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On the Landscape of Random Tropical Polynomials

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Abstract

Tropical polynomials are similar to classical polynomials, however addition and multiplication are replaced with tropical addition (minimums) and tropical multiplication (addition). Within this new construction, polynomials become piecewise linear curves with interesting behavior. All tropical polynomials are piecewise linear curves, and each linear component uniquely corresponds to a particular monomial. In addition, certain monomial in the tropical polynomial can be trivial due to the fact that tropical addition is the minimum operator. Therefore, it makes sense to consider a graph of connectivity of the monomials for any given tropical polynomial. We investigate tropical polynomials where all coefficients are chosen from a standard normal distribution, and ask what the distribution will be for the graphs of connectivity amongst the monomials. We present a rudimentary algorithm for analytically determining the probability and show a Monte Carlo based confirmation for our results. In addition, we will give a variety of different theorems comparing relative likelihoods of different types of tropical polynomials.

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Acknowledgments

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Chapter 1

Introduction

Compared to some other fields of mathematics, tropical geometry is fairly new. Although some of the work had appeared throughout the late 1900's, the consolidation of basic theorems and definitions surrounding the subject only really started to appear within the late 1990's.

The main founder of the field is Imre Simon, a prominent mathematician and computer scientist. Simon was a Hungarian-born Brazilian scientist who eventually moved to Ontario, Canada to obtain his Ph.D. in 1972 at the University of Waterloo. Shortly thereafter, he moved Brazil where he became a professor at the University of São Paulo. He then passed away in 2009 due to lung cancer shortly before he was 66 years old. Due to his major contributions to the field, the field was dubbed "tropical" due to Simon's Brazilian Heritage. However, it is uncertain who originally coined the term in the first place, or why the field was coined as "topical" geometry as opposed to "Simon" geometry.

Tropical geometry is the study of the geometry of polynomials, but using a "tropical" system where we replace addition with the minimum operator and multiplication with the addition operator. We can also view tropical geometry as a non-Archimedean algebraic geometry (Archimedian generally meaning that there is no infinitely large or infinitely small element), so it follows that tropical geometry is a subfield of algebraic geometry as opposed to an entirely new field. In addition, tropical polynomials contain some information about their more classical counterparts. For this reason, tropical geometry has major applications in algebraic geometry and some of its subfields. Most notably, tropical geometry has significantly influenced studies in mirror symmetry and Gormov-Witten theory, both of which are used extensively in the mathematical analysis of string theory. Other applications have been found in trade theory, optimization problems, biology, and more.

In general, a tropical polynomial appears to show up in one particular form out of a finite number of possibilities. Specifically, we note that because all tropical polynomials are piecewise linear curves and each linear component corresponds to a monomial in our tropical polynomial, we can combinatorially investigate tropical polynomials through the connectivity of their monomials. In this paper, we seek to understand what the distribution of tropical polynomial forms are when we create a tropical polynomial at random by allowing each of the coefficients to be chosen from a standard normal distribution.

In chapter 2, we will begin by discussing the finer details of tropical polynomials. In chapters 3-6, we discuss two mathematical objects (marked polytopes and cones) that will help us look at the problem in a more computationally reasonable way. In chapter 7, we will look at some example derivations for the types of tropical polynomials. Finally, in chapter 8 we discuss directions for future research in order to conclude the paper.

Chapter 2

The Tropical Semiring

At it's heart, geometry is essentially the study of polynomials. Tropical generally implies a change of addition and multiplication to tropical addition (minimums) and tropical multiplication (addition). Thus, as the name implies, tropical geometry is the study of polynomials in this new tropical setting. To formalize this idea, we will introduce a series of definitions to build up to tropical polynomials. First, we define the tropical semi-ring.

Definition 2.1. The *tropical semi-ring* is the set $\mathbf{R} \cup \{\infty\}$ (all real numbers with positive infinity) equipped with two binary operators $\oplus$ and $\otimes$ (referred to as *tropical addition* and *tropical multiplication*) defined such that for any values x and y,

$$x \oplus y = \min\{x, y\}$$
$$x \otimes y = x + y$$

As its name implies, the tropical semi-ring is indeed a semi-ring, meaning that it comes with all of the same properties as a ring, but without the guarantee that we will have additive inverses. For example, $3 \oplus 5 = 3$, but there is no value x such that $x \oplus 3 = 5$. This is due to the fact that $x \oplus 3 = \min\{x, 3\}$ which is always less than 5 regardless of the value of x.

Other properties that hold for rings still hold for the tropical semi-ring however. For example, multiplication is indeed distributive as $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$. In addition, once we remove the identity for tropical addition (which is ∞), then the remaining set of elements (which is just $\mathbb{R}$) is a group under tropical multiplication (which is addition).

Although this appears to be a unmotivated choice to newcomers, it makes sense within the context of exponential powers. Suppose a and b are two arbitrary real values. We note that in the limit as h goes to infinity,

$$e^{-ha} + e^{-hb} \to e^{-h \cdot \min\{a,b\}}, \quad \text{and} \quad e^{-ha} \cdot e^{-hb} \to e^{-h(a+b)}.$$

We note that in this sense if we let $''x''$ be the symbolic stand-in for e^{-hx}, we find that $''a'' + ''b'' = ''\min\{a, b\}''$, and that $''a'' \times ''b'' = ''a + b''$. This gives rise to the idea of tropical algebra, where we obtain that $a \oplus b = \min\{a, b\}$ and $a \otimes b = a + b$.

2.1 Tropical Polynomials

Our next step is to build up definitions so that we can precisely define a tropical polynomial. The next three theorems will describe integer powers, tropical monomials, and tropical polynomials in order.

Definition 2.2. Suppose k is a positive integer, and x is an element of the tropical semi-ring. We define x^k to be the tropical product of x by itself k times. More precisely:

$$x^k = \underbrace{x \otimes x \otimes \cdots \otimes x}_{k \text{ times}} = \underbrace{x + x + \cdots + x}_{k \text{ times}} = k \cdot x.$$

Definition 2.3. Suppose $x_1, \ldots, x_k$ are real-valued variables, $n_1, \ldots, n_k$ are non-negative integers, and a is an element of the tropical semi-ring. A tropical monomial is a function of the form $a x_1^{n_1} \cdots x_k^{n_k}$.

Definition 2.4. A tropical polynomial is the tropical summation of a collection of tropical monomials. For example, two such tropical polynomials are the functions

$$p(x) = 1 \oplus 3x \oplus 17x^2 \oplus (-3)x^3$$

$$q(x, y) = (-3) \oplus 3x \oplus 7x^2 \oplus (-2)y \oplus (0)xy \oplus 5y^2$$

From a holistic standpoint, we can think of a tropical polynomial as a regular polynomial, but where addition and multiplication have been replaced with their tropical counterparts. Some of the same properties and definitions that we use to describe classical polynomials still carries over to tropical polynomials. For example, we can define the degree of a tropical polynomial just as we would for classical polynomials:

Definition 2.5. The degree of a tropical polynomial is the smallest integer value d such that for each monomial $ax_1^{n_1} \cdots x_k^{n_k}$, we have that $d \geq n_1 + \cdots + n_k$.

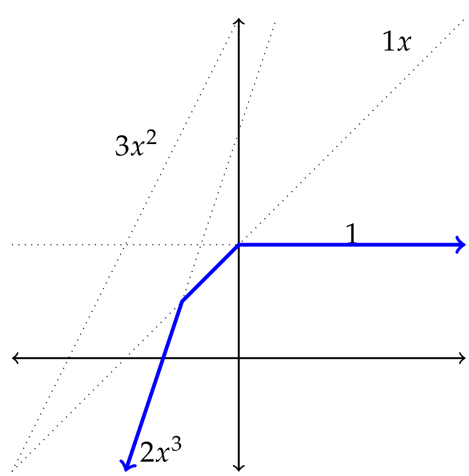


Figure 2.1 A plot of the tropical polynomial $p(x) = 1 + 1x + 3x^2 + 2x^3$. The actual function evaluation is shown in green, while the plots for 1, $1x$, $3x^2$, and $2x^3$ are presented as dotted lines.

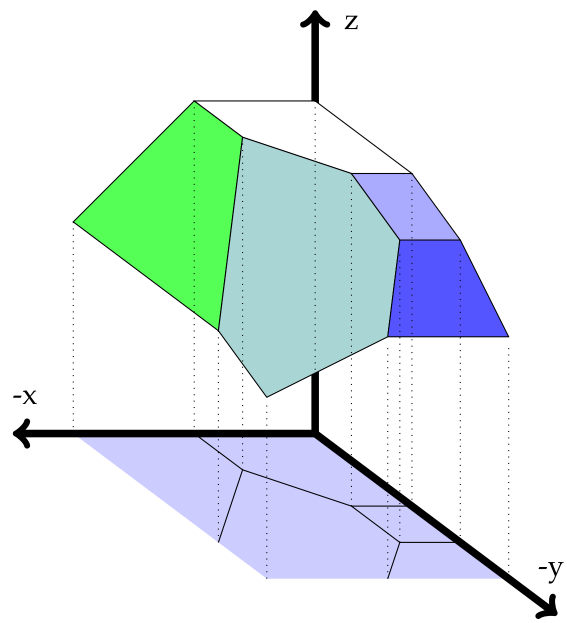


Figure 2.2 A plot of a tropical polynomial of two variables and of degree two. The polynomial is colored so that each region corresponds to a particular monomial: white for the constant, light blue for y, blue for y^2, teal for xy, and green for x^2. Below is the tropical curve, the plot of all points where two or more monomials are expressed at once.

Also like classical polynomials, we can plot tropical polynomials. However, we note that the plots of the two types of polynomials are very different. Plots of classical polynomials are very smooth in the sense that they are continuously differentiable. In contrast, tropical polynomials are piecewise-linear concave functions (Figures 2.1, 2.2). The linear nature comes from the fact that tropical monomials are tropical products or summations of a variety of different terms. The piecewise nature of tropical polynomials originate from the fact that tropical addition is a minimum operator.

2.2 Tropical Curves

One difficulty with plotting out tropical polynomials $p(x, y)$ of two variables is that they necessitate three dimensions to visualize: two for the variables x and y, one for the evaluation of $p(x, y)$. To approach this problem, we notice that $p(x, y)$ is locally linear for most values of x and y. Therefore, instead of viewing the entire plot of $p(x, y)$, we could instead just visualize the values for x and y where $p(x, y)$ is not linear at all. If we are dealing with a tropical polynomial of two variables (like $p(x, y)$), we won't need three dimensions to graph x, y, and the resulting value of $p(x, y)$. Instead, we only need a graph of two dimensions as we only need to plot the x and y coordinates where we see non-linearity. This generates the idea of a tropical curve:

Definition 2.6. Suppose $p(x_1, \ldots, x_k)$ is a tropical polynomial. The tropical curve of p is defined as the set of all points $(x_1, \ldots, x_k)$ such that two or more monomials in p such that the evaluation of the polynomial and the two monomials are equal at $(x_1, \ldots, x_n)$. (In other words, all of the points where at least two monomials are expressed at the same time.)

