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Newton, Maclaurin, and the Authority of Mathematics

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1. INTRODUCTION: MACLAURIN, THE SCOTTISH ENLIGHTENMENT, AND THE "NEWTONIAN STYLE." Sir Isaac Newton revolutionized physics and astronomy in his book *Mathematical Principles of Natural Philosophy* [**27**]. This book of 1687, better known by its abbreviated Latin title as the *Principia*, contains Newton's three laws of motion, the law of universal gravitation, and the basis of all of classical mechanics. As one approaches this great work, a key question is: How did Newton do all of this? An equally important question is: Can Newton's methods work on any area of inquiry? Newton's contemporaries hoped that the answer to the second question was yes: that his methods would be universally effective, whether the area was science, society, or religion. What are the limits of the Newtonian method? In 1687, nobody knew. But his followers wanted to find out, and they tried to find out by applying these methods to every conceivable area of thought.

In Great Britain, Newton's most successful follower was Colin Maclaurin (1698–1746). Maclaurin was the most significant Scottish mathematician and physicist of the eighteenth century, and was highly influential both in Britain and on the Continent. He was one of the key figures in what is called the Scottish Enlightenment, the eighteenth-century intellectual movement that includes philosophers like Francis Hutcheson and David Hume, scientists like James Hutton and Joseph Black, and social philosophers like Adam Smith [**3**], [**31**]. And I have come to think that Newton's method—what has been called "the Newtonian style"—is both the key to understanding what made Maclaurin tick intellectually and to understanding the nature of his influence. In this paper, I want to demonstrate these conclusions.

In particular, I want to describe how Maclaurin applied "the Newtonian style" to areas ranging from the actuarial evaluation of annuities to the shape of the earth. Maclaurin seems to have thought that using this Newtonian style could guarantee success in any scientific endeavor. And I have another point to prove as well. Maclaurin's career illustrates and embodies the way mathematics and mathematicians, building on the historical prestige of geometry and the success of Newtonianism, were understood to exemplify certainty and objectivity during the eighteenth century. Using the Newtonian style invokes for your endeavor, whatever your endeavor is, all the authority of Newton, of whom Laplace said, "There is but one law of the cosmos, and Newton has discovered it," of whom Alexander Pope wrote, "Nature and Nature's Laws lay hid in night; God said, 'Let Newton be!' and all was light," and of whom Edmond Halley stated, "No closer to the gods can any mortal rise." The key word here is "authority." Maclaurin helped establish that authority.

2. WHAT IS THE "NEWTONIAN STYLE"? The *Principia* presents Newton's methods in action. Of course I do not claim that nobody had ever used an approach in mathematical physics resembling his before. But our concern here is how Newton himself did his successful celestial mechanics in the *Principia*, because that is the source of what Maclaurin and many others internalized and applied. And the way Newton did his theoretical physics has best been described by I. Bernard Cohen, who first called this approach "the Newtonian style" [**5**, p. 132] [**6**, chap. 3].

In the *Principia*, according to Cohen, Newton first separated problems into their mathematical and physical aspects. A simplified or idealized set of physical assumptions was then treated entirely as a mathematical system. Then, the consequences of these idealized assumptions were deduced by applying sophisticated mathematical techniques. But since the mathematical system was chosen to duplicate the idealized physical system, all the propositions deduced in the mathematical system could now be compared with the data of experiment and observation. Perhaps the mathematical system was too simple, or perhaps it was too general and a choice had to be made. Anyway, the system was tested against experience. And then—this is crucial—the test against experience often required modifying the original system. Further mathematical deductions and comparisons with nature would then ensue. Success finally comes, in Cohen's words, "when the system seems to conform to (or to duplicate) all the major conditions of the external world" [**5**, p. 139]. Let me emphasize this: *all* the major conditions. What makes this approach nontrivial is the sophistication of the mathematics and the repeated improvement of the process. It is sophisticated mathematics, not only a series of experiments or observations, that links a mathematically describable law to a set of causal conditions.

In order to fully appreciate the process just described and thereby understand the argument of this paper, we need some examples. The best example of how the Newtonian style worked is the way Newton treated planetary motion in Book I of the *Principia*. He began with one body, a point mass, moving in a central-force field. But even the full solution to that problem is not adequate to the phenomena, so he continued. First he introduced Kepler's laws, then a two-body system with the bodies acting mutually on each other, then many bodies, then bodies that are no longer mass points but extended objects. The force that explained this entire system was universal gravitation, a force that Newton argued "really exists." So now we have found what causes both the fall of objects on the earth and the motions of the solar system.

