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On Contemplation in Mathematics

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Synopsis

In a section about research, we make the case that intentional, structured reflection on the mathematical research process, by mathematical researchers themselves, would result in better mathematicians doing better mathematics. As supporting evidence, we describe the *Flavors and Seasons* project. In a section about teaching, we describe the contemplative education movement and share personal experiences using meditation in the math classroom. We conclude with an explicit proposal for elucidating the experiential context of mathematics, in both research and teaching environments.

1. Introduction

Mathematicians like to think. We are trained to use extreme focus to solve problems, and to use incredible imagination and cleverness to discover/create new connections and abstractions. Sometimes I tell my students that thinking like a mathematician means blowing your own mind, over and over again. Given this heightened curiosity and imagination, flexibility and originality of thought, we are remarkably un-self-aware, particularly about mathematics itself. Indeed, there are many mathematics professors who pay no attention to the process of doing or teaching math, thus dehumanizing it.

Robert Kegan and Lisa Laskow Lahey, in a book about change and growth, have a nice technological metaphor: "True development is about transforming the operating system itself, not just increasing your fund of knowledge" [10, page 6]. For the most part, mathematicians devote their imaginations towards the files and programs, but rarely towards the operating system, the human doing the thinking. In the last year of my graduate program, I entertained myself by asking professors, "What makes a PhD? What tacit knowledge must I assimilate, what social norms must I adopt, what associations must I affiliate with, what mathematician-like things must I do, before I can truly join the community of mathematicians?" The answers were usually a combination of bewilderment and "Um, you need some results." Sure, results might be necessary, but are they sufficient?

This paper is about two ways that self-reflection can yield better mathematicians doing better mathematics – whether that means research or teaching or learning. It is a call to contemplate.

I will restrict myself to two specific definitions of contemplation. In the second section, about research, contemplation will mean self-reflection on the experience of doing research mathematics – reflecting on how you sit down with a pen and paper (or computer, or otherwise) to prove results. I make two claims. First, that reflecting on how you do research will help you be more effective and efficient at research. Second, that experiences that arise during research have common characteristics shared across time and person. To establish evidence toward these claims, I will describe the ongoing *Flavors and Seasons* project, which attempts to document and catalogue canonical research experiences.

In the third section, about education, contemplation will mean specific "contemplative practices," such as meditation. Indeed, millennia-old contemplative traditions see the mind as a tool, the control of which must be trained and honed. There is a movement in higher education to incorporate contemplative practices into the classroom, and there is a growing body of scientific literature establishing the benefits and limitations of doing so. I will briefly outline this movement and some of these studies, before describing my own experiences using meditation in my differential equations class.

There are other aspects of contemplation in mathematics that I will not consider here. Much of what is said in the second section, regarding self-reflection on the research experience, applies as well to the teaching experience. And the content of the third section is just one example of introducing metacognition into the classroom. I have chosen to focus on self-reflection in research because I think it is particularly lacking, and on meditation in the classroom because I think it is particularly novel and promising.

In the conclusion, I propose a strategy for encouraging self-reflection among practitioners of mathematics. Using a specific model set forth in my PhD thesis, I propose incorporating "Experiential Context" sections into written mathematics—whether published research papers or classroom assignments. This non-rigorous meta-data would force and induce contemplation, while closing the gap between the doing of mathematics and the artifacts of that doing.

2. Contemplation in Research

Here I will look at the role that reflection plays, or might play, in the mathematical research experience. In this section, by *contemplation* or *reflection*, I mean specifically self-reflection on the lived experience of doing research math. For example, one might contemplate the feeling of aesthetic pleasure induced by certain concepts and proofs. One might reflect on the emotional and cognitive context of a particular research work session. Or one might even contemplate the norms and tacit knowledge of the math research community.