Tropical curves allow us to determine where the tropical polynomial is and is not linear, which provides a useful way of looking at tropical curves of two variables. In addition, tropical curves are also useful because they allow us to start describing more appropriately what we mean by a "type" of a particular tropical polynomial. Generally, we want to say that two polynomials have the same "type" of tropical curve if their tropical curves generally look the same.

While this idea is great in theory, we may run into two separate issues when trying to compare the type. First, how do we tell the difference between two curves that actually share the same type, and two curves that just look like they have the same type? Secondly, how can we say that two tropical polynomials look the same if we have no reasonable way of graphing two different tropical polynomials? In response, we construct a formal definition for the type of a tropical polynomial.

Definition 2.7. Suppose $p(x_1, \ldots, x_n)$ is a tropical polynomial of n variables. Let V be the set of all points $(a_1, \ldots, a_n)$ such that the $x_1^{a_1} \cdots x_n^{a_n}$ term is a non-tivial monomial in $p(x_1, \ldots, x_n)$. Let E be the set of all pairs $\{(a_1, \ldots, a_n), (b_1, \ldots, b_n)\}$ such that there exists some $(n-1)$-dimensional set of real values $x_1, \ldots, x_n$ such that $p(x_1, \ldots, x_n)$ realizes the monomials $x_1^{a_1} \cdots x_n^{a_n}$ and $x_1^{b_1} \cdots x_n^{b_n}$ at the same time. We define the

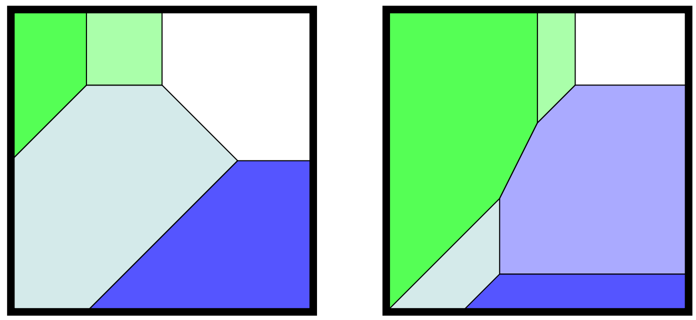


Figure 2.3 Two possible tropical curves (thin black lines) that we can have for tropical polynomials of two variables and of degree two. The linear regions represent the dominance of particular monomials within the polynomial: white for the constant, light-green for x, green for x^2, light-blue for y, blue for y^2, and teal for xy. We would say that two curves have different types.

type of $p(x_1, \ldots, x_n)$ to be the graph whose vertices are V and edges are E.

Note that the type of a tropical polynomial does depend on the coefficients of the polynomial, but it does not carry explicit information on the exact values for each of the coefficients.

One of the most immediate questions that we can ask about the types is about their relative frequencies, or how common or rare certain types can be. This leads us to the major fundamental question that we aim to answer within this paper:

Question 2.8. Fix positive integers d and n. Consider the polynomial

$$p(x_1, \ldots, x_n) = \bigoplus_{\substack{a_1,\ldots,a_n \in \mathbb{N} \\ a_1+\cdots+a_n \leq d}} c_{a_1,\ldots,a_n} x_1^{a_1} \cdots x_n^{a_n}$$

where each $c_{a_1,\ldots,a_n}$ is chosen identically and independently from the standard normal distribution. (In other words, we construct a random tropical polynomial of n variables and of degree d.) What is the resulting distribution of polynomial types?

Despite the straightforward nature of the question, the answer will surprisingly take us into a number of incredibly different areas of mathematics. We will discuss each of these areas as they come up in relationship to our tropical polynomials.

Chapter 3

Marked Polytopes

Although they are relatively simple objects, tropical polynomials are a bit hard to work with as is. Therefore, instead of talking directly about tropical polynomials, we will find it useful to think about them in a completely different framework. We are going to take a detour into marked polytopes and relevant terminology. We will start by defining a polytope, and moving along the discussion from there. Many theorems and definitions from this chapter come from Yang (2013).

Definition 3.1. A polytope is the convex hull of a finite collection of points in $\mathbb{R}^n$ for some arbitrary (but fixed) $n \geq 1$.

We can think of a polytope as a generalized polygon. All polytopes are identified by the fact that they a finite number of flat sides, are convex, and are closed objects. Within this polytope, we will find that it will be very useful to identify particular points of interest within the polytope. We will call give these kinds of polytopes a self-explanatory name and call them marked polytopes:

Definition 3.2. Let $\mathcal{A}$ be a collection of points in $\mathbb{Z}^n$, and let Δ be a convex polygon formed by the convex hull of $\mathcal{A}$. The pair $(\mathcal{A}, \Delta)$ of the polytope and the associated collection of points is known as a marked polytope.

With marked polytopes, we can imaging removing potentially irrelevant marked points. We could also "fracture" a marked polytopes along the marked points so that the new polytopes are convex hulls of subsets of the original set of marked points. We consider the objects resulting from the marked point removals and fracturings as "subdivisions."

Definition 3.3. A collection S of marked polytopes $(\mathcal{A}_1, \Delta_1), \ldots, (\mathcal{A}_k, \Delta_k)$ is considered to be a subdivision of a marked polygon $(\mathcal{A}, \Delta)$ if it satisfies the following three properties:

1. each $\mathcal{A}_i$ is a subset of $\mathcal{A}$, and each Δ_i is non-degenerate,
2. the intersection between Δ_i and Δ_j is either empty, or is a face of both and $\mathcal{A}_i \cap (\Delta_i \cap \Delta_j)$ is equal to $\mathcal{A}_j \cap (\Delta_i \cap \Delta_j)$.
3. The union of the convex hulls $\Delta_1 \cup \cdots \cup \Delta_k$ is the original convex hull Δ.

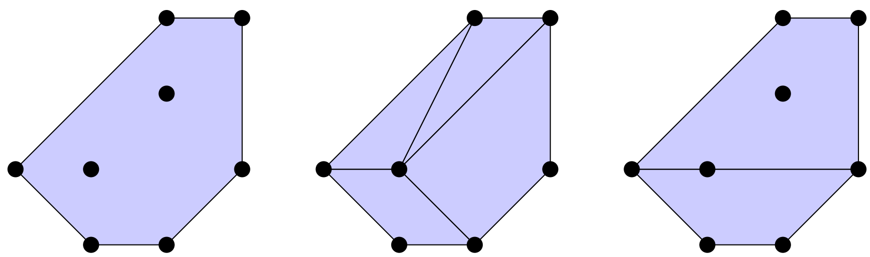


Figure 3.1 A marked polytope (left) and two possible subdivisions (middle and right).

In addition, we can consider some subdivisions that refine the marked polytope more than others. We call these former subdivisions "refinements" of the latter subdivisions.

Definition 3.4. Suppose S and S' are two subdivisions of $(\mathcal{A}, \Delta)$. We call S a refinement of S' if for each $(\mathcal{A}', \Delta')$ in S', the collection of all $(\mathcal{A}_i, \Delta_i)$ such that $\Delta_i \subset \Delta'$ forms a subdivision of S.

We also find that the "smallest" refinements, or subdivisions that have no refinements, have very particular geometric representations. We call these triangulations.

Definition 3.5. A triangulation S of $(\mathcal{A}, \Delta)$ is a subdivision such that each $(\mathcal{A}', \Delta')$ in S has the property that Δ' is a simplex, and $\mathcal{A}'$ is the three points that are the vertices of that simplex.

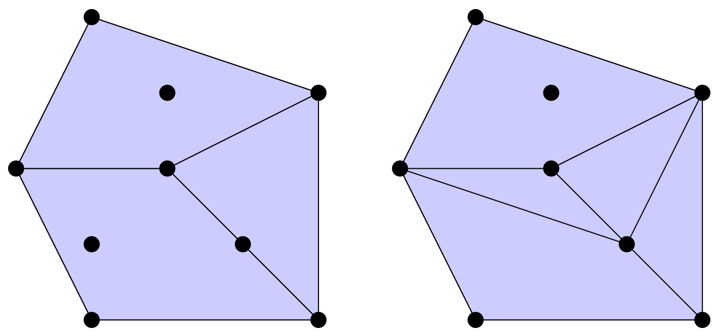


Figure 3.2 A subdivision (left) and a refinement of that subdivision (right).

3.1 Coherent Marked Polytopes and Tropical Polynomials

Now that we have discussed the nature of marked polytopes, we can now circle back to discuss the topic of how we can rigorously identify the type of tropical curve associated with a tropical polynomial.

Theorem 3.6 (Yang (2013))**.** *Suppose we have a marked polytope $(\mathcal{A}, \Delta)$. Consider and fix any function $\phi : \mathcal{A} \to \mathbb{R}$ (the function ϕ is sometimes known as the lifting vector). Let G_ψ be the set of all points $(a, \phi(a))$ where a is an element of $\mathcal{A}$.*

Now, define a function cc_ϕ to be a function $cc_\phi : \Delta \to \mathbb{R}$ such that cc_ϕ is the largest possible concave function such that $cc_\phi(a) \geq \phi(a)$ for all $a \in \mathcal{A}$. We find that cc_ϕ will be a piecewise linear function that acts as the lower convex hull of our set. For each linear region, define Δ' to be the set of all points x where $cc_\phi(x)$ lies on that linear region, and similarly define $\mathcal{A}'$ to be the set of all points in $\mathcal{A}$ such that $cc_\phi(a)$ lies on that linear region as well. The collection

S of all pairs $(\mathcal{A}', \Delta')$ from each linear region defines a subdivision of $(\mathcal{A}, \Delta)$.

Definition 3.7. Any subdivision S that can be created from the procedure in Theorem 3.6 is called a coherent subdivision.

Up to this point, we have been building up our intuition about subdivisions up to a point so that we can talk about coherent subdivisions. While this may seem arbitrary, they offer a very convenient basis for how we can actually approach tropical polynomials, as will be shown in the next theorem.