Here is another, simpler example of Newton's use of the Newtonian style. He considered bodies moving in circles. Once he had derived the law of centripetal force, Newton proved mathematically that the times taken to go around the circle vary according to the nth powers of the radii (that is, as R^n) if and only if the centripetal force varies inversely as R^{2n-1}. One consequence of this general mathematical relation is that the periods vary with the $3/2$ power of the radii (Kepler's Third Law) if and only if the force varies inversely with the square of the radii [**27**, Book I, Theorem 4 and Scholium, pp. 449–452]. Thus the test of the general mathematical theory for central forces against Kepler's specific observation establishes that the inverse-square force—which had been suggested by others before Newton—must be the right one. Even more important: the *causal* relationship between these two pre-Newtonian conclusions (Kepler's Third Law, the inverse-square force) is revealed by the mathematical system that includes Newton's laws of motion.

Maclaurin knew the Newtonian style intimately, and explained examples like those we have just described in his own exposition of Newton's work in physics. Maclaurin beautifully expressed his deep understanding of the Newtonian style when he wrote: "Experiments and observations ... could not alone have carried [Newton] far in tracing the causes from their effects, and explaining the effects from their causes: a sublime geometry ... is the instrument, by which alone the machinery of a work [the universe], made with so much art, could be unfolded" [**20**, p. 8].

Throughout his career, I shall show, this is is the methodology Maclaurin followed.

3. MACLAURIN'S FIRST USE OF THE NEWTONIAN STYLE. A sort of trial run of the Newtonian style was Maclaurin's youthful attempt—he was sixteen—to build a calculus-based mathematical model for ethics. In a Latin essay still (perhaps mercifully) unpublished today, "De Viribus Mentium Bonipetis" ("On the Good-Seeking Forces of Mind"), Maclaurin mathematically analyzed the forces by which our minds are attracted to different morally good things. Although he didn't publish this essay, he liked it enough to send it to the Reverend Colin Campbell, in whose papers it survives at the University of Edinburgh [**21**].

In "On the Good-Seeking Forces of Mind," Maclaurin postulated that the "forces with which our minds are carried towards different good things are, other things being equal, proportional to the quantity of good in these good things." Also, the attractive force of a good one hour in the future would exceed that of the same good several hours in the future. And so on. Maclaurin represented the total quantity of good as the area under a curve whose x-coordinate gives the duration and y-coordinate the intensity of the good at a particular instant. He said that one could find the maximum and minimum intensities of any good or evil using Newtonian calculus. Maclaurin graphed the total attraction of a good under various assumptions about how the intensity varies over time, and, by integration, derived equations for the total good. One conclusion supported by his mathematical models was that good men need not complain "about the miseries of this life" since "their whole future happiness taken together" will be greater. Maclaurin thus tested his mathematical model against the doctrine of the Church of Scotland, and found that the results fit. He had shown mathematically that the Christian doctrine of salvation maximized the future happiness of good men.

Of course, "maximizing" and "minimizing" are important techniques in the calculus. Applications of this technique abound in eighteenth-century physics, from curves of quickest descent to the principle of least action. In fact, it was Maclaurin who, in his *Treatise of Fluxions*, gave the first sophisticated account of the theory of maxima and minima, using Maclaurin series to characterize maxima, minima, and points of inflexion of curves in terms of the signs, or equality to zero, of first, second, third, and nth derivatives [**22**, pp. 694–703]. This work was highly praised by Lagrange, who gave a similar theory enriched by the Lagrange Remainder for the Taylor series [**19**, pp. 233–236] (see also [**12**, pp. 136–137, p. 217 n. 65]). Maclaurin also applied the techniques of finding maxima and minima in many novel situations and then compared his results with the best data: for instance, the best design for waterwheels and windmills, how bees build the three-dimensional cells in honeycombs, and the most economical way to build a barn [**20**, pp. 149, 172–178], [**22**, pp. 733–742], [**23**, pp. 386–391, 397–400].