It is evident that such reflection adds depth to the mathematical experience, and contributes to the enjoyment and meaning we find in doing math. Reflecting on the human dimensions of math might give us greater appreciation of the content, or might help us find our place in a community of peers. But is there a more direct way in which contemplation of the research experience might inform the research process itself, in a way that spurs on the development of new and good math?

Claim 2.1. *Contemplation of the research experience matters. Awareness of, and reflection on, the research process will result in more and better mathematical results.*

I claim that the more you understand how you do math, the more you will understand how you do your best math, and the more effective and efficient you will be at doing so. Below I will give more evidence to this claim, using specific examples from the *Flavors and Seasons* project.

But first, here's a simple example. Have you ever been working on a math problem, research or otherwise, and found yourself completely confounded, stuck, perhaps staring at a page or off into space, frozen in stalemate for, say, over a minute? How long did it take you to realize the mental wheels had stopped turning or were stuck in a rut? Some kinds of stuckness are productive, and indicate we're right on the verge of an insight. Other kinds of stuckness are unproductive and a waste of time. (Do you agree?) Can you distinguish between the two? Are there subtle cues, in how your thoughts have been proceeding, or in the quality of your awareness, or maybe even in your emotions and what your gut is telling you, that might indicate what to do next? Should you stick with the stuckness and work for a bit more, or get up and go for a walk? If you assess the situation correctly, you might have a breakthrough, but if you're wrong you might make yourself miserable and waste several hours.

Keep working or take a break? In such an example, I think it's clear that a deeper understanding of one's own process will help to make the right decision more often. If you decide to keep working, maybe there's a different approach or work technique you can try? If you decide to take a break, what sort of break do you need?

An analogy can be made by looking at the art community, where the value of articulating and discussing process is apparent. An art critique, among a group of painters for example, consists of an analysis of the aesthetics, themes, and ideas contained in a piece, and a discussion of the 'how' behind the work. How art is made, how art has been made, and how art could be made in the future are inextricably woven into the artistic discourse. Many, or most, artists are constantly contemplating their process and experimenting with different modes of working. Clearly, the experience of doing art is valued as an important dimension of art, worth thinking and talking about, just as much as the objects of art are.

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Reflecting on one's process can mean looking outward, inward, or both. One can look out at the ever-growing body of literature – by psychologists, cognitive scientists, philosophers, sociologists, and even mathematicians – attempting to understand the 'how' and 'why' of mathematical practice[1] These larger perspectives and empirical studies surely have a lot to offer to a working mathematician who has been trained via graduate apprenticeship. Such one-on-one apprenticeships, necessarily local and limited, teach you to find and solve problems without ever addressing interior experience, and teach you to act and live like a mathematician only by imitation. As

[1] I keep a reference list of my personal favorites at: http://tinyurl.com/cb7vrve, accessed January 26, 2013.

a graduate student, I organized a seminar on "The Mathematical Experience" that brought together undergrads, grad students, faculty, and staff to read and discuss such "meta-mathematical" literature. Despite departmental resistance to the seminar (and modest attendance), it was apparent that participants enjoyed and benefited from the chance to reflect.

Even more important and fruitful, I believe, is learning to look inward. Can you distinguish between the two types of stuckness described in the first example above? Can you check in with your state of mind, look back and observe the way you have just been thinking, establish some distance and assess your mood and emotions? How does one go about developing this type of self-awareness?

But wait. First we must address a deeper issue: do we even agree that there are two types of stuckness? Isn't this a personal, subjective distinction? Am I claiming that there is some amount of universality, reproducibility, and stability to mathematical experiences, so that we can characterize and categorize them meaningfully? Indeed, I am.