Theorem 3.8 (Yang (2013), Proposition 2.3)**.** *Suppose that $p(x_1, \ldots, x_k)$ is a tropical polynomial. Let T be the tropical curve of p.*

Now, let $\mathcal{A}$ be the set of all points $(n_1, \ldots, n_k)$ where p contains some monomial of the form $ax_1^{n_1} \cdots x_k^{n_k}$ where a is any arbitrary value. Next, let Δ be the convex hull of $\mathcal{A}$. Finally, let ϕ be a function where for each monomial $ax_1^{n_1} \cdots x_k^{n_k}$ in the tropical polynomial, $(n_1, \ldots, n_k)$ is mapped to a.

Let S be the coherent subdivision of $(\mathcal{A}, \Delta)$ induced by ϕ. If we treat the subdivision S and the tropical curve T as graphs, then S is the dual graph of T.

This provides us with an extremely strong tool. From an computational standpoint, it is much easier to work with marked polytopes and their corresponding subdivisions than it is to work with tropical polynomials themselves. Marked polytopes are easier to work with using elementrary tools than tropical polynomials, and some of the interactions between marked polytopes, refinements, and triangulations are intuitive. In addition, there are more well known results and a larger volume of literature to go along with marked polytopes. From a computational standpoint, tropical polynomials are harder to work with than coherent triangulations. While attempting to plot out the tropical polynomial is not straight forward, coherent triangulations are fairly straightforward. Finding a coherent triangulation essentially takes appending a new coordinate on a set of points, finding the faces of the lower convex hull, and then projecting those faces back down again.

Finally, this theorem also reveals a key insight to us. We do not have to create the plot of the tropical polynomial to find the connectivity of the terms. Instead, we observe that each monomial corresponds to a face in the tropical curve, and that the coherent subdivision that corresponds to a tropical curve is also its dual graph. Therefore, the coherent subdivision is the connectivity graph of the monomials in the original tropical polynomials. In other words, to answer our original question, we must be able to answer a more approachable, but more general question:

Question 3.9. Suppose $\mathcal{A}$ is an fixed collection of points in $\mathbb{R}^n$, and let Δ be the convex hull of $\mathcal{A}$. Let $\phi : \mathcal{A} \to \mathcal{R}$ be a function that is randomly chosen by independently assigning each value of $\phi(a)$ to a random value from a standard normal distribution. What is the resulting distribution of subdivisions?

Chapter 4

Cones and Fans

Coherent subdivisions are a convenient tool that will help us work with tropical polynomials. However, we also need to introduce a mathematical tool, called a cone. Again, many theorems in this chapter come from Yang (2013).

Definition 4.1. Suppose V is some vector subspace over $\mathbb{R}$. A cone σ is a closed collection of points such that for all vectors v_1 and v_2 in V and non-negative values λ_1 and λ_2, the vector $\lambda_1 v_1 + \lambda_2 v_2$ is also within σ. Cones are also sometimes known as positive linear spans, and these two terms will be used interchangeably.

Cones are actually extremely helpful for us. We note that if $(\mathcal{A}, \Delta)$ is a marked polytope. If we have two functions ϕ and ϕ' that map from $\mathcal{A}$ to $\mathbb{R}$ that generate the same subdivision, then their sum $\phi + \phi'$ will generate that same subdivision as well. In addition, for any positive constant $\lambda > 0$, we find that the subdivision corresponding to $\lambda\phi$ is the same as the subdivision for ϕ. Therefore, it is natural to think about cones in relationship to tropical polynomials and coherent subdivisions:

Definition 4.2. Suppose that S is a coherent subdivision of a marked polytope $(\mathcal{A}, \Delta)$. Define $\mathcal{C}(S)$ to be the set of all functions ϕ such that if the subdivision S' created by the procedure outlined in theorem 3.6 such that S is a refinement of S'. Furthermore, $\mathcal{C}(S)$ is also a finitely generated cone.

Theorem 4.3. *The set of all tropical polynomials of degree d and of k variables that share the same tropical curve is isomorphic to the interior of a cone.*

Thankfully, there are several nice properties about cones, particularly the cones corresponding to tropical polynomials and coherent subdivisions. We will proceed by giving out several definitions of various terms that will be of some use to us.

Definition 4.4. A cone σ is called finitely generated if there exists a finite number of vectors $v_1, \ldots, v_n$ such each vector in σ can be expressed as the positive linear sum $\lambda_1 v_1 + \cdots + \lambda_n v_n$ for some non-negative values $\lambda_1, \ldots, \lambda_k$. Such a cone is denoted either as cone$\{v_1, \ldots, v_n\}$ or as pos$\{v_1, \ldots, v_n\}$.

If each vector v_i can not be expressed as a positive linear sum of the other vectors, then σ is known as an n-cone and the vectors $v_1, \ldots, v_n$ are called the generators of σ.

Definition 4.5. A cone σ' is called a facet of another cone σ if either the generators of σ' is a strict subset of the generators of σ, or if σ is the trivial cone that contains only the vector 0. If σ' is a 1-cone, then it is referred to as an edge. Similarly, σ' is called a face if it is a 2-cone.

Definition 4.6. A fan Σ is a collection of cones where the intersection of any two distinct cones σ and σ' in Σ is a cone τ such that τ is both a facet of σ and σ' in addition to being an element of Σ as well.

Theorem 4.7 (Yang (2013), Proposition 2.1)**.** *Let $(\mathcal{A}, \Delta)$ be a marked polytope in $\mathbb{R}^n$. For all coherent subdivisions S of $(\mathcal{A}, \Delta)$, the cone $\mathcal{C}(S)$ of all functions that generate S is finitely generated. Furthermore, let Σ be the set of all $\mathcal{C}(S)$*

where S is a coherent subdivision of S. Then, Σ is a fan that covers all of $\mathbb{R}^{\mathcal{A}}$.

Cones help us immensely with organizing the set of all functions that create the same coherent subdivision, and allow us to talk about the type of tropical polynomial in terms of a singular object in linear algebra rather than just as a collection of different objects in tropical geometry. There are some nice attributes about the relationships between refinements of the marked polytope and the corresponding cones they generate.

Theorem 4.8. *Suppose that* $(\Delta, \mathcal{A})$ *is a marked polytope, and let* S *and* S' *be two coherent subdivisions. Then, if* S *is a refinement of* S'*, then* $\mathcal{C}(S')$ *is a face of* $\mathcal{C}(S)$.

4.1 Solid Angles

Another reason that we appeal to cones is that all cones have a property called a solid angle:

Definition 4.9. Let $\mathcal{V}$ be a vector space, and let σ be a cone in $\mathcal{V}$. Suppose that $f : \mathcal{V} \to \mathbb{R}$ is some function that is rotationally symmetric (that is, f depends only on the magnitude of x). We say that solid angle of σ over $\mathcal{V}$ is the fraction of integrals

$$\text{SolidAngle}_{\mathcal{V}}(\sigma) = \frac{\int_{x \in \sigma} f(x)\,dx}{\int_{x \in \mathcal{V}} f(x)\,dx}.$$

In the case that $\mathcal{V}$ is not provided, it is assumed that $\mathcal{V}$ is the span of σ. In this case, instead of saying "the solid angle of σ over $\mathcal{V}$", we simply say "the solid angle of σ", and we denote it as SolidAngle(σ).

Conceptually, we can think of a solid angle as the proportion of a sphere that a cone will cut out. In two dimensions, we can just think of this as just the angle between the two generating vector normalized between 0 and 1 instead of 0 and 2π. In addition, because the standard multinormal probability density function is $f(x) = (2\pi)^{-n/2} e^{-|x|^2/2}$, where n is the dimension of the space, is a rotationally symmetric function. Therefore, the solid angle of the cone can also be thought of as the probability that a point chosen randomly in the space (using a standard multinormal distribution) will be contained within the cone. Therefore, the solid angle can be used to extract the probability that a particular tropical curve will become one of a select variety of types.

There are a few formulas for solid angles of n-cones. In two dimensions, the solid angle is simply a rescaling of the angle between the two generating vectors.

Theorem 4.10. *Suppose σ is a 2-cone in $\mathbb{R}^2$ generated by the unit vectors v_1 and v_2 such that $v_1 \neq \pm v_2$. Then, $\cos(2\pi \cdot SolidAngle_{\mathbb{R}^2}(\sigma)) = v_1 \cdot v_2$. In addition, if θ is the angle between v_1 and v_2, then $SolidAngle(\sigma) = \theta/2\pi$.*

The formula for 3-cones is less intuitive, but there is thankfully a relatively simple formula for calculating their solid angles:

Theorem 4.11. *Suppose σ is a 3-cone in $\mathbb{R}^3$ generated by the unit vectors v_1, v_2, and v_3. Then,*

$$\tan\left(2\pi \cdot SolidAngle(\sigma)\right) = \frac{\left| det\left(\begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & v_1 \cdot v_3 \\ v_2 \cdot v_1 & v_2 \cdot v_2 & v_2 \cdot v_3 \\ v_3 \cdot v_1 & v_3 \cdot v_2 & v_3 \cdot v_3 \end{bmatrix}\right)\right|}{1 + v_1 \cdot v_2 + v_1 \cdot v_3 + v_2 \cdot v_3}$$

Unfortunately, there are no known simple expressions for calculating the solid angles of n-cones in $\mathbb{R}^n$ for any $n \geq 3$. There still do exist a variety of algorithms to calculate the solid angles of higher dimensional cones. For example, Ribando (2006) came up with a Taylor series polynomial that approximates the polynomial:

Theorem 4.12 (Ribando (2006), Theorem 2.2)**.** *Suppose σ is an n-cone generated by the linearly independent unit vectors $v_1, \ldots, v_n$. Let α_{ij} be the dot product between vectors v_i, and v_j, and let V be the matrix whose columns are the coordinates of these vectors. In addition, we denote $\mathbb{N}^{\binom{n}{2}}$ to be the set of all non-negative integer vectors whose coordinates are indexed by i and j such* *that $1 \le i, j \le n$. The solid angle of σ is equivalent to the Taylor series*

$$SolidAngle(\sigma) = \frac{|det(V)|}{(4\pi)^{n/2}} \sum_{a \in \mathbb{N}^{\binom{n}{2}}} \frac{(-2)^{\sum_{i<j} a_{ij}}}{\prod_{i<j}(a_{ij})!} \prod_i \left(\frac{1 + \sum_{m \neq i} a_{ij}}{2} \right) \cdot \prod_{i<j} \alpha_{ij}^{a_{ij}}$$

if the Taylor series converges.