Still, one wonders what could possibly have inspired Maclaurin to write that theological essay. I think it likely that the idea was suggested by the Scots mathematician John Craige's 1699 *Theologiae Christianae Principia Mathematica*, whose title translates as "Mathematical Principles of Christian Theology." In this work, Craige graphed the intensity of pleasures as various functions of time, calculated the total pleasure by integration, and concluded, after an argument so Newtonian that nineteenth-century readers called it "an insane parody of Newton's *Principia*," that one should forego the finite pleasures of this world in favor of the infinite pleasures of the world to come. But before we moderns laugh too readily, remember that this was the first generation after the *Principia*. Who knew for sure what the limitations of the Newtonian style were?

In fact the Newtonian style did not automatically produce valid results, even in the physical sciences, even for Newton himself. G. E. Smith has shown how Book II of the *Principia* was Newton's attempt to use the Newtonian style to find, from the phenomena of motion in fluid media, the resistance forces acting on bodies [**29**, p. 251].

But unlike the situation in the discovery of universal gravitation that I sketched earlier, in which each successive mathematical idealization dropped some assumption that had simplified the mathematics, Newton's attempt to model the inertial resistance to motion and its relation to viscosity did not yield to his approach. Indeed, some of the key quantities (e.g., the drag coefficient and the Reynolds number [the ratio of inertial to viscous effects in a flow]) still cannot be functionally related from theoretical principles. In establishing universal gravitation, the sequences from point mass to extended body, or from one-body to two-body systems, work because each approximation suggests unresolved questions that the next stage can address. But this did not work for resisted motion. In Smith's words, "this time ... the empirical world did not cooperate" [**29**, p. 288].

Still, for Maclaurin, the Newtonian style seemed to have a good track record, and Maclaurin completely committed himself to it. But there was more to Maclaurin's Newtonianism than what we have examined so far. Let us now turn to another Newtonian theme: the relationship between the authority of mathematical physics and religion.

4. RELIGION, AUTHORITY, AND MATHEMATICS FOR NEWTON AND MACLAURIN. A major theme in Maclaurin's work is the way the order of the universe demonstrates the existence and nature of God. Newton had written in the concluding section of his *Principia*, "This most elegant system of the sun, planets, and comets could not have arisen without the design and dominion of an intelligent and powerful being" [**27**, General Scholium, p. 940]. Maclaurin agreed, saying, "Such an exquisite structure of things could only arise from the contrivance and powerful influences of an intelligent, free, and most potent agent" [**20**, p. 388; compare p. 381]. Maclaurin took Newton to mean that it is the mathematically-based natural philosophy of the *Principia* that proves the existence of God, and that Newton's mathematically-based methods guarantee his conclusions about the nature of God's world.

Why was the authority of mathematics so attractive in the eighteenth century? Of course religion had authority, but there were theological disputes aplenty—Catholic versus Protestant, for instance. So for many eighteenth-century thinkers part of the authority of mathematics and science came, as Maclaurin said in the preface to his *Treatise of Fluxions*, from the belief that mathematical demonstration—unlike, say, theology or politics—produced universal agreement, leaving "no place for doubt or cavil" [**22**, p. 1].

George Berkeley, however, did have doubts, and attacked mathematics. Maclaurin felt obliged to defend it. Berkeley saw Newtonian science as entailing only a natural-order sort of God, and thus undermining the authority of Scripture. But besides criticizing the Newtonian philosophy in general, Berkeley, in *The Analyst, or a Discourse Addressed to an Infidel Mathematician* (1734), attacked the logical validity of the calculus. In particular, using well-chosen examples, Berkeley argued that, on the evidence of the way the calculus was actually being explained, mathematicians reasoned worse than theologians. And he ridiculed vanishing increments, which Newton had used to explain his calculus, as "ghosts of departed quantities" [**2**, sec. 35].

This is not the place to describe in detail how Maclaurin refuted Berkeley (see [**28**] and [**14**]). But I will say here that Maclaurin showed how rigorous Newtonian calculus could be made, and Maclaurin's improved algebraic and inequality-based understanding of Newton's limit concept played a role in the eventual complete rigorization of the calculus in the nineteenth century [**12**], [**14**]. For the present, the key point is that Maclaurin's refutation of Berkeley strongly reinforced the authority of mathematics in Britain.