Claim 2.2. *Mathematical research experiences have an intersubjective reality. You and I share common experience, and we should think about this and talk about it together.*

Intersubjectivity is defined as the sharing of subjective states by two or more individuals [19]. "The analysis of a corpus of descriptions of a same type of experience enables the researcher to identify regularities of structure, that make an intersubjective validation possible" [15, page 395]. Intersubjectivity is a middle ground between the comfortable third-person objectivity of classical science, and the purely subjective whims of an individual's heart and mind. In psychology and philosophy, a great deal of work has been done towards establishing "first-person methods" that allow us to apply the scientific method to interior states [26, 15, 4]. The challenges are evident—particularly regarding verifiability, falsifiability, interpretation, and reproducibility—but the progress is impressive. Introspection methodology has developed a means of measuring, for example using eye movements, the depth of experiencing in a subject that has been asked to evoke a particular experience for the purpose of describing it [8]. Practical methods have been developed for generalizing from descriptions of experience to categories of experience [14]. The degree to which realtime reflective verbalization interferes with purity of experience has been examined [20].

If we accept mathematics as an aspect of human culture [27], and mathematical training as embedded in a cultural context, it seems unavoidable that different practitioners will have assimilated common modes of thought and process. What are these modes, and how are they similar or different among mathematicians?

William Byers has looked at the intersubjective nature of mathematical thought, specifically the use of ambiguity, metaphor, and paradox in mathematical reasoning throughout history [3]. He makes the case that mathematical ideas themselves have an intersubjective reality – that we should think of ideas not only as objective, formal immutables but as intersubjective experiences.

Another in-depth investigation of intersubjectivity of the mathematical experience comes from Jacques Hadamard, an accomplished number theorist and differential geometer, who was intrigued specifically by the process of mathematical discovery. Hadamard built on the ideas and analysis of Poincaré [16] to suggest a common psychological process of discovery: immersive preparation, stuckness and dissonance, insight, and analytical follow-up. In his book [7], he establishes evidence, proposes reasons, and discusses exceptions to his theory. He relies on his personal experience, in-depth discussions with colleagues, and responses from a general survey, "An Inquiry into the Working Methods of Mathematicians," conducted in 1902. Although at the time, the field of psychology was in its infancy, Hadamard's perceptions and analyses, as a mathematician attempting to understand his own mind, are valid and compelling.

2.1. Case study: The Flavors and Seasons Project

I have been attempting to intentionally contemplate my mathematical experience for years, and use such reflections to do better math. Inspired by Hadamard's work, the *Flavors and Seasons* project grew out of a desire to standardize these reflections in a format that would be accessible to other mathematicians.

A *flavor* is a shorter math research experience, one that lasts for a few hours or days. A *season* is a longer research experience, lasting for weeks or months. The project is an attempt to document the flavors and seasons of research mathematics, to build a database of mathematical experiences. This is done using a list of standard questions that are to be answered as precisely and concretely as possible. Since most flavors repeat themselves, and the seasons last for some time, the emphasis during documentation is on those aspects of the experience that seem to repeat or endure. It is not claimed that these experiences are unique to mathematics, but in most cases some unique characteristics can be described.

2.1.1. Questions

The questions for each flavor are:

1. What is going on mathematically?
2. What is the emotional and logistical context?
3. What thoughts are there?
4. What quality of awareness?
5. What emotions?
6. What does it resolve to, after how much time?
7. How frequent is this flavor?
8. What are good and bad ways to change/follow it up?

The questions for each season are:

1. What mathematical activities? What level of rigor?
2. What relevant interactions with other mathematicians?
3. How does it feel, what is the mood?
4. What state of mind? stable vs. chaotic? focused vs. dispersed?
5. What type of self-reflection during the experience, and did it help?
6. An everyday metaphor for the experience?
7. An example of a good day and a bad day?
8. What did you do when you were stuck?
9. When and why did it end?

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Each flavor and season report is written up and posted on a website.[2] The project is ongoing, but at time of writing, there are twenty-two flavors and seven seasons posted. Some examples are included below. Reports will often include hyperlinked references to related writings of other mathematicians. Some are attracting comments from readers.

[2] http://flavorsandseasons.wordpress.com accessed January 26, 2013.

