What is So Negative About Negative Exponents?

Geoffrey D. Dietz
Gannon University

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Department of Mathematics, Gannon University, Erie, PA 16541
`gdietz@member.ams.org`

Synopsis

While teaching college-level mathematics (from College Algebra to Calculus to Abstract Algebra), I have observed that students are often uncomfortable using negative exponents in calculations. I believe the fault partially lies in the manner in which negative exponents are taught in Algebra 1 or Algebra 2 courses, especially in rigid instructions always to write answers using only positive exponents. After reviewing a sample of algebra texts used in the United States over the last two centuries, it appears that while attitudes toward negative exponents have varied from author to author over time, the current trend is to declare explicitly that an expression is not simplified if it contains negative exponents. I believe that this negative attitude toward negative exponents is at least somewhat to blame for students of Calculus and higher mathematics being less able to solve problems that require conversion between positive and negative exponents, as their algebraic instruction has only taught them to convert negative exponents to positive.

Which of the following algebraic expressions is simpler?

$$\frac{x^2 z}{x z^3} \text{ or } \frac{x}{z^2}$$

What about these two?

$$\frac{x^2 z}{x z^3} \text{ or } x z^{-2}$$

Now, here is the tough one. How about these two?

$$\frac{x}{z^2} \text{ or } x z^{-2}$$

It should be unanimous in the first two cases that the right-hand expressions are simpler. In the third case, a debate might erupt. Some may believe the left-hand side is simpler, while others may believe that they are just different notations for the same simplified expression. I firmly place myself in the latter camp, and in this article, I will try to demonstrate that too many textbook authors in the United States place themselves in the former camp.

Positive exponents may be more natural symbols for students who are already comfortable with fractions, but what is really more "simple" about using all positive exponents versus negative exponents? I believe that many (or is it most?) modern algebra textbooks are placing too much emphasis on positive exponents being "simpler" than negative exponents.

As we will see, the dominant trend in textbook exercises for Algebra 1 and Algebra 2 regarding the laws of exponents has been to require only positive exponents in final answers. I believe that this common requirement fails our students in two ways. First, when there are no corresponding exercises to convert exponents from positive to negative (say, to rewrite a fractional expression without a denominator) students do not receive practice in a skill that is commonly used throughout the Calculus curriculum for derivatives, integrals, and applications of the binomial theorem. Secondly, I believe that the emphasis on only using positive exponents in final answers unintentionally trains students to think that negative exponents are something to be avoided and eliminated from a problem, instead of being an alternative notation that has value in unifying statements on exponents, such as the power rule in Calculus. This attitude (which I have observed in my students many times) not only damages their abilities to solve problems in Calculus but can also lay a bad foundation for students of Linear Algebra or Group Theory where negative exponents are the preferred notation for multiplicative inverses.

1. A Brief History of Exponent Notation

Florian Cajori's *A History of Mathematical Notations* [4] thoroughly documents the history of exponent notation. A brief summary follows below.

Modern exponent notation evolved over time culminating with Descartes' use of raised superscipts in his *Géométrie* in 1637, but he only used positive exponents. His notation appears to be a compromise between that used by James Hume, who would write raised exponents with Roman numerals, and Pierre Hérigone, who would write coefficients on the left and exponents on

the right without superscripts. For example, while we and Descartes would write $5a^4$, Hume would write $5a^{iv}$, and Hérigone would write $5a4$.

It appears that the development of Calculus directly motivated the invention of negative and rational exponents. In the 1650s John Wallis suggested the use of fractional and negative exponents but never actually wrote them down. He referred to fractional and negative "indices" being useful for area calculations involving the curves $y = k/x^n$ and $y = k\sqrt[n]{x}$ or for describing families of sequences. Isaac Newton may have been the first (in his 1676 letter to the Royal Society of London announcing the binomial theorem) to use and explain the meaning of the symbol $\frac{m}{n}$ as an exponent, where the fraction denotes any rational number, positive or negative.[1]

2. Sampling and Evaluating Textbooks

Now that we have a sense of why and when negative exponents came about, we are ready to examine how they have been taught in the United States. In the following, I chose to sample algebra textbooks used in the United States over the period from 1825 to 2012.

