$ such that $a \oplus b = \min(a, b) = \infty$.

We now give some definitions and show that $\mathcal{R}$ equipped with $\oplus$ and $\otimes$ is indeed a semiring.

Definition 2.1.2. A *monoid* is a set M with a binary operation $+$ such that for all $a, b, c \in M$ the following hold:

1. There is some $0 \in M$ such that $0 + a = a + 0 = a$.

2. The operation $=$ is associative: $(a + b) + c = a + (b + c)$.

Note that a monoid satisfies all the conditions of a group except for the existence of additive inverses for each element. We are now ready to formally define the concept of a semiring:

Definition 2.1.3. A *semiring* is a set S with two binary operations $+, \cdot$ such that for $a, b, c \in S$ the following hold:

1. S when equipped with $+$ is a commutative monoid; in the context of this definition we will denote this identity as 0.

2. S when equipped with $\cdot$ is a monoid; in the context of this definition we will denote this identity as 1.

3. The following distributive properties hold:
$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$(b + c) \cdot a = b \cdot a + c \cdot a$$

4. $0 \cdot a = 0$

From the discussion above, we can now see $\mathcal{R}$ equipped with $\oplus, \otimes$ is indeed a semiring.

We can define polynomials in this setting, using $\oplus$ in place of addition and $\otimes$ in place of multiplication. These are tropical polynomials.

Definition 2.1.4. A tropical polynomial of degree n in k variables is an expression of the following form:
$$\bigoplus_{a_1 + \cdots + a_k \leq n} x_1^{a_1} \otimes \cdots \otimes c_{a_1, \ldots, a_k} x_k^{a_k}$$

where $c_{a_1, \ldots, a_k} \in \mathcal{R}$, and a_i are nonnegative integers. (Here the Σ denotes tropical addition) Here, $x_i^{a_i}$ is shorthand for $x_i^{\otimes a_i}$.

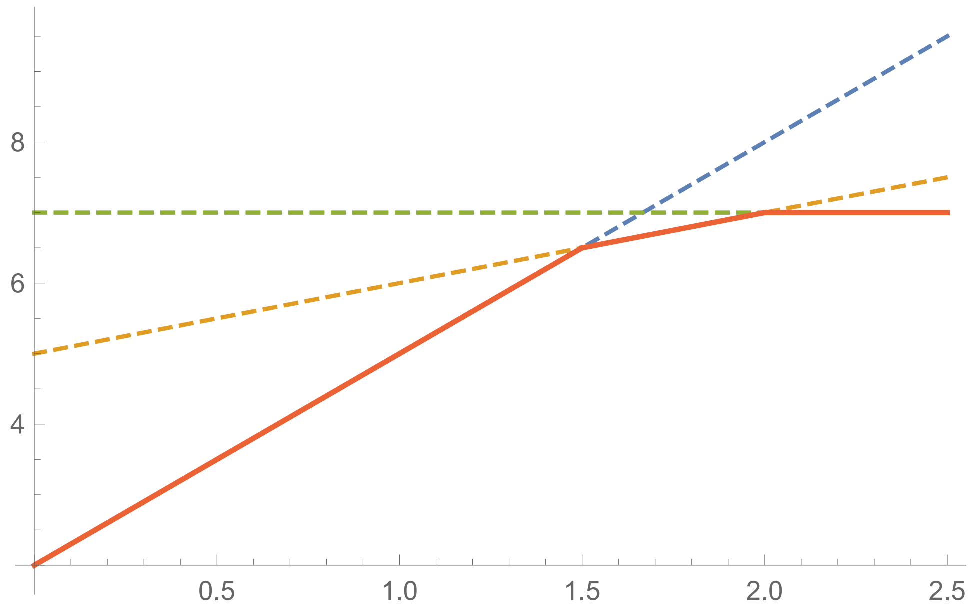

Figure 2.1 A plot of the tropical form of $p(x) = 2x^3 + 5x + 7$. Image created with Mathematica

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In this thesis, we will focus on tropical polynomials in one and two variables. To get an idea of what these tropical polynomials look like, we illustrate a a one-dimensional example and a two dimensional example:

Example 2.1.5. Consider the polynomial $p(x) = 2x^3 + 5x + 7$. In the tropical setting, we view this polynomial as

$$p(x) = (2 \otimes x^3) \oplus (5 \otimes x) \oplus (7) = \min(3x + 2, x + 5, 7)$$

A plot of the this function is shown in Figure **??**. We can see that this polynomial is a concave piecewise function composed of a finite number of line segments and rays.