The purpose of publicizing such reflections is to provide empirical evidence for Claims 2.1 and 2.2. Specifically, the answers to flavor question #8 and season question #5 often reveal concrete ways in which reflecting on process allows for improved math-doing. And by focusing on the recurring and more universal aspects of these experiences, the hope is that other mathematicians will find some overlap with their own process, and thereby affirm the intersubjective reality of our shared experience. If we can do this, then maybe we can start to embrace the intersubjective and experiential dimensions of our ideas, as Byers proposes. And then "mathematics" can mean so much more.

Here's a playful analogy. When a Hollywood director sits down to make a classic romantic movie, he or she might start with a list of canonical romantic experiences to include. Despite the vast diversity of human experience, we all live in a culture, and can easily recognize social norms of romance. In an American context, the flavors might include "the second date," "meeting the family," "cooking a meal together"; and some seasons might be "getting to know each other," "traveling together," and "long-distance." The following, then, is an incomplete list of themes that should appear in a summer blockbuster about the mathematical life.

2.1.2. Current flavors and seasons

The current list of flavors is:

Breakthrough I: Getting stuck.

Breakthrough II: Sticking with stuckness.

Breakthrough III: Rest.

Breakthrough IV: Insight.

Discussing with a colleague.

Dull mind.

Filling out results.

Finishing the PhD.

Getting your hands dirty, to clear up confusion.

Intentioned immersion.

Making a big mistake.

Math anxiety: "I should be doing more math."

Mathache.

Natural inspiration.
Preparing a (research) talk.
Proving myself wrong, via counterexample.
Stunned and sublime.
Translating a proof from one context to another.
Trying to explain math to a stranger.
Unintentioned immersion.
Using my math powers for evil.
Working through a proof.

The current list of seasons is:
Applying for jobs.
Building a theory / Imagining what could be.
Crescendo.
Doing computations.
Entering a field.
Pulling together and writing up.
Rest.

♢

2.1.3. Example flavor: Proving myself wrong, via counterexample.

The following is taken verbatim from the website.

1. What is going on mathematically?

After working towards proving something for a while, I find a counterexample. This involves an insight, followed by a verification.

2. What is the emotional and logistical context?

These counterexamples usually show up suddenly. The most dramatic and surprising cases are after working towards a particular result for weeks, because my expectation is that I've been getting closer and closer to a complete proof. So counterexamples hit when I'm hopeful, maybe even overly idealistic.

3. What thoughts are there?

The initial insight is a surprising "Aha" moment, accentuated by the fact that most counterexamples have a simplicity and necessity that seems to stab directly into the essence of the problem. This is immediately followed by some concerned analysis of the situation – does this mean I just wasted three weeks? is there a way to fix it? But before a complete reassessment, there's a careful verification, to prove that the counterexample is a counterexample, i.e. to prove myself wrong.

4. What quality of awareness?

It's like the rug has been pulled out from underneath me. There's a shock and surprise, grounded in certainty, that then trickles outward along logic pathways and finds a deserted city. Or worse, the city I thought I knew is now filled with people that speak a language I don't understand. On a deep level, it's an unsettled, shifting, almost paranoid wandering in this strange new city, searching for any familiar faces. But on the shallow level, there's a sharp certainty and cleanness, as my proved counterexample resonates within itself.

These are the times when I'm most aware of the non-logical, heuristic, mysterious "intuition" I have built up about the math I do. I had a mathematical worldview in which Proposition X was true – this sense of the way things work guided me, helped me make sense of it all. But now that I have found a counterexample, it's not just the statement of Proposition X, but the whole worldview, that needs to be adjusted.

5. What emotions?

Of course, I usually feel disappointed and frustrated, depending on the severity of the situation. At worst, it can devolve into fatigue and meaninglessness. (I'm fortunate that the most time I've thrown away on a false proposition is 2.5 weeks; I'm sure it gets much worse than that.) There's also an undeniable sense of finality, that comes with proving any result – "at least now I know for sure." It is a very strange feeling, to prove yourself wrong. This certainty is a feeling I almost only get from math, and for some reason I feel it more strongly when I've been proven wrong than when I've been proven right.