5. MACLAURIN'S MATURE USE OF THE NEWTONIAN STYLE.

The shape of the earth. We turn now to the heart of the current paper: how Maclaurin used the Newtonian style to become a successful scientist. Maclaurin's monumental *Treatise of Fluxions* is much more than an answer to Berkeley; it contains a wealth of important results both in mathematics and in physics. In particular, let us look at Maclaurin's treatment of the shape of the earth, for it exemplifies his use of the Newtonian style, especially the sophisticated use of mathematical models, to solve problems of great importance for physics and astronomy.

The earth is not a sphere, Newton argued in the *Principia*, since its rotation brings about real forces that cause it to be flattened at the poles and to bulge at the equator. He stated, though he did not prove, that the shape of the earth was an ellipsoid of revolution, and used the method of balancing columns to predict its dimensions assuming very small ellipticity [**27**, Book I, Prop. 91]. And he applied his theoretical discussion to the existing data on the period of pendulums at different latitudes [**27**, Book III, Prop. 19]. Maclaurin carried the theory of this subject substantially farther. He produced the first rigorously exact theory of homogeneous figures shaped like ellipsoids of revolution whose parts attract according to the inverse-square law. Among other things, he proved geometrically that a homogeneous ellipsoid, revolving around its axis of symmetry, is indeed a possible figure of equilibrium, and he established that the perpendicular columns balance. Maclaurin's geometric method was based on his deep knowledge of the conic sections and worked for figures whose ellipticities were finite as well as infinitesimal. He demonstrated how to calculate the gravitational attraction at any point on the surface of such an ellipsoid. Maclaurin also developed a theory of the attraction of such ellipsoids that were variable in density, in the process proving results about the gravitational attraction of confocal ellipsoids. He developed further, and exploited, many now-standard techniques (besides balancing columns, the idea of level surfaces) in his theory of these rotating bodies. All this was important and new and, especially through his correspondence with Alexis-Claude Clairaut, was fully integrated into what was taking place on the Continent [**15**, pp. 412–425, 585–601], [**22**, pp. 522–566], [**23**, pp. 342–344, 347–354, 359–370, 372–380].

Furthermore, besides the mathematical theory of rotating bodies, their shapes, and their attractions, Maclaurin took a keen interest in getting real data—the latest news from the expeditions to various parts of the world to measure the earth's shape, whether by his fellow Scotsman George Graham or by the various French academicians. Maclaurin, then, developed mathematically precise predictions about the earth's shape, and tested these predictions against the latest observations to refine his theory, from homogeneous ellipsoids of uniform density to stratified ones of variable density—an impressive and influential application of the Newtonian style [**22**, pp. 551–566]. Of course he did not solve all the problems in this subject, but he pioneered the serious mathematical study of the rotation of astronomical bodies, a subject carried further by Clairaut, d'Alembert, Lagrange, Jacobi, and on up to the twentieth century by Chandrasekhar (whose modern history of the topic, which devotes a full chapter to Maclaurin spheroids, can be consulted in [**4**]).

"Gauging": Finding the volumes of barrels. From the heavens, we move to earthly applications. Maclaurin, employing the fashionable rhetoric of his time that was based on the philosophy of Sir Francis Bacon, rejoiced that advances in science could produce useful knowledge and enthusiastically participated in that production. One of his contemporaries tells us that Maclaurin, before his untimely death in 1746, "had resolved... to compose a course of practical mathematics" [**24**, p. xix], the existing ones being of low quality. Perhaps as a first step toward this task, Maclaurin in 1745 published, with his own comments, a manuscript by David Gregory on finding the volumes of barrels, or as it was called at the time, "gauging."

Gauging was an important subject for eighteenth-century society, since the wooden barrel was the universal shipping container throughout the Atlantic economies of both Europe and America. And gauging was the subject of one of Maclaurin's most detailed contributions to applied science. In 1735, Maclaurin wrote a ninety-four-page memoir for the Scottish Excise Commission explaining the most accurate way to find the volume of molasses in the barrels in the port of Glasgow [**10**]. In 1998 I presented this story of Maclaurin's success in mathematically gauging molasses barrels as a case study in the use of mathematical authority to achieve consensus about a problem—taxation—clearly rife with disagreement between parties with vastly different interests [**13**]. But in the present context, Maclaurin's solution of the barrel-gauging problem is another example of his use of the Newtonian style.