Example 2.1.6. Consider the polynomial $p(x) = x^2 + xy + 3y^2 + 2x + y - 1$. In the tropical setting, we view this as

$$p(x) = (x^2) \oplus (x \otimes y) \oplus (3 \otimes y^2) \oplus (2 \otimes x) \oplus (y) \oplus (-1) = \min(x+2, x+y, 2y+3, x+2, y, -1).$$

A plot of this function is shown in Figure **??**. We can see that this polynomial is a concave piecewise function made up of a finite collection of pieces of planes.

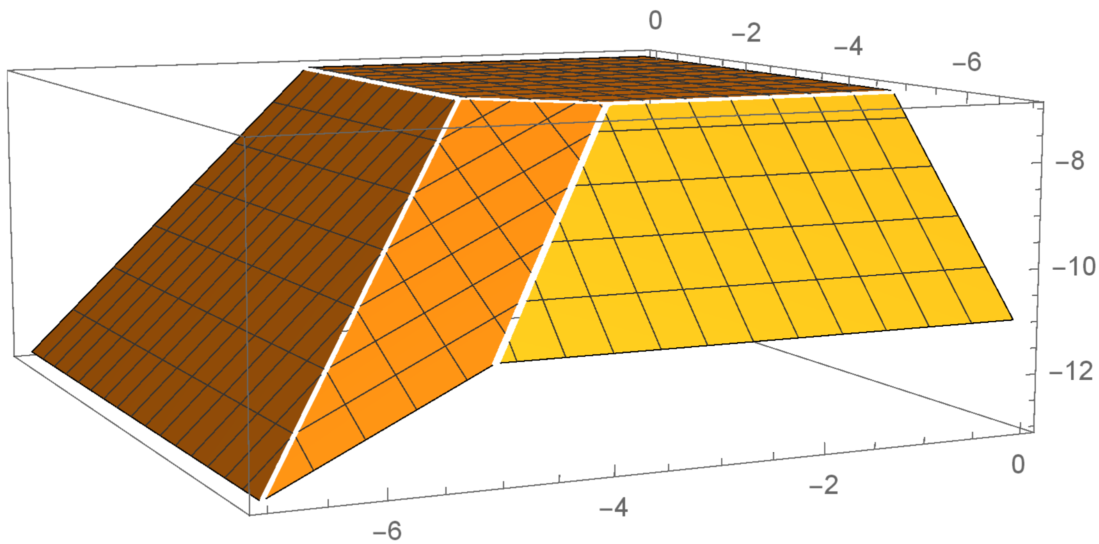


Figure 2.2 The plot of the tropical form of the polynomial $p(x) = x^2 + xy + 3y^2 + 2x + y - 1$. Image created with Mathematica.

By considering these and other examples, one may suspect that each tropical polynomial is a concave piecewise function with linear components. In fact, the following theorem gives us a stronger result [**?**].

Theorem 2.1.7. *The set of tropical polynomials on $x_1, \ldots, x_i$ is precisely the set of concave piecewise linear functions on $\mathbb{R}^n$ with integer coefficients.*

Thus, when we study tropical geometry, we are studying properties of certain piecewise linear functions.

2.2 Tropical hypersurfaces

In this subsection, the treatment of this material follows from **?**.

Classical algebraic geometry focuses on algebraic varieties, which are geometric objects that roughly correspond to the roots of polynomials. In tropical geometry, we focus on geometric objects that can be represented as the hypersurface of some tropical polynomial.

Definition 2.2.1. Note that every tropical polynomial p can be expressed as the minimum of a finite collection of distinct linear objects. The *hypersurface*

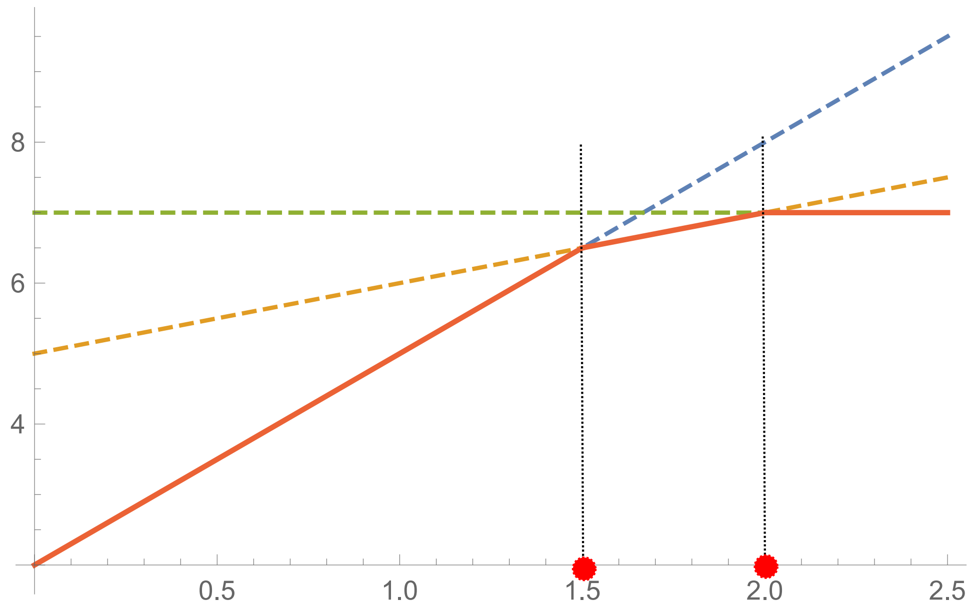


Figure 2.3 The polynomial $p(x) = 2x^3 + 5x + 7$ and its corresponding hypersurface. Image created with Mathematica.