I'll usually take a break from the problem for a bit, and then I feel some revulsion towards it. Maybe it's a feeling of being betrayed, but I don't want anything to do with the question. This goes away soon, though.

6. What does it resolve to, after how much time?

A good mathematician would say that in every counterexample there's new ideas to follow up. Maybe I just need to tweak my hypotheses; maybe the counterexample is pointing towards the essence of what's going on; maybe the fact that Proposition X fails is a "good" thing, that e.g. allows for more interesting behavior. I can usually start to pick up the pieces after a few hours.

7. How frequent is this flavor?

Oh, I'm such a bad research mathematician, this happens way too much.

8. What are good/bad ways to change or follow it up?

Bad: take it personally and get discouraged. Good: take a deep breath and get to work picking up the pieces. Mathematical intuition isn't built overnight, and without surprises math would be boring.

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2.1.4. Example season: Pulling together and writing up.

The following is taken verbatim from the website.

1. What mathematical activities? What level of rigor?

After a period of creative research (of days, weeks, months,. . .), it's necessary to consolidate and pull together results. This involves very carefully retracing steps, chronologically, and lining up ideas and proofs. Initially, results are scattered throughout my notes; or proofs haven't been written down; or some statements are wrong, or outdated, or improved upon.

Because the original path to the result is almost always not the most direct, everything must be restructured. The goal here is to present ideas and proofs in the conventional form, an explanation to a particular audience. So there is a pure logic component, of lining up arguments correctly, but also a conversational component, as I decide how much detail to include, how much exposition, how much rigor.

This task is relatively easy and straightforward. It can feel administrative at times, for example when compiling a list of references. Virtually all math is type-set in TEX, so writing up involves hours and hours of typing TEX code, which is not very intellectually gripping.

I try to keep a running list of random ideas or questions that pop into my head as I'm writing something up. But I won't pursue these until I've finished, since switching back and forth seems to make the writing up process less efficient.

2. What relevant interactions with other mathematicians?

This is maybe the most independent extended math experience I know. I might need to check some work with someone else, but presumably at this stage I've already solidified the results. It's helpful to ask for tips on TEX syntax. I might have someone check that I've included the right amount of justification and exposition for my target audience. When submitting a paper, there is a well-established process of refereeing, which involves recursive feedback and reworking, and this can drag out past any self-contained "writing up" experience.

3. How does it feel, what is the mood?

Pulling things together can be affirming, and satisfying. It feels good to solidify knowledge. Writing up can be relaxing, or mildly frustrating. It's unnerving when I find a mistake I made a long time ago, and have to fix it.

4. What state of mind? stable vs. chaotic? focused vs. dispersed?

Pulling together feels easy, methodical, and uniquely compartmentalized. I only need to worry about one proof or handful of ideas at a time, and can safely ignore the periphery. Of course, I try to stay open to the occasional random new idea or question, but I intentionally stop my mind from wandering too much from the concrete task at hand. Writing up results is conversational and performative – my mind traces through the ideas as though I were explaining them out loud, in real time.

This kind of math is also relatively easy to turn on and off. Sometimes while doing it my mind wanders from math, and I get lost in a daydream. It's maybe the closest that math comes to being a "day job."

5. What type of self-reflection during the experience, and did it help?

As mentioned above, I try to keep a balance between capturing any possibly-valuable peripheral thoughts, and not getting too distracted from finishing the write-up. So I allow myself to use the restructuring and reviewing as an opportunity to gain perspective, but this perspective only comes if I keep some distance and don't get wrapped up in following new leads. Maintaining this balance requires some self-reflection. In fact, it seems that the better I'm attuned to this balance, the closer I can get to simultaneously maximizing perspective and efficiency.

6. An everyday metaphor for the experience?

Pulling together and writing up is just like washing dishes. The goal is to sanitize all the mess of discovery, and to dry off any trace of the restructuring.