Maclaurin's memoir provided a set of clear rules for finding the volumes of real barrels. To derive these rules, Maclaurin began with the case where the barrels had precise mathematical shapes. Many eighteenth-century manuals on gauging treated barrels as solids generated by rotating conic sections about their axes. Further, the manuals often approximated the volumes of these solids of revolution by imagining them made up of slices perpendicular to the axis of rotation, with each slice approximated by a cylinder of the same height (typically about ten inches) such that the diameter of each cylinder was the diameter at the midpoint of the altitude of the corresponding slice. But this method was far from accurate. First, the precision of the approximation using cylindrical slices depends not only on the height of the slices but also on the type of conic section whose rotation generates the solid. Maclaurin was the first to calculate—and to prove—exactly what the differences are between these approximating cylinders and the corresponding slices of solids of revolution. (For Maclaurin's beautiful geometric result, see [**10**, p. 193]; for his derivation, using Newtonian calculus, see [**10**, pp. 229–235]; for his geometric proof, see [**22**, pp. 24–27].) Second, real barrels are not solids of revolution. In the eighteenth century, wooden barrels were assembled out of individual staves and hoops, with no two barrels being identical. So besides giving the mathematical theory of solids of revolution and working out how each type of solid differed from its approximation by cylindrical slices, Maclaurin showed how the actual barrels deviated from the mathematical solids of revolution, then showed how new calculations could deal with those deviations, and then even addressed deviations from those deviations (for an example, see [**10**, pp. 209–215]). Once again, Maclaurin used the Newtonian style. He began with a mathematical model and corrected it repeatedly according to observation to produce an authoritative, precise, and realistic solution. And the Excise officers in Scotland used his method for many years [**24**, p. xix]. That he was serving his society must have further reinforced his commitment to the Newtonian style.

Social agreement and scientific authority. Maclaurin recognized that scientists, whether pursuing pure research or solving society's problems, do not work alone. In fact he himself played a leading role in organizing the Philosophical Society of Edinburgh, Scotland's first real scientific society, in 1737. And social agreement and consensus—for Maclaurin, as we will see presently, these too come from Newtonianism—are part of the Society's rhetoric. The eighteenth-century ideal of consensus and universal agreement appears clearly in the Edinburgh Philosophical Society's stated rules: "Religious or Political Disputes" were forbidden, and members were warned that "in their Conversations, any Warmth that might be offensive or improper for Philosophical Enquiries is to be avoided" [**11**, p. 154].

But the avoidance of political disputes need not, and in the eighteenth century did not, mean a lofty lack of involvement in society. The Philosophical Society's members served as consultants to Scottish development agencies, from the British Linen Company to the Royal Bank of Scotland. They helped map Scotland and her coasts, made astronomical observations that could assist navigation and determine longitude, pumped water out of mines, designed and built canals, investigated the distribution of minerals in Scotland, and measured the forces of winds at sea. All these activities embody the eighteenth-century idea that science and technical expertise should be applied to solve problems of economic and social importance for the nation. Scientists were much sought after by progressive investors, whom the Scots called "improvers," both in Britain and on the Continent [**30**, p. 361]. And the scientists actually could help.

But why, in the eighteenth-century view, does science help? The crucial mark of the validity of the science to be applied, many eighteenth-century thinkers asserted, came from the fact that everybody could agree about it. Universal agreement gave eighteenth-century science its authority. It made science a collective, not just an individual, endeavor. And this universal agreement was, for Maclaurin, the achievement of Newton and of the Newtonian style. Maclaurin called the successful Newtonian style Newton's "right path." Newton, Maclaurin claimed, "had a particular aversion to disputes" (a hagiographical comment indeed) and "weighed the reasons of things impartially and coolly" [**20**, p. 13]. Moreover, just as Maclaurin had used the universal agreement historically possessed by mathematics to motivate his refutation of Berkeley, so he used his idealized picture of Newton to formulate his own version of the social achievement of eighteenth-century science. Maclaurin wrote [**20**, p. 62]: "We are now arrived at the happy aera of experimental philosophy; when men, having got into the right path, prosecuted useful knowledge... the arts received daily improvements; when not private men only, but societies of men, with united zeal, ingenuity and industry, prosecuted their enquiries into the secrets of nature, *devoted to no sect or system*" [italics added]. So it is the Newtonian style that lets scientists and scientific societies produce national prosperity.