of p, denoted $\mathcal{H}(p)$, is the set of points for which this minimum is achieved by at least two of the linear objects.

We can envision the tropical hypersurface as a sort of "spine" of the tropical polynomial. We now refer to our previous two examples to see what the corresponding hypersurfaces look like.

Example 2.2.2. Recall our polynomial $p(x) = 2x^3 + 5x + 7$ Figure **??** shows the places where this minimum is achieved twice. Thus, we see that $\mathcal{H}(p)$ consists of the points $x = 1.5$ and $x = 2$.

As this example suggests, tropical hypersurfaces in one variable always consist of a finite set of points.

Example 2.2.3. Recall our example

$$p(x) = (x^2)\oplus(x\otimes y)\oplus(3\otimes y^2)\oplus(2\otimes x)\oplus(y)\oplus(-1) = \min(x+2, x+y, 2y+3, x+2, y, -1).$$

Figure **??** shows where this minimum is acheived twice.

Note: We may refer to tropical hypersurfaces on two variables as *tropical curves*. Note that in the previous example, the hypersurface can be veiwed a set of bounded and unbounded edges meeting at vertices. It turns out that hypersurfaces in two dimensions must satisfy certain structural conditions [**?**].

Theorem 2.2.4. *Let $p(x, y)$ be a tropical polynomial in two variables. $\mathcal{H}(p)$ can be viewed as a graph embedded in $\mathbb{R}^2$ with the following properties*:

- *$\mathcal{H}(p)$ has a finite number of vertices and edges.*
- *$\mathcal{H}(p)$ must contain a nonzero number of unbounded edges. It may also contain bounded edges.*
- *Each edge has rational slope.*
- *The collection of edges emanating from each vertex satisfy the following balancing condition*: *Let $v = (v_1, v_2)$ be a vertex of the hypersurface. Let $e_1, \dots, e_n$ be the edges emanating from v. For each $1 \leq i \leq n$, let (x_i, y_i) be the closest lattice point on e_i to v.* (*Note that such a point must exist because the edges have rational slopes*) *Let u_i be the vector $(x_i - v_1, y_i - v_2)$. Then*

$$u_1 + u_2 + \cdots + u_n = 0.$$

The main takeaway here is that tropical hypersurfaces are fundamentally combinatorial objects. In the next section, we will introduce another type of combinatorial object that we can naturally associate with tropical hypersurfaces, relating these objects to polyhedral geometry.