We present a stack of dry, clean, glistening ideas, full of order and necessity, untouched by humans. These spotless ideas are complete in themselves, but sit ready to be used and rearranged as vessels and tools for someone else's new mess.

The dish-washing process is narrative (to me), relaxing, and mechanical. I can let my mind wander, to some extent. There are definitely more efficient and less efficient ways to do it.

7. An example of a good day and a bad day?

A good day ends with a few new pages of nice, clean TeXed math. On a bad day, I find a gap or hole, and can't fix it.

8. What did you do when you were stuck?

Getting stuck might mean finding a gap in some argument; this needs to be fixed. Or it might mean that I lost or can't find some proof, so I have to reprove it. Or it might be that I don't know the TeX syntax for the symbol I want, which means I have to hunt through TeX documentation.

9. When and why did it end?

It ends when the results are typed up and clean.

3. Contemplative Education

Let's return to a question posed earlier: How does one go about developing self-awareness? Cultures throughout the world have been asking and answering this question for millennia, and today we have many paths to this goal. Any such method is called a *contemplative practice*; examples are meditation, tai chi, yoga, journaling, improvisational art, or deep listening[3]. In this section, I would like to look at the use of contemplative practices in the classroom, including my differential equations class.

I will focus on meditation, which is perhaps the most common and most studied contemplative practice. There are hundreds of meditation techniques, but for the most part they fall into a few main families: mindfulness, insight, and lovingkindness. The basic mindfulness instructions are: sit still and notice whatever unfolds, without judgement. Insight meditations are more explicitly aimed at transforming one's sense of self and relationship to the world. Lovingkindness meditations broadly aim to cultivate universal compassion.

[3] A more complete list can be found at: `http://www.contemplativemind.org/practices`

Meditation has been gaining in popularity in the United States since the 1950s, and seems to have reached a critical mass. In recent years, we've seen meditation used in sports [13, 22], business [29, 24], prisons [18, 5], politics [17], and even the military [23]. There have been numerous studies exploring the potential for using meditation in medicine, to help everything from chronic pain and depression, to cancer and aging[4]. Advances in brain imaging technology and brain science have allowed for empirical studies on meditators' brains, and the results are significant and intriguing. These include, for example, evidence of thicker cortical regions associated with attention and sensory processing [11], or heightened activity in the regions that detect emotional cues [12], in the brains of expert meditators. On the other hand, after only five days of meditating for twenty minutes a day, Chinese undergraduates showed better regulation of the stress hormone cortisol [25]. In fact, the amount of research on mindfulness meditation has exploded in the last decade. According to the constantly updated Mindfulness Research Guide[5], in the 1990s there were roughly a dozen research articles published each year about mindfulness. In 2002, there were 44; in 2007, there were 167; and in 2011, there were 397.

Many of the potential benefits of meditation have a direct relevance to education. In 2008, the Center for Contemplative Mind in Society put out "Toward the Integration of Meditation into Higher Education: A Review of Research" [21]. With references to specific studies, it concludes that mindfulness meditation "may improve ability to maintain preparedness and orient attention" and "to process information quickly and accurately." Mindfulness meditation "may decrease stress, anxiety, and depression; and [it] supports better regulation of emotional reactions." Furthermore, the studies mentioned present evidence that meditation can help the development of creativity, interpersonal skills, and empathetic responses.

Inspired by these empirical studies and by firsthand experience with the benefits of meditation, the contemplative education movement aims to explore the possibilities of incorporating meditation and other contemplative practices into the classroom. The pioneer in this direction was certainly Naropa University, in Boulder, CO, a "Buddhist-inspired contemplative liberal arts institution" [1] founded in 1974 by Oxford-educated Tibetan Bud-

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[4]See [28] for a good 2008 review of research.