Actuarial science. Now let us turn to an episode that combines our main themes, the mathematician's authority and the Newtonian style, applied together to serve society: Maclaurin's actuarial work for the Scottish Ministers' Widows' Fund in 1743. First, consider Maclaurin's authority. Robert Wallace, Moderator of the General Assembly of the Church of Scotland, wrote Maclaurin that, when objections to the soundness of the pension scheme were raised in Parliament, "I answered them that the Calculations had been revised by you... this entirely satisfied them" [**1**]. Or, as Maclaurin's contemporary biographer Patrick Murdoch put it, "The authority of [Maclaurin's] name was of great use... removing any doubt" [**24**, p. xix]. The calculations as revised by Maclaurin were indeed satisfactory; the fund continued even into the twenty-first century to provide for some of the "widows and fatherless children of ministers of the Church of Scotland, the Free Church of Scotland and some of the professors of the four old Scottish Universities" [**9**, p. xii].

The Church's goal in developing the Fund was to keep its ministers' widows and orphans out of poverty by providing them an annuity. Schemes like this had been tried before, but the funds tended to run out of money. Everyone involved in the 1743 scheme recognized that making it work required finding data and creating a mathematical model based on that data. As it turned out, the task also involved checking and refining data, and then revising the model to fit that better data accurately. The men who did the first two steps, finding data and making a model, were Alexander Webster and Robert Wallace. Maclaurin, by carrying out the two last steps—checking theory against data and revising the model to deal with the discrepancies—improved their scheme and made it work. The Newtonian style is apparent even from the title of the printed account of the Fund: "Calculations, with the Principles and Data on which they are instituted: relative to a late Act of Parliament, intituled, an Act for raising and establishing a Fund for a Provision for the Widows and Children of the Ministers of the Church, and the Heads, Principals, and Masters of the Universities of Scotland, shewing The Rise and Progress of the Fund." Maclaurin thus used the Newtonian style once again, this time to become a pioneer of actuarial science.

To start planning for the Fund, Webster collected the relevant data by sending questionnaires to every parish of the Church, asking how many ministers there were, how many widows there were, how many orphans there were, how old these people all were, and so on. Wallace then made a mathematical model predicting the changes in the widow and orphan populations. He sent a copy of the proposed scheme to Maclaurin for possible criticism. And he got it.

Using Wallace's model, Maclaurin prepared tables to predict the future of the scheme according to probability theory. Maclaurin showed that the fund would run out of money unless it reduced the children's benefits. Why? Because Wallace, after consulting mortality tables, had assumed that $1/18$ of the widows would die each year, a mathematically tractable assumption since his model began with eighteen widows, but an assumption that ignored the varying age distribution of the group.

Maclaurin sought empirical data to check Wallace's assumption and found it by a study, more careful than Wallace's, of the mortality tables from Breslau published by Edmond Halley in 1673. According to Maclaurin's refined model based on his analysis of the age distribution in this data, the average annual mortality rate of the widows would at first be lower, and thus more payouts would be needed. Even after the first year, the two models yield different predictions. Wallace's model has 35 widows; Maclaurin's has 35.42. After only thirty years, the predicted number of widows differs by almost 20 percent: 257 according to Wallace's model, 307 according to Maclaurin's [**7**, p. 53], [**17**, p. 63], [**23**, pp. 108–109].

Actuarial historians David Hare and William Scott assert that Maclaurin's recalculations were "the earliest actuarially-correct fund calculations ever carried out" because Maclaurin had used "a realistic and accurate life table" [**17**, p. 57, 68]. And, not incidentally, it worked; what actually happened corresponded remarkably well to the calculated figures.