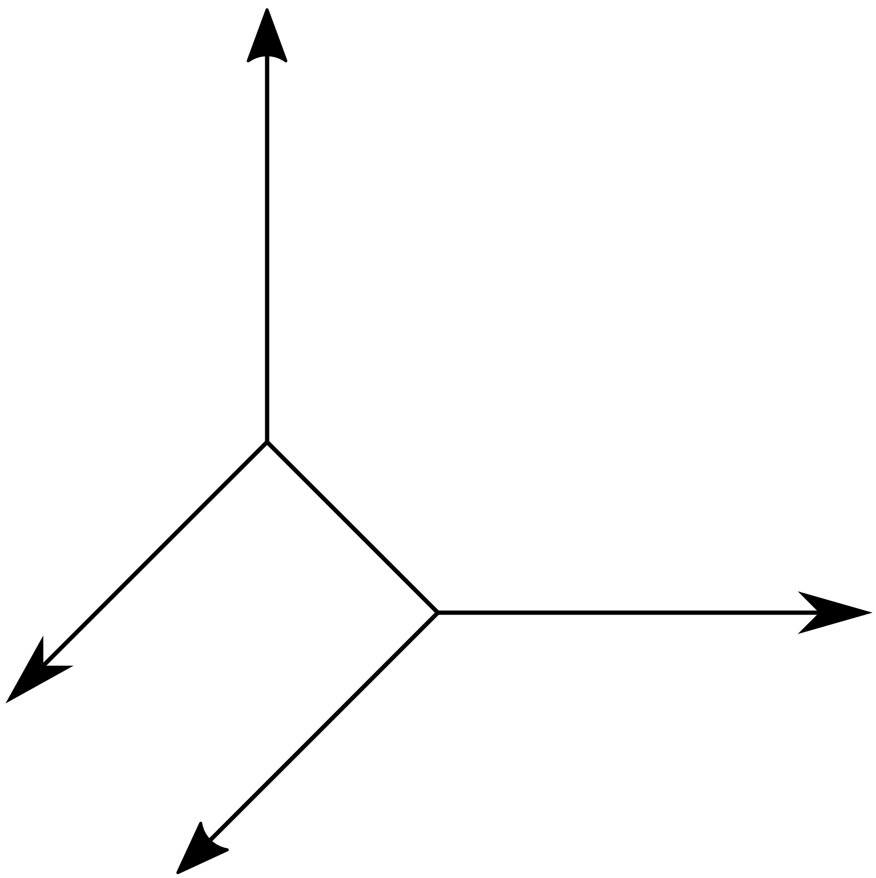


Figure 2.4 The hypersurface corresponding to $p(x) = x^2 + xy + 3y^2 + 2x + y - 1$

Chapter 3

Newton Polytopes

In this chapter we define the combinatorial type of a tropical polynomial. It turns out that there is a correspondence between tropical curves and objects called Newton polytopes, and the combinatorial type can be found through this correspondence. In this section, we will introduce these objects and outline a method of how to find the combinatorial type of a given polynomial using these objects.

Definition 3.0.5. A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite collection of points in $\mathbb{R}^n$.

For each tropical polynomial, we can construct the following polytope.

Definition 3.0.6. Let $p(x, y)$ be a tropical polynomial. For each monomial $a_{i,j}x^i y^j$ of p, for $a_{i,j} \neq \infty$, let $P_{i,j} = (i, j) \in \mathbb{R}^2$. The corresponding *Newton polytope* to p is the convex hull of all such $P_{i,j}$. We denote this polytope as newt(p):

$$\text{newt}(p) = \overline{\{P_{ij}\}} = \text{conv}(\{P_{ij}\}),$$

where conv(A) denotes the convex hull of a set of points A.

Furthermore, we can construct a specific subdivision of newt(p) based on the coefficients of $p(x, y)$. First, we more formally define a subdivision in this polytope.

Definition 3.0.7. Let A be a finite set of points and let Q be conv(A). A *subdivision of* (Q, A) is a subdivision of Q into polytopes whose vertices are elements of A.

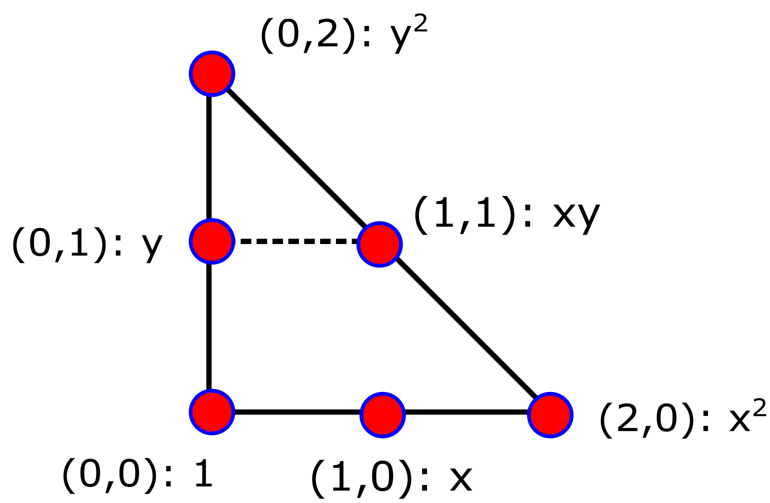


Figure 3.1 The polytope corresponding to $p(x) = ax^2 + bxy + cy^2 + dx + ey + f$ and a possible subdivision

Here, we will typically let Q be newt(p), and let A be all (i, j) such that $a_{i,j}x^i y^j$ is a monomial of p where $a_{ij} \neq \infty$. For each monomial of $a_{i,j}x^i y^j$ of p such that $a_{i,j} \neq \infty$, we let $Q_{i,j}$ be the point $(i, j, a_{i,j})$. In the subdivision corresponding to p, we include the segment between $P_{i,j}$ and $P_{m,n}$ if and only if the following condition holds:
The segment between $Q_{i,j}$ and $Q_{m,n}$ does not lie above any of the points in the convex hull of the set of $Q_{i,j}$.

Thus, one can specify inequalities involving the coefficients of $p(x, y)$ that determine whether specific edges exist in this subdivision of $newt(p)$.

Example 3.0.8. Consider some generic conic

$$p(x) = ax^2 + bxy + cy^2 + dx + ey + f$$

Figure **??** shows the $newt(p)$ along with a possible subdivision, depending on the coefficients.