Now that we have covered the main mathematical objects, we can now find the probability of a particular type of tropical polynomial. In general, the solid angle of the cone corresponding to a type of tropical polynomial is the probability that a randomly generated polynomial will become that type of tropical polynomial. This gives us the basis for the following theorem:

Theorem 4.13. *Suppose that $p(x)$ is a tropical polynomial with corresponding marked polytope subdivision $(\mathcal{A}, \Delta)$. The probability that a randomly generated tropical polynomial will have the same type as $p(x)$ is given by $SolidAngle_{\mathbb{R}^{\mathcal{A}}}(\mathcal{C}(S))$.*

Already, we can use solid angles for their benefit because it tells us that the probability of particular types of tropical polynomials will be very negligible.

Theorem 4.14. *Suppose $p(x)$ is a tropical polynomial whose corresponding subdivision is not a coherent triangulation. Then, the probability of creating a tropical polynomial of the same type as $p(x)$ is zero.*

proof. The probability of generating a polynomial of the same type as $p(x)$ is equal to the solid angle of the cone corresponding to the tropical subdivision. If the corresponding subdivision is not coherent or is not a triangulation, then the cone cannot be maximal. Therefore, the solid angle of the cone in $\mathbb{R}^{\mathcal{A}}$ is zero. Therefore, the probability that a random polynomial will have the same type as $p(x)$ is zero. ◈

Therefore, this tells us that the only major tropical polynomials that we need to pay attention to are the tropical polynomials corresponding to triangulations of marked polytopes.

However, solid angles present us with two significant issues that we need to address. First, for a particular type of tropical polynomial and its corresponding coherent subdivision, what vectors actually generate the cone? Secondly, even if we do have the generating vectors of a cone, how do we find its solid angle, especially when it goes beyond three dimensions? We will now discuss approaches to these two challenges.

Chapter 5

Normal Cones and Secondary Polytopes

First, we need a reliable way of determining the cone corresponding to a coherent subdivision, particularly the vectors that generate the cone. One way to do it is through secondary polytopes and by normal cones. Therefore, before we directly explain the process of finding the generating vectors, we must first conver normal cones and secondary polytopes. Some of the theorems and ideas in this chapter come from Davis (2016).

5.1 Normal Cones

Definition 5.1. A hyperspace-union $Q(A, b)$ is the collection of all vectors x that satisfy $Ax \leq b$. When, bounded hyperspace-unions $Q(A, b)$ are equivalent to polytopes.

As the name implies, a hyperspace-union is the union of multiple half spaces. Each half space is defined by a particular normal vector a_i that defines its orientation and a constant b_i that defines its position. We can tell if a point x lies on the correct side of the half space if $a_i x \leq b_i$. Therefore, we can consider a hyperspace-union as the set of all x that satisfy $a_i x \leq b_i$ for all i. However, we can write this more compactly. If we let A be the matrix whose rows are the a_i terms, and b be the column vector whose coordinates are the b_i's, then $Ax \leq b$ becomes a more compact statement of "the union of half spaces."

With hyperspace-unions (and convex sets in general), we can also have "normal cones" to points within our collection of points:

Definition 5.2. Suppose that C is a convex set, and let x be any point within C. The normal cone $N_C(x)$ is the set of all points y that satisfy $y \cdot (z - x) \leq 0$ for all $z \in C$.

Theorem 5.3 (Davis (2016), Lemma 2)**.** *Suppose $Q(A, b)$ is a hyperspace-union. The set of all distinct cones $N_{Q(A,b)}(x)$ where $x \in Q(A, b)$ is a fan. If $Q(A, b)$ is bounded, then the fan spans the entire space that $Q(A, b)$ lives within.*

The phrase "normal cones" comes from the fact that the set is both a cone (which is straightforward to show) and some of the faces of the polytopes and the normal cone meet each other at ninety degree angles. Thankfully, if we know all of the hyperplanes involved in the hyperspace-union, then calculating the normal cone turns out to be very straightforward.

Theorem 5.4 (Davis (2016), Theorem 3)**.** *For any hyperspace-union $Q(A, b)$ and point x within the polytope,*

$$N_{Q(A,b)}(x) = \{A^T y | y \geq 0, y^T (Ax - b) = 0\}.$$

proof. Let Y be the set $\{A^T y | y \geq 0, y \cdot (Ax - b) = 0\}$. We will show that $Y = N_{Q(A,b)}(x)$.

First, we will show that $Y \subset N_{Q(A,b)}(x)$. Consider any $A^T y \in Y$ and $z \in Q(A, b)$. It directly follows that

$$\begin{aligned} A^T y \cdot (z - x) &= y^T A(z - x) \\ &= y^T Az - y^T Ax \\ &= y^T Az - y^T b && \text{(Since } y^T (Ax - b) = 0.\text{)} \\ &= y \cdot (Az - b) \end{aligned}$$

Since all components of y are non-negative and all components of $Az - b$ are non-positive, it follows that the dot product between y and $Az - b$ is at most zero. Therefore $A^T y \cdot (z - x) \leq 0$ as well. Therefore, $A^T y$ must be within $N_{Q(A,b)}$, so all elements within Y are within $N_{Q(A,b)}$ as desired.

Next, we will show that $N_{Q(A,b)}$ is contained within Y. Suppose to the contrary that there exists a point p in $N_{Q(A,b)}$, but p is not in Y. Since Y and $\{p\}$ are both convex sets, it follows by the hyperplane theorem that there exists some vector $\widehat{a}$ and constant $\widehat{b}$ such that

$$\widehat{a} \cdot p > \widehat{b} > \widehat{a} \cdot A^T y$$

for all $A^T y \in Y$. We note that by setting $y = 0$, then $\widehat{b} > \widehat{a} \cdot A^T 0 = 0$. Therefore, $\widehat{b}$ is strictly positive.

Our goal now is to show that for a small enough $\epsilon > 0$, we have $z(\epsilon) = x + \epsilon\widehat{a}$ is in $Q(A, b)$. Let a_i be the i^{th} row of A, and let b_i be the i^{th} element of b. We will prove that for all i that $a_i z(\epsilon) - b_i \leq 0$. We have one of three major cases: $a_i^T \cdot x = b_i$, $a_i^T \cdot x < b_i$ and $a_i^T \cdot \widehat{a} < 0$, or $a_i^T \cdot x < b_i$ and $a_i^T \cdot \widehat{a} > 0$.

Suppose that for our choice of x that $a_i^T \cdot x = b_i$. Therefore, we find that $a_i^T \cdot x - b_i = 0$. This implies that if e_i is the i^{th} basis vector, that $e_i^T(Ax - b) = 0$, since the e_i is picking out the i^{th} row of A and the i^{th} component of b. Since $e_i \geq 0$, it follows that $A^T e_i = a_i^T$ is an element of Y. This implies that λa_i^T is also within Y for all $\lambda > 0$, and thus $\widehat{b} > \widehat{a} \cdot \lambda a_i^T$. Rearranging, we find $\frac{1}{\lambda}\widehat{b} > \widehat{a} \cdot a_i^T$. In the limit as λ grows to infinity, we can deduce that $\widehat{a} \cdot a_i^T \leq 0$. Therefore, $z(\epsilon) \cdot a_i^T = x \cdot a_i^T + \epsilon\widehat{a} \cdot a_i \leq a_i x = b_i$.

Now, assume that $a_i^T \cdot x < b_i$ and $a_i^T \cdot \widehat{a} < 0$. It follows directly that $z(\epsilon) \cdot a_i^T = x \cdot a_i^T + \epsilon\widehat{a} \cdot a_i^T < b_i$.

Finally, consider the case that $a_i^T \cdot x < b_i$ and $a_i^T \cdot \widehat{a} > 0$. Then, if $\epsilon < (b_i - x \cdot a_i^T)/(\widehat{a} \cdot a_i^T)$, it follows that

$$\begin{aligned} z(\epsilon) \cdot a_i &= x \cdot a_i^T + \epsilon\widehat{a} \cdot a_i^T \\ &< x \cdot a_i^T + \frac{b_i - x \cdot a_i^T}{\widehat{a} \cdot a_i^T}(\widehat{a} \cdot a_i^T) \\ &< x \cdot a_i^T + (b_i - x \cdot a_i^T) \\ &< b_i \end{aligned}$$

Thus, $z(\epsilon) \cdot a_i^T < b_i$.

Observe that if we set ϵ to be less than $(b_i - x \cdot a_i)/(\widehat{a} \cdot a_i)$ for all i such that $a_i x < b_i$ and $a_i \cdot \widehat{a} > 0$, we know that $a_i \cdot z(\epsilon) \leq b_i$ for all i, and thus $Az(\epsilon) \leq b$. Therefore, $z(\epsilon)$ is indeed an element within $Q(A, b)$.

Now note that because $z(\epsilon)$ is in $Q(A, b)$, and since p is in $N_{Q(A,b)}(x)$, it must follow that $p \cdot z(\epsilon) \leq 0$. However, observe that

$$\begin{aligned} p \cdot \big(z(\epsilon) - x\big) &= p \cdot (x + \epsilon\widehat{a} - x) \\ &= p \cdot \epsilon\widehat{a} \\ &> \epsilon\widehat{b} && \text{(By the hyperplane definition)} \\ &> 0 && \text{(As } \epsilon, \widehat{b} > 0.\text{)} \end{aligned}$$

Therefore, p cannot actually be an element of $N_{Q(A,b)}(x)$. We can then conclude that $N_{Q(A,b)}(x)$ is a subset of Y. Since we have shown that $N_{Q(A,b)}(x) \subset Y$ and $Y \subset N_{Q(A,b)}$, we have proven that $N_{Q(A,b)}(x) = Y$ as desired. ◈

The implication of the proof is fairly simple, if a tad verbose. Suppose we have a polytope identified by the union of half planes, so it is the set of all x such that $a_1 \cdot x \leq b_1$, $a_2 \cdot x \leq b_2$, $\ldots$, $a_n \cdot x \leq b_n$ for some series of vectors a_i and constants b_i. The proof states that if we have some point x within the polytope, the normal cone to x is the positive linear span of all a_i such that $a_i \cdot x \leq b_i$ reaches equality. The positive linear span comes from the fact that all elements within the normal cone must be of the form $A^T y = \begin{bmatrix} a_1^T & a_2^T & \cdots & a_k^T \end{bmatrix} y = a_1^T y_1 + \cdots + a_n^T y_n$ where all components of y are non-negative. The fact that its only the positive linear span of the a_i only when we reach equivalence in $a_i^T \cdot x \leq b_i$ comes from the requirement that $y^T(Ax - b) = 0$. When we reach equivalence in $a_i^T \cdot x \leq b_i$, then we find that $a_i x - b_i = 0$, and when we don't reach equivalence in $a_i^T \cdot x \leq b_i$, then we force $y_i = 0$. Both of these are encapsulated by the fact that $y_i(a_i^T \cdot x \leq b_i)$ for all i. We can notate this effect for all i with the more elegant (albiet less self-explanatory) statement that $y^T(Ax - b) \leq 0$.