[5]`http://mindfulexperience.org`, accessed January 26, 2013.

dhist Chögyam Trungpa. The movement went national in 1997, when the Center for Contemplative Mind in Society began offering Contemplative Practice Fellowships, to begin a more systematic exploration of what became known as *contemplative pedagogy*[6]. The most common technique is simply a short mindfulness meditation at the beginning of class. Now, the Association for Contemplative Mind in Higher Education (ACMHE) has a network of 600 university-level educators, and there are similar such networks for primary and secondary teachers. Each year sees a range of regional meetings and workshops, where one can hear anecdotes and discuss challenges. On the ACMHE website one can find syllabi that incorporate contemplative pedagogy, in fields ranging from music to law, architecture to history. Unsurprisingly perhaps, mathematics and the hard sciences are poorly represented in the movement.

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In the remainder of this section, I will describe my own experiments with contemplative pedagogy. My methods were ad hoc and unscientific, but the results were promising.

I've had a personal meditation practice for six years now, which involves daily sitting meditation. In 2010 I began my first contemplative education experiment: I would meditate for 30 minutes in the hour before teaching. During class I felt more relaxed and spontaneous, more attuned to the classroom dynamic, and more flexible to adapt my lecture midstream. I thought I was able to give better explanations of the material, and maintain better peripheral awareness of time remaining and spatial use of the board. I think I ended up making more jokes, because the rapport with the students improved until we were often laughing and just generally having a little more fun. There were only a handful of times, when I left the meditation until immediately before class started, that I think I was noticeably spaced-out for the first few minutes of class. Still, the results have been positive overall, and I've continued to do this since, in classes on differential equations, linear algebra, and multivariable calculus.

The second experiment brought meditation into the class period. It was an introductory differential equations class, with about 45 students, meeting

[6] See [2] for a good history of contemplative education in the US.

for 50 minutes three times a week. The eleven-week course began conventionally, but starting in week six, I began devoting a few minutes in the beginning of class to mindfulness meditation. Before doing this the first time, I explained what marvelous benefits they might expect, as confirmed by empirical studies! Each time I would read from a short prompt that I had constructed from various resources. For example, the first week's prompt was the following. (Between each line there was a quiet pause of some length.)

Sit in a comfortable upright position with your feet planted flat on the ground. Rest hour hands on your thighs or on your desk. Eyes open or closed, doesn't matter...

Now just breathe. Refreshing, comfortable and natural breaths...

Don't worry about technique, just allow relaxed breaths to enter deeply and exhale fully...

For the next 60 seconds, your task is to focus all your attention on your breathing. When you're inhaling, notice and feel the inhalation – in your nose, throat, and belly. When you're exhaling, notice and feel the exhalation – in your nose, throat, and belly...

Don't worry about whether you're doing it right...

When your mind wanders, patiently and without judgement let go of your thoughts and return your awareness to breathing. You can think those thoughts later; now your job is just to focus on the sensations of breathing...

I changed the prompt only each week, so students had a chance to get a little familiar with the technique. Each new week, the prompt changed in a way that built on previous weeks, and was lengthened slightly. In the beginning, we spent one minute and 15 seconds on it; by the end of the term we were spending two minutes.

Although I made it optional, during these short meditations there were usually only three or four students that didn't participate, choosing instead to sit idly or spend the time with their phone. The rest of the class was an ocean of stillness and contemplation.

From the beginning of the term, part of their homework assignment was to complete a weekly online survey, which had two questions. Basically: (1) What didn't make sense this week? and (2) Any feedback about the course?

This realtime feedback helps me to adapt, and helps the classroom dynamic. Although the survey is not anonymous, there is always an option through the course website for anonymous feedback. Once we started the meditation experiment, I encouraged them to use the survey to let me know what they thought of it. I especially asked for any negative feedback or concerns that might have arisen.