Maclaurin's success with the Widows' Fund had great influence. In 1761, the first real life insurance plan in the United States, devised by the Presbyterian Church in Philadelphia, was based on it [**8**, pp. 19–20]. Likewise, the Fund's example influenced the first Scottish life insurance company to be established, as that company's name, Scottish Widows, indicates [**7**, p. 24]. Furthermore, the importance of this kind of successful mathematical modeling of society was enormous for the social sciences. Both Webster and Wallace later applied quantitative methods to study populations in broader contexts. For instance, Webster again used his parish survey methods to produce his *Account of the Number of People in Scotland*, published in 1755, a work whose focus on the "political arithmetic" of the nation makes it an important step toward the modern census. Wallace constructed mathematical models to study human populations in the books *A Dissertation on the Numbers of Mankind* (1753) and *Various Prospects of Mankind, Nature and Providence* (1761), which are often recognized as forerunners of the work of Thomas Malthus. And the mathematical models of agricultural growth (linear) and population growth (exponential) made Malthus conclude in his *Essay on*

Population (1798), also in the Newtonian style, that the finite empirical world means that human populations cannot continue to grow exponentially forever. These conclusions became quite important, not least in their role in Darwin's argument for his population-based theory of evolution by natural selection.

"Method" and authority. From these examples, let us turn back again to the relationship between Maclaurin and Newton. Maclaurin's most influential public explanation of Newtonianism was his book, published posthumously but based on his lectures at Edinburgh, *An Account of Sir Isaac Newton's Philosophical Discoveries* [**20**]. In this book, Maclaurin sought to consolidate Newtonianism and establish its authority beyond all doubt, "in order to proceed with perfect security, and to put an end forever to disputes" [**20**, p. 8]. Book I of the *Account* begins with a chapter on the scientific method done Newton's way, and then shoots down any differing approaches. Book II introduces elementary classical mechanics. Then the major books, Books III and IV, expound the Newtonian system of the world in the Newtonian style. Look at the titles of these books: Book III is called "Gravity demonstrated by analysis" and Book IV is entitled "The effects of the general power of gravity deduced synthetically." But to see the Newtonian style in action one more time, we need to explain the terms "analysis" and "synthesis" to which Maclaurin gave such prominence.

Newton defined his own version of "analysis" in natural science by referring to the Greek usage, where problem solving in mathematics is done by "analysis," literally "solution backwards." We discover how to construct a line, for instance, by assuming that the line has been constructed and working backwards from that until we find something we already know how to construct. Only afterwards can we *prove* that the original construction can be made, and the proof is obtained by reversing the order of the steps in the "analysis." If "analysis" was the method of discovery, the method of proof was called "synthesis," or, in Latin, "composition."

What, then, might "analysis" be in *science*, or as people in the eighteenth century called it, "natural philosophy"? Newton had addressed this question in his *Opticks*. After saying "As in Mathematicks, so in Natural Philosophy, the Investigation of difficult Things by the Method of Analysis, ought ever to precede the method of Composition" he explained that in natural philosophy "This Analysis consists in making Experiments and Observations, and in drawing general Conclusions from them by Induction, and admitting of no Objections against the Conclusions, but such as are taken from Experiments, or other certain Truths. For Hypotheses are not to be regarded in Experimental Philosophy." As for synthesis in natural philosophy, it "consists in assuming the Causes discover'd, and establish'd as Principles, [like the three Laws of Motion, which Newton actually called "axioms"] and by them explaining the Phaenomena proceeding from them, and proving the Explanations" [**26**, 31st Query, pp. 404–405].

This passage from Newton's *Opticks* is one of the most often quoted statements of Newton's methodology. Scholars often say that Newton was contrasting his more Baconian method, which requires extensive empirical observation and experiment, to what he regarded as the premature jumping to conclusions of Descartes's physics. But whatever Newton's reasons were for wording the passage as he did, Maclaurin gave it a different slant.

In his own gloss on this passage, Maclaurin emphasized the certainty that was achieved by obeying Newton's precepts. And to demonstrate this certainty, he identified the success of Newtonian physics with its following the methods of mathematics. It was "in order to proceed with perfect security, and to put an end for ever to disputes," wrote Maclaurin, that Newton says to us, "as in mathematics, so in natural philosophy, the investigation of difficult things by the method of analysis ought ever to precede the method of composition, or synthesis. For in any other way, we can never be *sure* that we assume the principles *which really obtain in nature*" [**20**, pp. 8–9; italics added]. According to Maclaurin, Newton's scientific achievement came from impartiality, not disputation; mathematical proof of causes, not hypotheses; and above all, success in finding the true laws of nature by the right use of mathematics.