The structure of the proof is to start by claiming that Y is the positive linear span of all the a_i where we reach equivalence in $a_i x - b_i \leq 0$. We aim to show that Y is equivalent to the normal cone. We show this by demonstrating that all elements of Y are in the normal cone, and anything not in Y is not in the normal cone. The first statement is fairly straightforward, and is derived from the definitions we initially have. The second statement takes surprisingly much more effort. We begin by showing that if a point p is in the normal cone, but not in Y, there is some plain (defined by vector $\widehat{a}$ and constant $\widehat{b}$), that comes between our new point p and the cone Y. We use this new plane in order to construct a new point $z(\epsilon)$ such that $z(\epsilon)$ falls inside of the polytope, and has a positive dot-product. To show that it falls inside of the polytope, we have to check that $a_i^T \cdot z(\epsilon) \leq b_i$ for all i.

Although the theorem is incredibly involved, thankfully the implication is fairly straightforward. From a computational standpoint, we note that we can express the convex hull of the marked polytope as the union of half-planes, and we can computationally find the convex hull in the form $Q(A, b)$ for a known A and b. Then for any point in the convex hull, we can then use the previously stated theorem to find the normal cone to that point. This is one of the two components that we need in order to find the generating vectors of a cone $\mathcal{C}(S)$ corresponding to a triangulation S of an arbitrary marked polytope.

5.2 Secondary Polytopes

In addition to normal cones, we must also discuss secondary polytopes, which come from a very curious and non-intuitive definition:

Definition 5.5. Consider any marked polytope $(\mathcal{A}, \Delta)$. For each triangulation $S = \{(\mathcal{A}_1, \Delta_1), (\mathcal{A}_2, \Delta_2), \ldots\}$ of the marked polytope, define a point $x \in \mathbb{R}^{\mathcal{A}}$ indexed by points of $\mathcal{A}$ where for each $p \in \mathcal{A}$, we let x_p be the sums of the volumes of each Δ_i where x_p is included in $\mathcal{A}_i$. Define $\mathcal{B}$ as the set of all such points created from all triangulations of $(\mathcal{A}, \Delta)$. The convex hull of $\mathcal{B}$ is a hyperplace-union, and is known as the secondary polytope of $(\mathcal{A}, \Delta)$.

The secondary polytope of a marked polytope is a physical representation of the ways that we can subdivide the marked polytope. From the definition, we note that the vertices of the secondary polytope. But more than that, it turns out that for any subdivision S of the original marked polytope, S is represented in the secondary polytope as the convex hull of all vertices corresponding to the triangulations that refine S! We can also consider the secondary polytope also as a graph of how related particular triangulations are to one another.

One resource for looking at the relationships between convex hexagons and their corresponding secondary polytopes is provided at *hexahedria.com/associahedron/*.

5.3 Generating Vectors of a Triangulation

Secondary polytopes and normal cones together form a very surprising result about how we can find the generating vectors of a particular triangulation.

Theorem 5.6 (Gelfand and Zelevinsky (1994), Chapter 7, Theorem 2.4)**.** *Let $(\mathcal{A}, \Delta)$ be a marked polytope, and $Q(A, b)$ be the corresponding secondary polytope. If S is a triangulation of the marked polytope and x is the corresponding point in the marked polytope, then*

$$\mathcal{C}(S) = N_{Q(A,b)}(x).$$

In other words, the cone of all lifting vectors on $\mathcal{A}$ that realize a particular triangulation coincides with the normal cone of the secondary polytope at the point corresponding to the triangulation.

This theorem tells us that in order to find the generating vectors for $\mathcal{C}(S)$, where S is a triangulation, we must first find all triangulations. Then, we construct the secondary polytope, and then find the normal cone to the vertex corresponding to the triangulation of interest via our theorem about normal cones and hyperspace-union polytopes.

Chapter 6

Conic Manipulations

As we will see later on, the generating vectors may not interact with our solid angle formulas very well. We will frequently see that many of our cones will be of the form $\text{pos}\{v_1, \ldots, v_n\} \oplus \text{span}\{u_1, \ldots, u_m\}$, while our solid angle formulas only operate on cones of the form $\text{pos}\{w_1, \ldots, w_k\}$ and are only exact for $k \leq 3$. Although we will not completely resolve this issue, we will show in this chapter that any cone that is a direct sum of a cone and a linear subspace has a corresponding cone that is just a positive linear span, and has the same solid angle.

We start by showing that all cones of the form $\text{pos}\{v_1, \ldots, v_n\} \oplus \text{span}\{u_1, \ldots, u_m\}$ can be expressed in a different way:

Theorem 6.1. *Suppose that a cone σ is given by $pos\{v_1, \ldots, v_n\} \oplus span\{u_1, \ldots, u_m\}$. Then, σ is also given by $pos\{v_1 + cu_1, \ldots, v_n\} \oplus span\{u_1, \ldots, u_m\}$ for any value of c.*

proof. Suppose that x is a point in σ. This implies for some positive values of λ_i and real values of μ_j,

$$\begin{aligned} x &= \lambda_1 v_1 + \sum_{i=2}^{n} \lambda_i v_i + \mu_1 u_1 + \sum_{j=2}^{m} \mu_j u_j \\ &= \lambda_1 (v_1 + cu_1) + \sum_{i=2}^{n} \lambda_i v_i + (\mu_1 - \lambda_1 c) u_1 + \sum_{j=2}^{m} \mu_j u_j \end{aligned}$$

We observe that since $\mu_1 - \lambda_1 c$ is a real number, it follows that x must be within pos$\{v_1 + cu_1, \ldots, v_n\} \oplus \text{span}\{u_1, \ldots, u_m\}$.

By this logic, we can also determine that all x within pos$\{v_1 + cu_1, \ldots, v_n\} \oplus \text{span}\{u_1, \ldots, u_m\}$ must also be within pos$\{v_1, \ldots, v_n\} \oplus \text{span}\{u_1, \ldots, u_m\}$ (by adding $-cu_1$ to the first term in the positive linear span). Therefore, we find that σ is also defined by pos$\{v_1+cu_1, \ldots, v_n\} \oplus \text{span}\{u_1, \ldots, u_m\}$. ◈

We notice that this theorem states that if we have a cone that is the direct product of a positive linear span and a span, that adding multiples of the span's generating vectors to the generating vectors of the positive linear span does not change the cone itself. This gives us the intuition for the following corollary:

Corrolary 6.2. *Suppose that a cone σ is given by pos$\{v_1, \ldots, v_n\} \oplus$ span$\{u_1, \ldots, u_m\}$. Then, if $w_1, \ldots, w_n$ are vectors in span$\{u_1, \ldots, u_m\}$, it follows that the cone is also generated by pos$\{v_1 + w_1, \ldots, v_n + w_n\} \oplus$ span$\{u_1, \ldots, u_m\}$.*

proof. Note that w_1 can be expressed as $c_1u_1 + \cdots + c_mu_m$. Use Theorem 6 to iteratively add each c_iu_i to v_1, until the generator v_1 is changed to $v_1 + w_1$. Repeat for each of the remaining v_i. Since adding each c_iu_i component to each of the v_i terms does not change the cone, the initial cone generated by pos$\{v_1, \ldots, v_n\} \oplus \text{span}\{u_1, \ldots, u_m\}$ is the same as pos$\{v_1 + w_1, \ldots, v_n + w_n\} \oplus \text{span}\{u_1, \ldots, u_m\}$. ◈

From this corollary, we note that we are always allowed to adjust generators in the positive linear span by generators in the linear span. From this, we can change all of the generators in the positive linear span so that they are all orthogonal to all of the generators in the linear span.

Corrolary 6.3. *Suppose that a cone σ is given by pos$\{v_1, \ldots, v_n\} \oplus$ span$\{u_1, \ldots, u_m\}$. For each v_i, define v_i' as the component of v_i that is orthogonal to the subspace span$\{u_1, \ldots, u_m\}$. Then, σ is also defined by pos$\{v_1', \ldots, v_n'\} \oplus$ span$\{u_1, \ldots, u_m\}$.*

proof. Note that for all i, $v_i' - v_i$ is an element of $\text{span}\{u_1, \ldots, u_m\}$. Therefore by Corrolary 6.2, this implies that σ is also defined by $\text{pos}\{v_1 - (v_1' - v_1), \ldots, v_n - (v_n' - v_n)\} \oplus \text{span}\{u_1, \ldots, u_m\}$, which gives us the theorem statement as desired. ◈

Now, whenever we work with a cone of the form $\text{pos}\{v_1, \ldots, v_n\} \oplus \text{span}\{u_1, \ldots, u_m\}$, we can change all of the v_i's so that they become linearly independent from the u_i terms. Now, we observe that there is an additional property that we can appeal to with respect to cones that can be decomposed into two orthogonal components.

Corrolary 6.4. *Suppose σ and σ' are two cones in a subspace $U \subset R^n$ and its orthogonal complement $U^\perp$ respectively. Then, $\sigma \oplus \sigma'$ is a cone, and it's solid angle is given by*

$$SolidAngle_{\mathbb{R}^n}(\sigma \oplus \sigma') = SolidAngle_U(\sigma) \cdot SolidAngle_{U^\perp}(\sigma').$$

proof. Consider any two vectors in $\sigma \oplus \sigma'$ and any two non-negative constants c and d. We know that the two vectors must be of the form $u + u'$ and $v + v'$ where $u, v \in \sigma$ and $u', v' \in \sigma'$. Therefore, it follows that $cu + dv$ is in σ and $cu' + dv'$ is in σ'. Thus, we find that $(cu + dv) + (cu' + dv') = c(u + u') + d(v + v')$ is in $\sigma \oplus \sigma'$. Thus, $\sigma \oplus \sigma'$ is indeed a cone.