We use this subdivision to define combinatorial type:

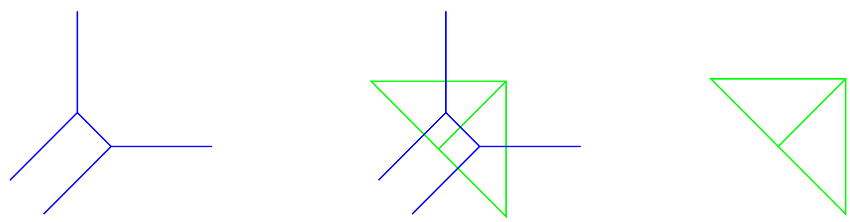


Figure 3.2 A tropcial hypersurface and the corresponding subdivision of the Newton polytope.

Definition 3.0.9. Two tropical polynomials p and q of the same degree have the same combinatorial type if and only if they result in the same subdivision of the newton polytope through following the steps above.

The following result shows that this subdivision does indeed encode some important combinitorial information about $\mathcal{H}(p)$ [**?**].

Theorem 3.0.10. *Let G be the graph formed by* $newt(P)$ *and subidivision obtained by following the above steps. The dual graph of G is equivalent under graph isomorphism to* $\mathcal{H}(p)$*.*

Example 3.0.11. Recall our polynomial $p(x) = x^2 + xy + 3y^2 + 2x + y - 1$ from a previous example. Figure **??** shows how the subdivision of $newt(p)$ and $\mathcal{H}(p)$ are dual graphs to each other.

This correspondence allows us to enumerate all possible combinatorial types for polynomials of a given degree, since there a finite number of subdivisions of a given newton polytope that can arise from the above procedure. **?** uses this technique to find relationships between the coefficients of the polynomials that determine the combinatorial type. We use this paper in more detail in the next chapter.

Chapter 4

Prior work

4.1 Explorations in One Variable

We recall that in one dimension, tropical hypersurfaces consist of finite sets of points. Here, we consider two polynomials to be of the same combinatorial type if and only if their hypersurfaces contain exactly the same number of points. [**?**] explores this question of finding the number of roots of a randomly generated tropical polynomial. In this section we briefly describe their techniques and results.

The main result of this paper is the following:

Theorem 4.1.1 (Theorem 1 in **?**). *Let X be a continuous distribution, supported on $(0, \infty)$. Assume $F(y) \sim Cy^a + o(y^a)$ as $y \to 0$ for some constants $C, a > 0$. Let Z_n be the number of points in $(H)(p)$, where $p(x)$ is tropical polynomial of degree n with each coefficient taken from F. Then as $n \to \infty$,*

$$\frac{Z_n - \frac{2a+2}{2a+1}\log(n)}{\sqrt{\frac{2a(a+1)(2a^2+2a+1)}{(2a+1)^3}}\log(n)} \to \mathcal{N}(0,1)$$

This result tells us that for certain distributions that decay like polynomials near 0, as the degree of random polynomials increases, the distribution given by the number of zeros of the polynomials approaches a normal distribution. The scaling and shifting from the standard normal depends on the distribution and the degree of the polynomials. When I first considered the main research question of this thesis, I envisioned drawing the coefficients from identical uniform distributions. However, this paper suggests that it may be enlightening to consider drawing the coefficients of our polynomials from other distributions, and see what results are preserved and what changes.

In order to reach this result, Baccelli and Tran use the idea of subdivisions of newton polytopes in one dimension. In this dimension, the polytope is the real line, and the subdivision is a collection of points that break up the real line. The coefficients determine points in $\mathbb{R}^2$, whose lower envelope determines the subdivision. Thus, we can answer the original question by considering the following question: Suppose we have some randomly chosen points in the unit square, their convex hull forms a polytope, how many vertices does this polytope have?

One thing that might make our exploration have a slightly different flavor from this exploration is that in the one-dimensional case, we can associate a number to each of the combinatorial types, since these types are simply the number of points in the zero set of the polynomial. Since we have a number associated with each type, we can think about the result in terms of probability distributions. Thus, in this case it seems that there is more immediate context in which to place the results. However, in our case we do not seem to have a natural way of assigning unique values to our combinatorial types, so it seems that we cannot really express our results in terms of probability distributions in exactly the same way. We will address this further in the methodology section of this paper.

4.2 Classification of Tropical Conics

In [**?**], Ellis defines a classification of all tropical conics in two variables. She partitions the set of tropical conics into several categories and for each category gives a necessary and sufficient set of restrictions on the coefficents. To find these restrictions, she uses the , simplifying some of the required casework by considering the set of linear morphisms on tropical space.