6. RELIGIOUS AUTHORITY REVISITED. But for Maclaurin, there were even higher goals. Maclaurin said that natural philosophy's chief value is to lay "a sure foundation for natural religion and moral philosophy" [**20**, p. 3]. Natural religion proves God's existence arguing from the order of Nature; moral philosophers inquire into human and divine nature to support conclusions about ethics. The mature Maclaurin was much more sophisticated about how to do this than he was when he wrote that essay at age sixteen.

Maclaurin wanted this "sure foundation" for religion and morality, but this required a natural philosophy that was true. How to achieve this? A naive empiricism, for Maclaurin, cannot produce the true laws of nature. All we can obtain from experiments and observations alone, he said, are natural history and description; this does not take us from observed phenomena "to the powers or causes that produce them" [**20**, p. 221]. For that, we need something more.

Let us, then, return to an earlier quotation, which gives Maclaurin's answer to the question raised by philosophers like Thomas Hobbes and David Hume as to how a set of correlated observations could produce any understanding of causal connections: that is, true natural laws capable of supporting science, technology, moral philosophy, and theology. Recall how Newton had used mathematics to derive the causal connection between the centrally-directed, inverse-square force law and the three laws of Kepler. And now again read Maclaurin [**20**, p. 8]: "Experiments and observations," he wrote, "could not alone have carried [Newton] far in tracing the causes from their effects and explaining the effects from their causes: a sublime geometry was his guide in this nice and difficult enquiry." The "sublime geometry," the mathematics, was for Maclaurin, "the instrument, by which alone the machinery of a work [the universe], made with so much art, could be unfolded." Only the Newtonian style could reveal the true nature of the universe that God had made.

7. CONCLUSION. The "Newtonian style" seemed to Maclaurin to guarantee success, not only in physics, but also in areas ranging from gauging barrels to insurance to theology. Maclaurin's successes and reputation helped others expect comparable achievements from approaches that were or claimed to be Newtonian, from the quantified ethics of Maclaurin's Glasgow classmate Francis Hutcheson, to the geological world-machine of Maclaurin's student James Hutton, to the optimal economic outcomes of Hutcheson's student Adam Smith.

Maclaurin's work embodied the following key aspects of his Newtonianism: Newtonianism's mathematical prowess, its sophisticated use of mathematical models to solve problems, its links with particular social classes to advance and stabilize a particular system of government, Newtonianism's harmony with religion, and, above all, its authority. The mathematical core of the Newtonian style helped inspire Maclaurin's quest to gain, for Newtonian natural philosophy, all the authority of mathematics. The story as I have told it is set in Scotland. I could tell a different story for the Continent, with the key players including Euler, Lagrange, and Laplace, but the overall outcome there was in many ways the same [**16**] [**18**]. At the core of eighteenth-century science are Newtonian physics put in modern mathematical dress and the mathematization of many new areas. The perceived success of eighteenth-century science vastly and permanently increased the authority of mathematical methods in *all* endeavors. The success and prestige of modern science reflects and embodies the triumph of the Newtonian style.

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A Proof of the Irrationality of $N^{1/k}$

Let N and k be positive integers such that $N^{1/k}$ is not an integer. We claim that under this assumption the number $N^{1/k}$ is irrational. To see this, put

$$a_n = (N^{1/k} - \lfloor N^{1/k} \rfloor)^n,$$

where $\lfloor x \rfloor$ denotes the floor function of x. By assumption

$$0 < N^{1/k} - \lfloor N^{1/k} \rfloor < 1,$$

so $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$. Furthermore we can express a_n as

$$a_n = c_{n1} + c_{n2}N^{1/k} + c_{n3}(N^{1/k})^2 + \cdots + c_{nk}(N^{1/k})^{k-1},$$

where $c_{n1}, c_{n2}, \ldots, c_{nk}$ are integers. If we can write $N^{1/k} = p/q$ for positive integers p and q, then

$$a_n = (c_{n1}q^{k-1} + c_{n2}pq^{k-2} + \cdots + c_{nk}p^{k-1})/q^{k-1} \geq 1/q^{k-1}$$

for each n. This is a contradiction. Therefore $N^{1/k}$ is irrational.

Remark. Let α be a real algebraic integer. By definition, α is a root of a monic polynomial $P(x)$ with integral coefficients. Let k be the degree of $P(x)$. We assume that α is not a rational integer. Then replacing $N^{1/k}$ with α in the foregoing proof, we can establish the irrationality of α.

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