Now, we will prove the equivalence amongst the solid angles. Note that if we set $f(x) = e^{-|x|}$, then if $x = u + v$ where u and v are orthogonal to one another, we find that $f(u + v) = e^{-|u+v|} = e^{-|u|-|v|} = f(u) \cdot f(v)$. Observe that

$$\begin{aligned}
&\text{SolidAngle}_{\mathbb{R}^n}(\sigma \oplus \sigma') \\
&\quad = \int_{x \in \sigma \oplus \sigma'} f(x)dx \Big/ \int_{x \in \mathbb{R}^n} f(x)dx \\
&\quad = \int_{u \in \sigma} \int_{v \in \sigma'} f(u + v)dvdu \Big/ \int_{u \in U} \int_{v \in U^\perp} f(u + v)dudv \\
&\quad = \int_{u \in \sigma} \int_{v \in \sigma'} f(u) \cdot f(v)dvdu \Big/ \int_{u \in U} \int_{v \in U^\perp} f(u) \cdot f(v)dudv
\end{aligned}$$

$$= \left(\int_{u \in \sigma} f(u)du \Big/ \int_{u \in U} f(u)du \right) \left(\int_{v \in \sigma'} f(v)dv \Big/ \int_{v \in U^{\perp}} f(v)dv \right)$$
$$= \text{SolidAngle}_{U}(\sigma) \cdot \text{SolidAngle}_{U^{\perp}}(\sigma')$$

Therefore, we have shown that $\text{SolidAngle}_{\mathbb{R}^n}(\sigma \oplus \sigma') = \text{SolidAngle}_{U}(\sigma) \cdot \text{SolidAngle}_{U^{\perp}}(\sigma')$ as desired. ◈

We use the previous two theorems to help us simplify the process of finding the solid angles of cones that are the direct sum of a positive linear span and a linear span.

Corrolary 6.5. *Suppose σ is the cone in $\mathbb{R}^n$ given by pos$\{v_1, \ldots, v_n\} \oplus \mathcal{U}$ where $\mathcal{U}$ is a linear subspace and span$\{\sigma\} = \mathbb{R}^n$. Let P be the projection operator that maps vectors in $\mathbb{R}^n$ to $U^{\perp}$. Then,*

$$\textit{SolidAngle}(\sigma) = \textit{SolidAngle}(\textit{pos}\{Pv_1, \ldots, Pv_n\})$$

proof. For all i, we note that $Pv_i - v_i$ is an element of U. Therefore, the cone σ generated by pos$\{v_1, \ldots, v_n\} \oplus$ span$\{U\}$ is also generated by pos$\{v_1 + (Pv_1 - v_1), \ldots, v_n + (Pv_n - v_n)\} \oplus$ span$\{U\}$, which simplifies to pos$\{Pv_1, \ldots, Pv_n\} \oplus$ span$\{U\}$. In addition, we note that

$$\begin{aligned} \text{SolidAngle}_{\mathbb{R}^n}(\sigma) &= \text{SolidAngle}_{\mathbb{R}^n}(\text{pos}\{Pv_1, \ldots, Pv_n\} \oplus \text{span}\{U\}) \\ &= \text{SolidAngle}_{U^{\perp}}(\text{pos}\{Pv_1, \ldots, Pv_n\}) \cdot \text{SolidAngle}_{U}(U) \\ &= \text{SolidAngle}_{U^{\perp}}(\text{pos}\{Pv_1, \ldots, Pv_n\}) \end{aligned}$$

We know that span$\{v_1, \ldots, v_n\} \oplus \mathcal{U}$ must span all of $\mathbb{R}^n$. Thus, we find that $\mathcal{U}^{\perp} = P(\text{span}\{\sigma\})$. Thus,

$$\begin{aligned} \mathcal{U}^{\perp} &= P(\mathbb{R}^n) \\ &= P(\text{span}\{\sigma\}) \\ &= P(\text{span}\{\text{pos}\{v_1, \ldots, v_n\} \oplus \mathcal{U}\}) \\ &= P(\text{span}\{v_1, \ldots, v_n\} \oplus \mathcal{U}) \\ &= \text{span}\{Pv_1, \ldots, Pv_n\} \oplus P\mathcal{U} \\ &= \text{span}\{\text{pos}\{Pv_1, \ldots, Pv_n\}\} \end{aligned}$$

Therefore, we are allowed to drop the subscript, and it follows directly that

$$\text{SolidAngle}(\sigma) = \text{SolidAngle}(\text{pos}\{Pv_1, \ldots, Pv_n\})$$

as desired. ◈

Note that since U is a linear subspace of $\mathbb{R}^n$, so is its orthogonal complement $U^\perp$. Since $U^\perp$ is a linear subspace, trivially it is also a vector space. Therefore, it does make sense to consider the solid angle of a cone embedded in $U^\perp$, and not to just restrict ourselves to $\mathbb{R}^n$.

The main point of this theorem is to show that for any cone that is composed of a direct sum between a positive linear span and a linear span, we find an alternative way of generating the cone where the positive linear span component is independent of the linear span component.

6.1 Pseudoinverses

At this point, it may be useful to discuss how to extract the orthogonal projection operator. Namely, given a vector v and a subspace U given by the span of span$\{u_1, \ldots, u_n\}$, how do we find the projection of v into $U^\perp$? The main operator that will help us out is the Moore-Penrose inverse, which is much more widely known as the pseudoinverse of a matrix:

Definition 6.6. Suppose that U is any matrix, not necessarily invertible or square. The pseudoinverse U^+ is a matrix such that if U^* is the conjugate transpose,

- $UU^+U = U$,
- $U^+UU^+ = U^+$,
- $(UU^+)^* = UU^+$, and
- $(U^+U)^* = U^+U$.

In the case that V has linearly independent columns, then V^+ can be computed as $V^+ = (V^*V)^{-1}V^*$. Note that in this case, $V^+V = I$, so V^+ is a left inverse, but V^+ is only an inverse if and only if V has full rank. In addition, a pseudo-inverse is a unique matrix. That is if B and B' are both pseudoinverses of A, then $B = B'$.

Pseudoinverses are ocassionally used to solve programs within linear programming. For example given a matrix A and vector b, suppose that we want to find the vector x such that $Ax = b$. If A is invertible, then we can quickly find the unique x that solves the equation by using the formula $x = A^{-1}b$. However, for certain matrices A and b, such a unique solution might not be possible because there could be either an infinite number of solutions or there could be no solutions whatsoever.

In these cases, we can use pseudoinverses to help us look at the "next best" possibilities. When $Ax = b$ has no solutions, the vector x that minimizes the difference between Ax and b turns out to be A^+b. When there are infinite number of solutions, then all solutions x are of the form $A^+b + (I - A^+A)w$ where w is any vector of the same dimension as b. It should be noted that when A does have an inverse, both of these cases turns out to be true. The vector x that minimizes the error between Ax and b is indeed $A^+b = A^+b$ (the solution minimizes the error). In addition, all solutions that satisfy $Ax = b$ are of the form $A^+b + (I - A^+A)w = A^{-1}b + (I - A^{-1}A)w = A^{-1}b$, so there is just one distinct solution.

The main use of pseudoinverses, at least within this paper, is the fact that they work well with describing projections.

Theorem 6.7. *Suppose that $\mathcal{U}$ is a subspace of $\mathbb{R}^n$, and is equal to the span of U. The projection mapping that takes any vector v from $\mathbb{R}^n$ into $\mathcal{U}^\perp$ is given by*

$$Pv = (I - UU^+)v = (I - U(U^*U)^{-1}U^*)v$$

We will use this extensively in conjunction with corollary 6.5 when proceed to simplify down the cone so that we can use our solid angle formulas.

Chapter 7

Example Derivations for the Distributions for Tropical Polynomials

Here, we will show various examples of how these results can be used to extract the analytic distributions for various types of tropical polynomials.

7.1 Tropical Quadratics of One Variable

The simplest example of the procedure is for calculating the distribution for the types of tropical quadratics (i.e. degree 2) of one variable. We recall that all types with a non-negligible probability correspond to a triangulation of the appropriate marked polytope. In this case, the marked polytope $(\mathcal{A}, \Delta)$ is given by $\mathcal{A} = \{0, 1, 2\}$, which represents the powers of the tropical polynomial, and $\Delta = [0, 2]$, which is the convex hull of $\mathcal{A}$. We note that the two triangulations of this marked polytope are $S_1 = \{(\{0, 2\}, [0, 2])\}$, which corresponds to tropical quadratics of the form ax^2+c, and the subdivision $S_2 = \{(\{0, 1\}, [0, 1]), (\{1, 2\}, [1, 2])\}$, which corresponds to tropical quadratics of the form $ax^2 + bx + c$.

For the first triangulation S_1, we note that the corresponding point in the secondary polytope is $(2, 0, 2)$, since the sum of the sub-polytopes that the first, second, and third points are part of are 2, 0, and 2 respectively. Similarly, the corresponding point in the secondary polytope for the second triangulation S_2 is $(1, 2, 1)$. These two diagrams are shown in figure 7.1.

Since (2,0,2) and (1,2,1) are the only two points corresponding to triangu-

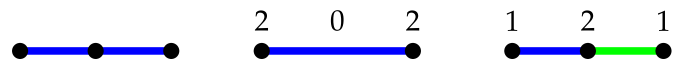


Figure 7.1 Representative diagrams of the original marked polytope (left), and the two triangulations S_1 (middle) and S_2 (right). For the two triangulations, the sum of volumes for each sub-polytope a point is part of is written above the corresponding point.

lations, it follows that the secondary polytope is the convex hull of $(2, 0, 2)$ and $(1, 2, 1)$. We can deduce that the marked polytope can be written in the form $Q(A, b)$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ -1 \\ -4 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$$

We observe that for the point (2,0,2) corresponding to the triangulation S_1, we find that

$$A \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} - b = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, we can deduce that the condition $y \geq 0$ and $y^T(Ax - b) = 0$ is satisfied exactly when y_1, y_3, y_4, y_5, y_6 are all non-negative values and y_2 is zero when $x = [2, 0, 2]^T$. Therefore,

$$\begin{aligned} N_{Q(A,b)}([2,0,2]^T) &= \{A^T y | y \geq 0, y^T(A[2,0,2]^T - b) = 0\} \\ &= \{A^T y | y \in (\mathbb{R}^+)^6, y_2 = 0\} \\ &= \text{pos}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

$$= \text{pos}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\} \oplus \text{span}\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}.$$

Now, we need to calculate the projection of $[1,0,0]^T$ into the orthogonal complement of the span. Observe that

$$P\begin{bmatrix}1\\0\\0\end{bmatrix} = (I - UU^+)\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

In addition, we note that

$$\text{span}(U^\perp) = \text{span}(I - UU^+) = \text{span}\left(\begin{bmatrix}1\\-2\\1\end{bmatrix}\right).$$

Therefore, we determine that

$$\begin{aligned}
&\text{SolidAngle}_{\mathbb{R}^3}\left(N_{Q(A,b)}\left(\begin{bmatrix}2\\0\\2\end{bmatrix}\right)\right)\\
&= \text{SolidAngle}_{\mathbb{R}^3}\left(\text{pos}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\} \oplus \text{span}\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}\right)\\
&= \text{SolidAngle}_{\mathbb{R}^3}\left(\text{pos}\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\} \oplus \text{span}\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}\right)\\
&= \text{SolidAngle}_{U}\left(\text{span}\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}\right) \cdot \text{SolidAngle}_{U^\perp}\left(\text{pos}\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}\right)\\
&= 1 \cdot \text{SolidAngle}_{\text{span}([1,-2,1]^T)}\left(\text{pos}\left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}\right)\\
&= \frac{1}{2}.
\end{aligned}$$

Therefore, the probability of creating a tropical polynomial of the form $ax^2 + c$ is exactly one half (i.e, the x term is trivial). Since there are only two kinds of tropical polynomials, the one where the x term is trivial and the one where it is non-trivial, it follows that the probability of generating a tropical polynomial where the x term is non-trivial is also one half. It should be noted that the same procedure we just used can be used to show that the probability of generating a polynomial of the form ax^2+bx+c is also one half.