A summary of her results in Table 5 of this paper. This table provides the following information for each of her classifications: A representative picture of the curve, a picture of the corresponding subdivision of the newton polytope, and list of restrictions on the coefficients. Her classification scheme is finer than our classification by combinatorial types. Each of her classifications corresponds to exactly one possible subdivision of the Newton polytope of a generic conic, but there are several cases in which different subdivisions correspond to the same combinatorial type. However, we can combine subsets of her classifications to recover the classification we want.

Thus, this paper provides us with a good starting point in our research for several reasons. First of all, it gives us a classification of conics that we can use to calculate probabilities, and furthermore it provides us with a methodology that we may use to explore sets of higher higher degree polynomials. Perhaps most importantly, this paper lead me to think about my research problem in terms of the moduli space of coefficients of tropical polynomials, and which regions of this space correspond to various combinatorial types.

Chapter 5

Methodology

In the last section, we mentioned the difficulties that might arise if we tried to use a similar method that Baccali and Tran employed; we then briefly mentioned the possibility of approaching our main research question by considering what parts of the moduli space of coefficients. In order to do this, we will use an object called the secondary cone corresponding to each combinatorial type.

Before we discuss secondary cones and fans, we first briefly define some of the objects involved. In this discussion of cones and fans, we use definitions and results from **?**.

Definition 5.0.1. A *cone* is a subset of a vector space that is closed under taking positive linear combinations of its elements.

Definition 5.0.2. A *face* of a cone is an intersection of the cone with a hyperplane that includes no interior points of the cone. 0-dimensional faces are called *vertices* and 1-dimensional faces are *edges*.

Definition 5.0.3. A *fan* is a finite set F of cones such that the following properties hold:

- If $\sigma \in F$, then every face of $\sigma \in F$.
- If $\sigma, \sigma' \in F$, then $\sigma \cap \sigma'$ is a face of both σ and σ', and is thus in F.

Definition 5.0.4. A *maximal* cone in a fan is a cone that is not contained in any other cone in the fan, and is thus not the face of any other fan.

Fans and cones naturally arise in this project because they keep track of parts of the moduli space of tropical polynomial coefficients using combinatorial data. (This technique is used in **?** to determine which classes of metric graphs of a certain genus correspond to tropical polynomials.) We now discuss how we can use cones to help us keep track of which portions of this space correspond to certain combinatorial types of tropical polynomials.

Definition 5.0.5. Let Q be a convex polytope and $A \supseteq$ be a finite set of points $\{a_1, \ldots, a_n\}$ in $\mathbb{R}^k$. A subdivision of Q is a *regular* subdivision if there exist some $\ell_1, \ldots, \ell_n$ such that the subdivision can be realized through the following steps:

1. Take the lower convex hull of $(a_1, \ell_1), \ldots, (a_n, \ell_n) \in R^{k+1}$.

2. Project the facets of this convex hull back onto $\mathbb{R}^k$.

We refer to $(\ell_1, \ldots \ell_n)$ as a *lifting vector* corresponding to the subdivision.

From the above definition, we see that a subdivision is $regular$ if and only if it has a corresponding lifting vector. It thus seems seems natural to consider the set of all lifting vectors that correspond to a specific regular subdivision of Q. This leads us the the concept of a secondary cone:

Theorem 5.0.6. *Given a regular subdivision of a convex polytope Q on a finite set of points $A = \{a_1, \ldots, a_n\}$ that includes the vertices of Q. The set of all lifting vectors corresponding to this subdivision forms a cone.*

Theorem 5.0.7. *Given a convex polytope Q, the secondary fan with respect to a finite set A of n points is a complete fan on $\mathbb{R}^n$. In other words, each point in R^n is contained in a secondary cone corresponding to a subdivision of Q.*

Note that because of the way we construct the subdivisions in Chapter 3, the secondary cone of a specific subdivision consists exactly of lifting vectors corresponding to the coeffecients of polynomials that give rise to that subdivision. Thus, each secondary cone encodes all the polynomials of a specific combinatorial type. In the previous theorem, we see that the secondary fan of a polytope partitions Euclidean space into a finite number of regions each corresponding to different combinatorial type. To compare the likeliness of different combinatorial types, we need some way of comparing the size of these regions. We first consider the case in which these regions are of different dimensions, beginning with some background information.

Definition 5.0.8. A *regular triangulation* of an n−dimensional polytope Q is a regular subdivision of Q such that every region in the subdivision has $n + 1$ vertices.

Theorem 5.0.9 (?). *The maximal cones of the secondary fan of a polytope Q correspond to the regular triangulations of Q.*

Using this result, we can come to the following conclusion:

Corollary 5.0.10. *If the coefficients of a polynomial in two variables are chosen from a uniform distribution, the probability that the corresponding tropical curve contains a vertex with degree larger than three is negligible*

Proof. We first note that the curves with a vertex with degree larger than three correspond to subdivisions of the newton polytope that are not triangulations. Thus, as a result of the above theorem, the corresponding secondary cones are faces of cones of a higher dimension.