Relishing my unscientific methodology, I compiled survey responses that addressed the meditation: there were 13 positive responses and no negative responses. Here are verbatim examples of positive responses, answering the second survey question:

– *I actually liked the breathing exercise that we did. I noticed that I was more calm and focused.*
– *breathing exercise was fun:)*
– *so focused after the exercises*
– *Mindfulness meditation are very useful for me, I think we should keep doing this.*
– *like the mind activity before class.*
– *The 2 minute relaxation thing we do in class is a really good idea, it helps me release stress and have a fresh mind to bare with some of the difficult integrals.*
– *I like the meditation before class! Pretty soothing!*

For anyone interested in contemplative education, I would recommend starting, if you haven't already, with a personal contemplative practice of some kind, meditation or otherwise. The consensus seems to be that one must have some degree of grounding in a practice before beginning to impose it on students. Then the internet can quickly point you to organizations, summer workshops, syllabi, and other resources. The Mind and Life Institute and the Association for Contemplative Mind in Higher Education are good resources.

Finally, it's important to point out that, although I've presented meditation as a means towards an end, and a tool towards better performance, it is really a whole journey unto itself. True, there are some documented benefits to meditation, but these studies hardly capture the real, profound consequences, that can be recounted firsthand by any regular practitioner. One reconnects with the simple joys of "curiosity, creativity, pondering, wondering, exploring, not knowing, ... cherishing openness, [and] a sense of awe, without presupposing or seeking correct results and predetermined answers" [6]. Gradually, there is, or may be, an expansion of awareness and self-sense, a natural inquiry into values and ethics, and an opening of the heart. These, of course, are the cornerstones of a liberal arts education.

4. Conclusion

In 1993, Arthur Jaffe and Frank Quinn proposed adjusting our publication system to allow for conjectural mathematics [9]. They were interested in speeding up the transmission of mathematical ideas, complete or incomplete, and finding a way to acknowledge the importance of good conjectures and not-yet-concise notions. In some ways, the arXiv, MathOverflow, and Polymath have met some of these needs. Towards a similar goal, but with a more experiential bent, I would like to propose a new publication convention that provides a place for public reflection on ideas, their experience, and their development.

Specifically, I propose that written mathematics—papers, books, assignments—should include a concluding section, entitled "Experiential Context." The author should address four things:

1. where and when the ideas arose,
2. what the key insights and central organizing principles are,
3. conceptual metaphors and mental images used to reason about the ideas, and
4. the process of development of the ideas and results.

I would like to think of this as meta-data for the ideas and results, that attaches a bit of humanity to the objective representations and reasoning. Including this information closes the gap between the practice of mathematics and the artifacts of that practice. Of course, good mathematical exposition weaves in some or all of these topics, in the introduction, between proofs, or during the flow of the arguments. But good mathematical exposition is rare, and I see a specific benefit to isolating these less rigorous components. By devoting a separate section, we can clearly distinguish between conventional writing, which needs to be as rigorous as possible, and the new section that will necessarily be more subjective and personal. In the Experiential Context section, then, one is much more free to reflect, to struggle to articulate one's process, and to express one's perspective. But it must be clear the intention is not to remove rigor, just augment it, and perhaps re-prioritize it.

To date, there is but one published mathematical work that has adopted the convention of including Experiential Context sections, as proposed above: they appear at the end of each rigorous chapter of my PhD thesis [30].

In my teaching, I've found that students often struggle with math literature that seems stripped of humanity. By introducing a section devoted to the experiential context of mathematical ideas, we show a little more of the human face of math. Math is rigorous, but that's not all it is. Too many students give up before they realize this. Imagine if in college or high school you had studied from books and articles that included descriptions of "how you might think of this" or "what's really going on" or mentioned a few of the historical stops along the way to the formal definitions. Imagine if, as part of learning to do mathematics, you were also asked to write about the experiential context of your homework assignments. *Imagine if reflecting on process was part of your process.* A more explicit and thorough understanding of mathematical practice among the math community can lead to only a more explicit and thorough training in best practices. The consequence would be better mathematicians doing better mathematics.

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