7.2 Tropical Cubics of One Variable

We now move onto tropical cubics of one variable. Our first step is to realize that tropical cubics of one variable have four terms total: the constant, the x term, the x^2 term, and the x^3 term. Therefore, types of tropical cubics correspond to subdivisions of the polytope $[0,3]$ marked by the points $\{0,1,2,3\}$. We illustrate all four possible subdivisions in figure 7.2.

Type of Polynomial	Corresponding Subdivision	Vector of Volume Sums
ax^3+bx^2+cx+d		$[1,2,2,1]^T$
ax^3+bx^2+d		$[2,0,3,1]^T$
ax^3+cx+d		$[1,3,0,2]^T$
ax^3+d		$[3,0,0,3]^T$

Figure 7.2 The four different tropical conics of one variable, along with their corresponding subdivision and the associated vector of volume sums.

Now, our next step is to find the convex hull of the vector of volume sums, creating the secondary polytope. Afterwards, we must find the normal cone to each point corresponding to a triangulation. Next, we express it as $\text{pos}\{v_1,\ldots,v_n\}\oplus\mathcal{U}$, where $\mathcal{U}$ is a linear vector space. Finally, we determine $\text{pos}\{Pv_1,\ldots,Pv_n\}$ where P is the projection vector for $\mathbb{R}^n$ into $\mathcal{U}^{\perp}$. We then use the solid angle of $\text{pos}\{Pv_1,\ldots,Pv_n\}$ to find the probability of the type of tropical polynomial associated with the original triangulation.

Instead of performing all of these calculations ourselves, we give the task to a function in R. So for example, if we want to find the cone $\text{pos}\{Pv_1, \ldots, Pv_n\}$ corresponding to the normal cone of $[1,2,2,1]^T$ in the secondary polytope, we use the following code:

```
> vectorOfAreaSums <- rbind( c(1,2,2,1),
+                           c(2,0,3,1),
+                           c(1,3,0,2),
+                           c(3,0,0,3))
> vectorOfInterest <- c(1,2,2,1)
> findNormalCone(vectorOfAreaSums, vectorOfInterest)
     [,1] [,2] [,3] [,4]
[1,]  0.2 -0.1 -0.4  0.3
[2,]  0.3 -0.4 -0.1  0.2
```

The function `vectorOfAreaSums` can be found in the file "findNormalCone.R" within the GitHub repository located here.

This means that the solid angle of the normal cone to $[1,2,2,1]^T$ in the secondary polytope is equal to the secondary polytope of the cone generated by $[0.2,-0.1,-0.4,0.3]^T$ and $[0.3,-0.4,-0.1,0.2]^T$. From here, we can use the dot product of these two vectors to determine the angle separating them, and thus the solid angle of the cone they generate. The results are recorded in figure 7.3.

7.3 Tropical Quartics of One Variable

We can implement the same procedure as with the prior sections to find the vector of volume sums and the solid angle of the normal cone to the vector in the secondary polytope (which is also the probability of the type of tropical polynomial). The eight types of tropical quartics and their corresponding probabilities calculated from this method are listed in figure 7.4

7.4 Tropical Quadratics of Two Variables

Again, we follow the same procedure in order to find the probability of the types of tropical quadratics of two variables. We will list the corresponding subdivision, the vector of area sums, and the probability of the subdivision

Type of Polynomial	Vector of Volume Sums	Solid Angle of Normal Cone
$ax^3 + bx^2 + cx + d$	$\begin{bmatrix}1\\2\\2\\1\end{bmatrix}$	$\frac{1}{2\pi}\arccos\left(\frac{2}{3}\right) \approx 13.39\%$
$ax^3 + bx^2 + d$	$\begin{bmatrix}2\\0\\3\\1\end{bmatrix}$	$\frac{1}{2\pi}\arccos\left(\frac{-1}{3\sqrt{3}}\right) \approx 28.50\%$
$ax^3 + cx + d$	$\begin{bmatrix}1\\3\\0\\2\end{bmatrix}$	$\frac{1}{2\pi}\arccos\left(\frac{-1}{3\sqrt{3}}\right) \approx 28.50\%$
$ax^3 + d$	$\begin{bmatrix}3\\0\\0\\3\end{bmatrix}$	$\frac{1}{2\pi}\arccos\left(\frac{-2}{7}\right) \approx 29.61\%$

Figure 7.3 The types of tropical cubics of one variable, the vector of volume sums of the corresponding to the marked polytope subdivisions, and the solid angle of the cone based off of the volume sum vector in the secondary polytope.

and type of tropical polynomial. The vector of area sums for each subdivision corresponding to the fourteen types of tropical polynomials are listed in figure 7.5, and the resulting probabilities are listed in figure 7.6.

Type of Polynomial	Probability of Type
$ax^4 + bx^3 + cx^2 + dx + e$	$\frac{1}{2\pi}\arctan\left(\frac{\sqrt{35}}{21+8\sqrt{7}}\right) \approx 2.22\%$
$ax^4 + cx^2 + dx + e$	$\frac{1}{2\pi}\arctan\left(\frac{\sqrt{5}}{3+\sqrt{7}}\right) \approx 6.00\%$
$ax^4 + bx^3 + dx + e$	$\frac{-1}{2\pi}\arctan\left(\frac{\sqrt{10}}{2\sqrt{2}-6}\right) \approx 12.48\%$
$ax^4 + bx^3 + cx^2 + e$	$\frac{1}{2\pi}\arctan\left(\frac{\sqrt{5}}{3+\sqrt{7}}\right) \approx 6.00\%$
$ax^4 + dx + e$	$\frac{-1}{2\pi}\arctan\left(\frac{3\sqrt{70}}{-28+7\sqrt{2}+2\sqrt{14}}\right) \approx 18.63\%$
$ax^4 + cx^2 + e$	$\frac{1}{2\pi}\arctan\left(\frac{2\sqrt{35}}{7}\right) \approx 16.50\%$
$ax^4 + bx^3 + e$	$\frac{-1}{2\pi}\arctan\left(\frac{3\sqrt{70}}{-28+7\sqrt{2}+2\sqrt{14}}\right) \approx 18.63\%$
$ax^4 + e$	$\frac{-1}{2\pi}\arccos\left(\frac{2\sqrt{10}}{\sqrt{14}-6}\right) \approx 19.54\%$

Figure 7.4 The types of tropical cubics of one variable and their corresponding likelihoods.

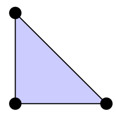
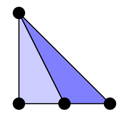
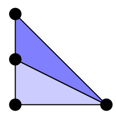
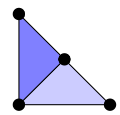
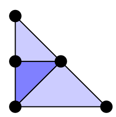
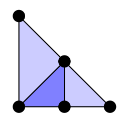
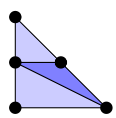
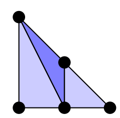
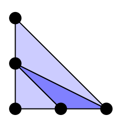
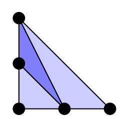
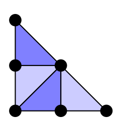
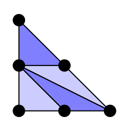
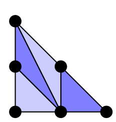
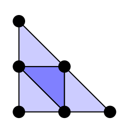
Corresponding Subdivision	Vector of Area Sums	Corresponding Subdivision	Vector of Area Sums
	$\begin{bmatrix} 2 & & \\ 0 & 0 & \\ 2 & 0 & 2 \end{bmatrix}$		$\begin{bmatrix} 2 & & \\ 0 & 0 & \\ 1 & 2 & 1 \end{bmatrix}$
	$\begin{bmatrix} 1 & & \\ 2 & 0 & \\ 1 & 0 & 1 \end{bmatrix}$		$\begin{bmatrix} 1 & & \\ 0 & 2 & \\ 2 & 0 & 1 \end{bmatrix}$
	$\begin{bmatrix} \frac{1}{2} & & \\ 1 & 2 & \\ 1\frac{1}{2} & 0 & 1 \end{bmatrix}$		$\begin{bmatrix} 1 & & \\ 0 & 2 & \\ 1\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$
	$\begin{bmatrix} \frac{1}{2} & & \\ 2 & 1 & \\ 1 & 0 & 1\frac{1}{2} \end{bmatrix}$		$\begin{bmatrix} 1\frac{1}{2} & & \\ 0 & 1 & \\ 1 & 2 & \frac{1}{2} \end{bmatrix}$
	$\begin{bmatrix} 1 & & \\ 2 & 0 & \\ \frac{1}{2} & 1 & 1\frac{1}{2} \end{bmatrix}$		$\begin{bmatrix} 1\frac{1}{2} & & \\ 1 & 0 & \\ \frac{1}{2} & 2 & 1 \end{bmatrix}$
	$\begin{bmatrix} \frac{1}{2} & & \\ 1 & 2 & \\ 1 & 1 & \frac{1}{2} \end{bmatrix}$		$\begin{bmatrix} \frac{1}{2} & & \\ 2 & 1 & \\ \frac{1}{2} & 1 & 1 \end{bmatrix}$
	$\begin{bmatrix} 1 & & \\ 1 & 1 & \\ \frac{1}{2} & 2 & \frac{1}{2} \end{bmatrix}$		$\begin{bmatrix} \frac{1}{2} & & \\ 1\frac{1}{2} & 1\frac{1}{2} & \\ \frac{1}{2} & 1\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Figure 7.5 The corresponding subdivisions and the associated vector of area sums for each of the fourteen distinct types of tropical quadratics of two variables.