The textbooks from the 19$^{\text{th}}$ century were selected based on guidance from John Nietz's survey of American secondary school textbooks [24]. Nietz also cites books going back into the 18$^{\text{th}}$ century, but I relied only on books which I was easily able to obtain online or through interlibrary loan.

The choices for textbooks throughout most of the 20$^{\text{th}}$ century were guided by the NCTM's *A History of School Mathematics* [22]. Like Nietz, the texts cited were highlighted because they were either highly popular books in their day or were influential on future textbooks.

The textbooks that I review from the 1980s up through 2012 represent a convenience sample, obtained by visiting two local high school algebra teachers in the Erie, Pennsylvania, area, and asking one of my colleagues at Gannon University.

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[1] The use of negative exponents for inverse functions arose much later. The notation $f^{-1}(x)$ for the inverse function appears in the works of John Herschel in 1813–1820 dealing with inverse trigonometric functions, paralleling the use of $d^{-n}V = \int^n V$ as notation for an iterated antiderivative. Herschel credits German analyst Burmann in first using the notation d^{-n} for antiderivatives but notes that Burmann does not appear to extend the idea to inverse functions.

Attitude	Criteria
very tolerant	exercises require or clearly accept negative exponents as answers
tolerant	example answers use negative exponents, but exercises do not specify
neutral	mixed or indeterminate attitude
intolerant	only positive exponents are used in examples and exercises
very intolerant	there is a clear statement that simplification requires positive exponents

Table 1: The rating scheme for the perceived attitude of a book toward negative exponents.

The sampled textbooks were evaluated based on their perceived attitude toward negative exponents as described in Table 1 above. I will grant to anyone that this rating system is subjective but hope that my evaluation of the texts gives a reasonable measure of attitude.

2.1. 19th Century Textbooks

We begin with three algebra textbooks published between 1825 and 1892.

Colburn's *An Introduction to Algebra Upon the Inductive Method* [7] first appeared in 1825 and had gone through twenty editions by 1848. Relevant pages are shown in Displays 1, 2, and 3.[2] Based on the use of negative exponents in statements of exercises and the listing of both positive and negative exponents in answer lines, I rate Colburn's text as **tolerant** in its attitude toward negative exponents.

Thomson's 1844 text, *Elements of Algebra* [29], was based on the 1814 book, *Introduction to Algebra*, by Jeremiah Day. While Day, a Yale professor, wrote his book primarily for college students, he later commissioned Thomson to write an abridged version for secondary school use. According to Nietz [24], both texts were very successful. Based on the sample pages in Displays 4 and 5, I rate Thomson as **tolerant**.

Milne's *High School Algebra* from 1892 [20] was adapted from his 1881 text *Inductive Algebra*. This text is the first in our sample that has exercises that clearly require the student to convert positive exponents to negative exponents and is thus a **very tolerant** textbook. See Display 6.

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[2]All displays referred to are in the article supplement; see http://scholarship.claremont.edu/jhm/vol4/iss1/8.

2.2. *Early 20th Century Textbooks*

We now review five textbooks published between 1902 and 1962.

Academic Algebra by Beman and Smith [2] is from 1902. See Displays 7 and 8. While the examples demonstrate a tolerant attitude toward negative exponents (especially the reference to an *integral form* and the commentary that $x^{-1}y$ and $\frac{y}{x}$ are equivalent), the exercises require only positive exponents. I rate this text as **neutral**.

Elementary Algebra by Wentworth [32] from 1906 was one of the final books published by Wentworth. Nietz [24] states that Wentworth's texts "somewhat dominated" the mathematics market from the 1880s through the early 1900s. Based on examples and exercises that require writing expressions without denominators (see Display 9), I give a **very tolerant** rating.

First-Year Mathematics For Secondary Schools by Myers [21] comes from 1907. This text's attitude is difficult to assess as no examples are given in the section on negative exponents, but the instructions in the exercises seem to indicate that positive exponents are necessary for final answers. See Display 10. Overall, this appears to be the first **intolerant** text in the sample.

The text *Progressive High School Algebra* by Hart [17] first appeared in the 1920s. We review the 1943 edition due to availability. There are few examples, some exercises that require only positive exponents, and some that ask for calculations without any comment on simplification. See Displays 11 and 12. With a little doubt, I will rank this text as **neutral**, but the second batch of exercises indicate a degree of tolerance as well.