Thus, if you randomly choose a point in the moduli space of tropical polynomials, the probability that you choose a point corresponding to such a curve is negligible. □

However, in most cases we need to find some other way to compare the sizes of secondary cones, since they have infinite volume. In order to do this, we will turn each maximal cone into a polytope, and for each of these polytopes compute a polynomial that gives a sense of the original secondary cone.

The first step in finding a polytope that corresponds to the size of a given secondary cone is to put our secondary fan into a simpler form that maintains the information that is necessary to us. In order to do this, we first define the lineality space of a secondary fan:

Definition 5.0.11. The *lineality space* L of a fan is the common intersection of all of its cones.

Given the secondary fan F of a polytope, we construct a fan F' as follows: We map two points in F to the same point in F' if and only if their projection onto L is the same. Thus, we find F' by quotienting F out by its lineality space. Taking the quotient of F and its lineality space is used by in **?** to simplify later computations, so we take inspiration from these sources when we use this technique here.

We are now ready to find polytopes corresponding to each secondary cone of F, and thus to each combinatorial type. For each secondary cone in F, we find the corresponding cone $C' \in F'$. We construct a polytope $P(C)$ by taking the convex hull of the lattice points in C' closest to the origin along each edge of C', together with the the origin. Now that we have a polytope corresponding to each combinatorial type, we can compare their sizes in the following way:

Definition 5.0.12. Let P be a convex polytope on some lattice A. For each nonnegative integer t, $E(P,t)$ give the number of lattice points contained in the t-fold dialation of P. For a given polytope P, E is in fact a polynomial in t, and we refer to it as the Ehrhart polynomial.

The Erhart polynomial encodes the relationship between the area of the polytope and the number of lattice points that it contains. Recall that we wish to compare the size of cones with an infinite volume. Note that if we dialate $P(C')$ by larger and larger n, we obtain polytopes that sit inside C' and fill larger and larger portions of C'.] Thus, if we have two secondary cones C_1 and C_2, then

$$\lim_{t\to\infty} \frac{E(P(C_1),t)}{E(P(C_2),t}$$

gives a measure of the ratio of the sizes of C_1 and C_2, and thus the moduli space of tropical polynomial coefficients that correspond to each.

In summary, our overall strategy for comparing the prevalences of combinatorial types of tropical curves of a given degree is as follows:

1. We first find the Newton polytope of a general poynomial of degree n, $p(x) = \sum_{i+j\leq n} a_{i,j}x^i y^j$. Note that such a polytope will look like a right triangle of leg length n.

2. Find the secondary fan, F, of this polytope by finding the secondary cone of each possible regular triangulation of newt(p). Note that subdivisions that are not triangulations will have secondary cones that are faces of the cones corresponding to triangulations.

3. Compute F' by taking the quotient of F and its lineality space.

4. For each maximal cone $C' \in F'$, we construct a polytope $P(C)$ by taking the convex hull of the lattice points in C' closest to the origin.

5. Compute the Ehrhart polynomial, $E(P(C),t)$ for each $P(C)$. This gives a measure of the size of each $P(C)$, and thus the prevalence of the corresponding combinatorial type.

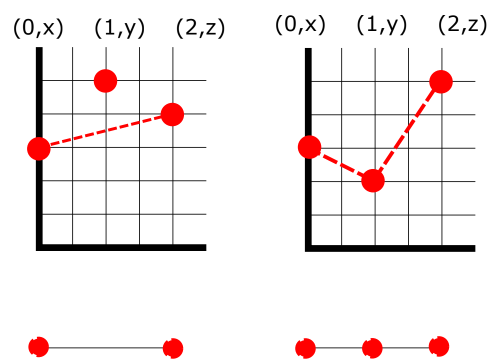
6. Compare the prevalences of combinatorial types corresponding to secondary cones C_1 and C_2 by computing:

$$\lim_{t\to\infty} \frac{E(P(C_1),t)}{E(P(C_2),t}.$$

Chapter 6

Quadratics in One-Dimension

We now apply the process outlined in the following section to the set of tropical quadratics in one variable[1]. The general form for such a polynomial is $p(x) = ax^2 + bx + c$, so newt(p) is the convex hull of $0, 1, 2 \in \mathbb{R}$, so just the closed interval from 0 to 2. There are exactly two subdivisions of this segment: We can subdivide it by including the point one, or we could not. We will refer to the unsubdivided segment as S_0, and the subdivided segment as S_1.