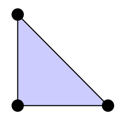
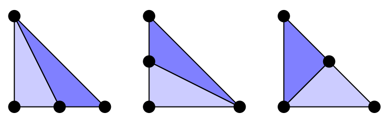
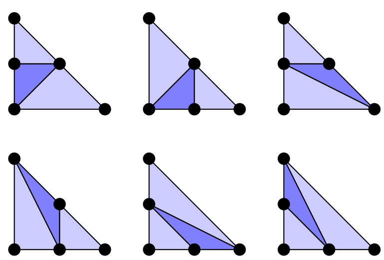
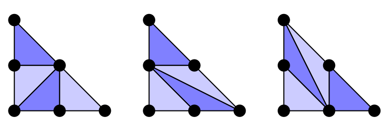
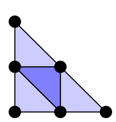
Corresponding Subdivisions	Probability of the Type
	$\frac{1}{2\pi}\arctan\left(\frac{2\sqrt{35}}{7}\right) \approx 16.50\%$
	$\frac{1}{2\pi}\arctan\left(\frac{\sqrt{35}}{7}\right) \approx 11.17\%$
	$\frac{1}{2\pi}\arctan\left(\frac{\sqrt{35}}{\sqrt{35}+6}\right) \approx 5.58\%$
	$\frac{1}{2\pi}\arctan\left(\frac{\sqrt{35}}{3\sqrt{35}+14}\right) \approx 3.34\%$
	$\frac{1}{2\pi}\arctan\left(\frac{\sqrt{15}}{9}\right) \approx 6.48\%$

Figure 7.6 The corresponding subdivisions and the associated vector of area sums for each of the fourteen distinct types of tropical quadratics of two variables.

Chapter 8

Future Work

Although we have laid down the fundamental work for this research, there are a few directions that need to be further explored in order to make this procedure viable for the general case. In addition, there are a few potential areas that this research could focus on even more than it currently has. We will present a higher level understanding here, and provide any additional details whenever possible.

8.1 Finding Triangulations

One of the major steps within the presented research is the task of finding triangulations of the marked polytope corresponding to the form of the polynomial of interest. In particular cases, we have noted that finding these triangulations are fairly simple. With single variable tropical polynomials, all of the triangulations correspond to partitions of an integer length into integer length pieces. With second degree tropical polynomials, the triangulations then corresponded to triangulations of a fairly simple triangle with only six marked points.

The challenge however is in using tropical polynomials of at least two different variables and of much higher degree. When the tropical polynomial has only two variables, it may be difficult to find all of the coherent triangulations, but we have the benefit of being able to draw each one out. However, when we go up to three different variables or more, then we have to find triangulations of figures that are not easily visualized, making the procedure much harder.

Ultimately this boils down into the fundamental tribulation associated with triangulations: given a marked polytope and a series of marked points, what are all of the possible triangulations?

8.2 Solid Angles in Dimensions Four and Higher

As discussed earlier, solid angles are another vital part of the procedure we have presented thus-far. However, we have also noticed that many of the solid angle formulas are also in their infancy.

Convenient solid angle formulas only exist for n-cones within dimensions one, two, and three. In one dimension, cones can only have three different solid angles: 0, $1/2$, or 1. For two dimensions, the solid angle directly corresponds to our notions of angles between vectors. When it comes to three dimensions, although the solid angles for 3-cones is not very apparent, they have been well studied due to the historic popularity of spherical trigonometry.

Past three dimensions, the intuition beyond starts to fall apart, and it appears that most of the research on solid angles has stagnated, mostly due to the lack of applicability of solid angles past three dimensions. However, there is a possibility that a formula, or at least a more numerically accurate procedure, could be established.

Suppose that we create a 3-cone in $\mathbb{R}^3$ where $v_1 = [1, 0, 0]$, $v_2 = [0, 1, 0]$, and v_3 is an arbitrary vector where v_1, v_2, and v_3 are unit vectors that are linearly independent. What are the equivalence classes of v_3 where the solid angles are constant? It turns out that the projection of all of the vectors v_3 that share the same solid angle on the $x - y$ plane is a partial circle that circumscribes the vectors $-v_1$, $-v_2$ and the projection of v_3. For evidence, we see that in figure 8.1, the boundary between the projections of v_3 that lead to a solid angle of more than 0.5 and less than 0.5 is a circle that begins at $-v_1$ and ends at $-v_2$. This implies that the set of all vectors v_3 that share the same solid angle is always the intersection of a sphere and a cylinder (of potentially infinite radius) that is perpendicular to the xy plane.

It turns out that when we allow v_1, v_2, and v_3 to be general linearly independent unit vectors, we still see the same phenomenon. That is, if we fix v_1 and v_2, the set of all v_3 that share the same solid angle is always realized as the intersection of the unit sphere and the surface of a cylinder (of potentially infinite radius) containing $-v_1$, $-v_2$, and v_3, and is perpendicular to plane spanned by v_1 and v_2.

Because of this elegance, perhaps there exists a similar attribute for cones

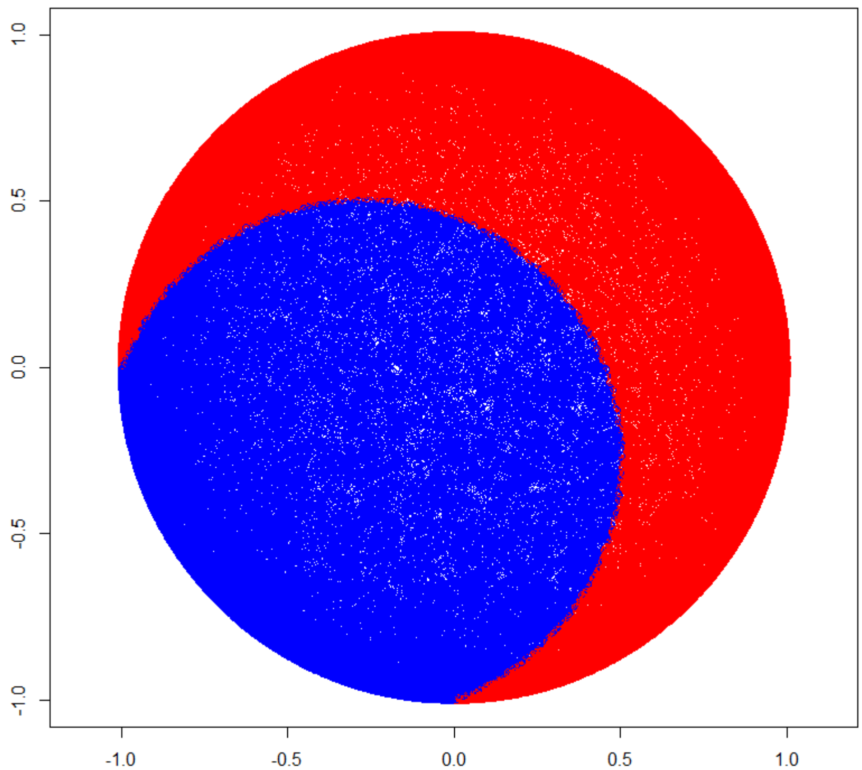


Figure 8.1 Projections of vectors in the xy plane that form a cone of solid angle less than 0.5 (red) or greater than 0.5 (blue) with the vectors $[1, 0, 0]$ and $[0, 1, 0]$.

in higher dimensions. For example, maybe for the 4-cone generated by $v_1 = [1,0,0,0]$, $v_2 = [0,1,0,0]$, $v_3 = [0,0,1,0]$, and arbitrary unfixed unit vector v_4, perhaps the set of all vectors v_4 that share the same solid angle is also a geometrically nice object. If this were to be the case, we could exploit the symmetry in order to find a better way of calculating the solid angle, and a more numerically accurate method of generating the solid angle.

8.3 Viro Patchworking and Classical Polynomials

Tropical polynomials share many connections to their classical counterparts. One example is a method called Viro Patchworking. In this framework, tropical polynomials of two variables (by their corresponding subdivisions) can be transformed into the zero sets of their classical polynomial counterparts. One particular advantage of this is that the information gathered by the method that we have established could potentially transfer over very well into classical polynomials. In otherwords, our results could be used to prove the probabilities that the zero set of some arbitrary polynomial is of a particular shape, like the chance that the zero set of a quadratic polynomial of two variables will create a loop.

There are two main challenges in this approach. First, we must determine how much information is transfered over by Viro Patchworking, and how viable it is do perform Viro Patchworking. The second main challenge is in generalizing Viro Patchworking so that it works in the general case for tropical polynomials of any number of variables, not just those of two variables.

8.4 Generalizations of Tropical Polynomials

Nothing precludes us from introducing alternative kinds of functions that are very similar to the tropical polynomials we have worked with so far. For example, when defining tropical powers, we said that the tropical power x^k for any integer k is equal to k times x. Therefore, we could create "tropical polynomials" with non integer polynomials, like $2 \oplus 3x^{2.1} \oplus -1x^{\pi}$. The methods shown here appear to generalize exceptionally well non-integer power polynomials. Similarly, we can also forcefully exclude certain monomials from our expressions. For example, instead of considering tropical polynomials of the form $ax^4 + bx^3 + cx^2 + dx + e$, we could remove the x^3 term to look at tropical polynomials of the form $ax^4 + cx^2 + dx + e$. Finally, we could also look at the distributions of "rational tropical polynomials" that are the difference between two tropical polynomials.

8.5 Alternative Distributions

Appealing to marked polytopes and secondary polytopes allows us to find cones that perfectly coincide with the set of all tropical polynomials that share the same combinatorial type. From here, we used the fact that the coefficients of the polynomial are from a normal distribution. Part of the reason we use the normal distribution is because the probability density function of the multinormal distribution has some nice properties when we try to integrate over cones. Specifically, we can appeal to the fact that these integrals are equivalent to solid angles.

A possible extension for this project is to look at the distributions of curves when we appeal to other functions that might be easier to integrate over cones. For example, a uniform distribution might prove to be an interesting route of analysis. Since the probability density function of a multivariable uniform distribution is constant, integrating it over the cone reduces to finding the hyper-volume of the intersection between the cone and the bounds of the uniform distribution. If the bounds of the uniform distribution are simple enough, then the probabilities would be straightforward to calculate with the aid of computers, even if the dimension of cone is arbitrarily high.

Bibliography

Davis. 2016. Orie 6300 mathematical programming i: Lecture 6. URL https://people.orie.cornell.edu/dsd95/teaching/orie6300/lec06.pdf.

Gelfand, Kapranov, and Zelevinsky. 1994. *Discriminants, Resultants, and Multidimensional Determinants*. Birkhäuser.

Ribando. 2006. Measuring solid angles beyond dimension three. *Discrete and Computational Geometry* 36:479–478.

Yang. 2013. Secondary fans and tropical severi varieties .