The final text for this time period is *Algebra - Book Two* by Welchons and Krickenberger. Copies of this text from 1949 [30] and 1962 [31] were reviewed. All examples and exercises require positive exponents in both editions. Thus, the texts are rated as **intolerant**.

2.3. *Dolciani Textbooks*

An entire section is devoted to the textbooks written by Mary Dolciani and her collaborators. Her textbooks have managed to be very successful and very highly regarded for 50 years. Even if they are not as greatly used in schools today, many parents still seek out older editions of these texts to help their children learn. These texts are also highlighted here because I

was able to obtain four editions: 1963, 1977, 1986, and 2000. These editions appear to illustrate my hypothesis that modern textbooks are growing more intolerant in their attitudes toward negative exponents.

In the 1963 edition of *Modern Algebra and Trigonometry: Structure and Method Book 2* [11], we see a **very tolerant** view of negative exponents. In the exercises shown in Displays 13 and 14, the teacher's edition solutions make it very clear that negative exponents are to be used in the final answers.

The 1977 edition of the same text [12] is similarly **very tolerant** as can be seen in Displays 15 and 16. The teacher's edition solutions show that final answers are considered simplified when all cancellation has occurred, but variables may contain positive or negative exponents.

The strange part of the story is that the 1986 edition of this text [13], as well as the 2000 edition [14] of *Algebra: Structure and Method Book 1*, has become **intolerant** toward negative exponents. (The topic of negative exponents moved from Algebra 2 to Algebra 1 somewhere between these editions.) All exercises in these editions require only positive exponents in final answers. See Display 17. This change from the earlier editions is even more interesting as the 2000 edition is stamped as "The Classic" on its cover.

2.4. Modern Textbooks

The final books under review come from a convenience sample containing textbooks published since 1983.

Foster's *Algebra 2 with Trigonometry* [16] from 1983 receives an **intolerant** rating as it requires positive exponents for all answers in examples and exercises. The text *Algebra* by Corcoran [8] was published in 1984. A single example leaves x^{-3} as a final solution when introducing negative exponents, but all exercises require only positive exponents. Thus, the text is rated as **neutral**. *Algebra 2* by Coxford from 1987 [9] receives an **intolerant** rating for only using positive exponents for final answers in examples and exercises. Also from 1987, *Algebra 2 with Trigonometry* by Dilley [10] appears to be **very tolerant**. See in Display 18 that examples and exercises require conversion from positive to negative exponents.

The text *Algebra and Trigonometry* by Foerster [15] comes from 1994. This text appears to be **very tolerant** as well. An example from page 245 of the text shows the following simplification:

$$\frac{(6x^{2/7}y^{-4}z^0)^3}{9x^2y^5z^{-8}} = \cdots = 24x^{-8/7}y^{-17}z^8.$$

Not only does this example end in negative exponents, but page 247 contains a batch of exercises asking the student to "[w]rite the answer as a product of powers, with no variables in the denominator," a request that will require negative exponents in the final answer. Finally, this text contained no exercises that required only positive exponents in the answer.

Saxon's *Algebra 2: An Incremental Development* from 1997 [25] also manges to be a **very tolerant** textbook. The student exercises stress conversion between positive and negative exponents as well as writing expressions with no denominators. See Display 19.

The text *Algebra 2* from Schultz, et. al. [26] in 2003 requires only positive exponents in all examples and exercises, and so it is classified as **intolerant**. The 2007 edition of this text [3], with Burger as its new lead author, becomes the first case of a **very intolerant** text in our sample. This text earns this rating by not only requiring all positive exponents in examples and exercises but for also going the extra step of declaring a blanket rule that simplified always means using only positive exponents.

The slightly older text *Algebra 1* by Bellman, et. al. [1] from 2004 is also **very intolerant**. Like [3], this textbook has elevated the preference for positive exponent answers of some authors to what sounds like a universal rule of mathematics: "An algebraic expression is in simplest form when it is written with only positive exponents." [1]

The last three texts all come from the same publisher. The Glencoe series of Algebra 1 and Algebra 2 texts ([18], [19], and [6]) from 2008 and 2012 all continue the trend of not only requiring positive exponents only but also making explicit declarations that simplified means only positive exponents. The Algebra 1 text [18] states on page 369:

> An expression is simplified when it contains only positive exponents.