From the above figure, we can see that the set of lifting vectors (x, y, z) corresponding to S_0 if and only if $2y \geq x + z$. Thus, this halfspace forms the cone C_0 that is the secondary cone corresponding to S_0. We can also see that (x, y, z) corresponds to S_1 if and only if $2y \leq x + z$. This half space forms the cone C_1 that is the secondary cone corresponding to S_1 We can see

[1]The strategy and results here were outlined by M. Chan

that the lineality space L is the plane $x + z = 2y$, which are spanned by the vectors $(1, 0, -1)$ and $(0, 1, 2)$. If we use the basis $(1, 0, 0), (1, 0, -1), (0, 1, 2)$ for $\mathbb{R}^3$, when we project onto L we are left with R^1, with the positive half line correspnding to C_1 and the negative half line correspondint to C_0. Note that $P(C_0)$ is the segment $[0, 1]$ and $P(C_1)$ is the segment $[-1, 0]$. Thus, the corresponding ehrhart polynomials are both the polynomial $x + 1$, since dilating a unit segment by n results in segment of length n that covers $n + 1$ lattice points.

This suggests that the two combinatorial types of quadratics in one dimension have about the same likeliness, since they have the same Ehrhart polynomial.

Chapter 7

Quadratics in Two Dimensions

In this section, we apply the process outlined in the Methodology section to the set of tropical quadratics in two variabes[1]. The general form for such a polynomial is $p(x) = ax^2 + bxy + cy^2 + dx + ey + f$, so newt(p) is the convex hull of $(0,0),(1,0),(2,0),(1,1),(0,2),(0,1) \in \mathbb{R}^2$. This is a 2 by 2 right triangle. Note that because of **??**, the only combinatorial types that have substantial probability are those that correspond to regular triangulations of newt(p). There are only three regular triangulations of newt(p), and these are shown in [ref]. We will refer t0 these three triangulations as S_1, S_2 and S_3 as labeled in the diagram.

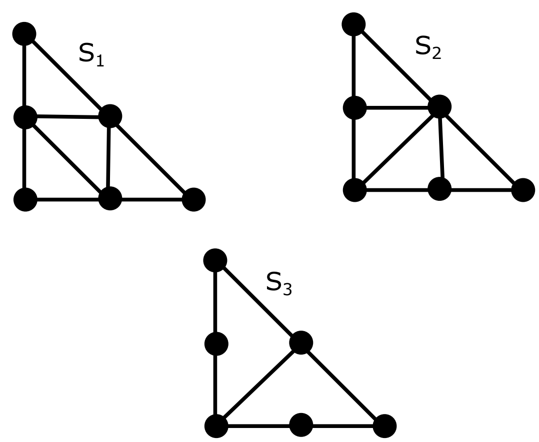


Figure 7.1 The three possible triangulations of newt(p).

[1]Again, the strategy here were outlined by M. Chan

Unlike the one-dimensional case, here each secondary cone is a subse tof $\mathbb{R}^6$, making it much more difficult to have a spatial intuition of how the secondary cones form a fan. Thus, to compute the Ehrhart polynomials of the secondary cones corresponding to S_1, S_2, S_3 we rely on the Gfan package of the open source software Polymake. This package can handle computations involving Groebner bases, Groebner fans, and secondary fans. Using this software, we obtain the following secondary cones corresponding to each S_i:

$$C_1' = \{a[0,-1,0,1,2,0] + b[0,0,0,0,1,0] + c[0,-1,0,-1,0,0] : a,b,c \geq 0\}$$

$$C_2' = \{a[0,0,0,0,1,0] + b[1,0,0,0,1,1] + c[0,0,0,0,2,1] : a,b,c \geq 0\}$$

$$C_3' = \{a[-2,0,0,0,0,-1] + b[0,0,0,0,0,1] + c[-2,0,2,0,0,-1] : a,b,c \geq 0\}$$

The above cones have already been modded out by the lineality space, since polymake automatically performs this step. The following are the Ehrhart polynomomials corresponding to each C_i', also computed with Polymake.

$$E(P(C_1),t) = \frac{1}{3}t^3 + \frac{3}{2}t^2 + \frac{13}{6}t + 1$$

$$E(P(C_2),t) = \frac{1}{6}t^3 + t^2 + \frac{11}{6}t + 1$$

$$E(P(C_3),t) = \frac{2}{3}t^3 + \frac{5}{2}t^2 + \frac{17}{6}t + 1$$

We can now make the following comparisons:

$$\lim_{t\to\infty} \frac{E(P(C_3),t)}{E(P(C_1),t} = \frac{2/3}{2/3} = 2$$

$$\lim_{t\to\infty} \frac{E(P(C_1),t)}{E(P(C_2),t} = \frac{1/3}{1/6} = 2$$

Thus, these results suggest that the when drawn from a uniform distribution, the combinatorial type corresponding to S_3 is roughly twice as likely as the type corresponding to S_1, which in turn is twice as likely as the type corresponding to S_1.