The Algebra 2 text [19] similarly states on page 312:

> To simplify an expression containing powers means to rewrite the expression without parentheses or negative exponents.

Each is rated as **very intolerant**.

Given the history of the negative exponent notation and its origins in the history of Calculus, it is surprising to note that in the newest editions of Stewart's Calculus texts, for example [27], the new diagnostic test in algebra at the front of the text also only tests students on their abilities to convert negative exponents to positive exponents. One could argue that the opposite conversion is far more important to master in Calculus.

2.5. Summary

A scatterplot of the ratings versus publication dates does not show much of a global trend. See Figure 1. The cluster of intolerant and very intolerant ratings for the 21st century textbooks, though, may trouble anyone who agrees with the stance that positive exponents are not necessarily simpler than negative exponents and that we should not train students to shy away from using negative exponents.
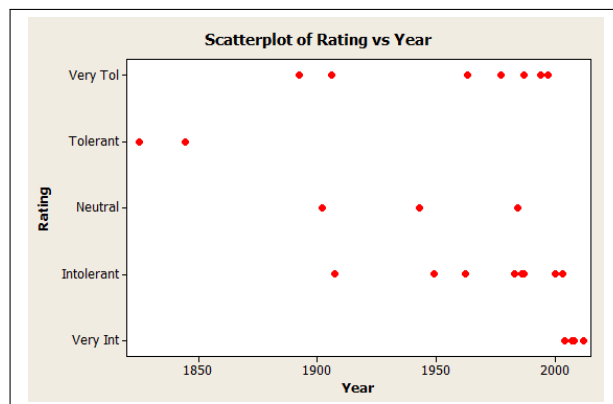

Figure 1: Scatterplot of Rating Versus Year

3. Common Core Standards and AP

The bias against negative exponents in many textbooks does not appear to be a result of modern national standards set for mathematics education. In the recently released Common Core State Standards [23], the only mention of negative exponents comes in 8th grade and is neutral on the issue.

> **8.EE.1** Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \cdot 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.

The standard makes no mention of only using positive exponents. Of course, the standard also only refers to numerical expressions and not general algebraic expressions.

The emphasis on positive exponents also does not come from the writers of the AP Calculus exams. Free response answers on the AP Calculus exams do not require students to simplify answers at all, let alone use only positive exponents [28].

> Unless otherwise specified, answers (numerical or algebraic) need not be simplified.

4. Conclusions and Questions

I am not sure where misuse of exponent rules ranks among the great "sins against algebra" committed by students, but I still wonder: Given the utility and convenience of negative exponents in Calculus (and other fields of mathematics) and the increasing number of students taking Calculus in college or high school, why would textbook authors and publishers go out of their way to imply that there is something "negative" about negative exponents? This bias against negative exponents may or may not lead to future mathematical errors; still it seems plausible to me that it might encourage students to hesitate when changing a positive exponent into a negative exponent. While the strongest students have little trouble with these issues, somewhat weaker students may occasionally fall into a mode of thinking that negative exponents are taboo. A recent study [5] also points out that college students tend to have persistent misconceptions when it comes to manipulating algebraic expressions containing negative signs in the base or exponent.

This investigation leaves a number of questions to ponder (many of which have no objective answer): Why do many textbook authors (and teachers) place so much emphasis only on conversion from negative to positive exponents? Is there some intrinsic value in writing final answers using only positive exponents so that the use of an equivalent expression that includes negative exponents should be penalized? Or is this just an aesthetic issue?

Is it done for the convenience of students who find unique answers to problems more comforting? Is it done for the convenience of teachers who find unique answers to problems easier to grade? Are aesthetics, comfort, or ease of grading good enough reasons to bias students against negative exponents despite the future risks to their mathematical abilities?

Without conducting large-scale surveys of the major publishers, textbook authors, and algebra teachers, I am unable to provide firm answers to any of these questions. I believe, however, that these are important questions to ask.

Acknowledgments

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