Brownian Motion and Planar Regions: Constructing Boundaries from $h$-functions

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In this thesis, we study the relationship between the geometric shape of a region in the plane, and certain probabilistic information about the behavior of Brownian particles inside the region. The probabilistic information is contained in the function $h(r)$, called the harmonic measure distribution function.

Consider a domain $\Omega$ in the plane, and fix a basepoint $z_0$. Imagine lining the boundary of this domain with fly paper and releasing a million fireflies at the basepoint $z_0$. The fireflies wander around inside this domain randomly until they hit a wall and get stuck in the fly paper. What fraction of these fireflies are stuck within a distance $r$ of their starting point $z_0$? The answer is given by evaluating our $h$-function at this distance; that is, it is given by $h(r)$.

In more technical terms, the $h$-function gives the probability of a Brownian first particle hitting the boundary of the domain $\Omega$ within a radius $r$ of the basepoint $z_0$. This function is dependent on the shape of the domain $\Omega$, the location of the basepoint $z_0$, and the radius $r$.

The big question to consider is: How much information does the $h$-function contain about the shape of the domain’s boundary? It is known that an $h$-function cannot uniquely determine a domain, but is it possible to construct a domain that generates a given $h$-function? This is the question we try to answer.

We begin by giving some examples of domains with their $h$-functions, and then some examples of sequences of converging domains whose corresponding $h$-functions also converge to the $h$-function. In a specific case, we prove that artichoke domains converge to the wedge domain, and their $h$-functions also converge. Using another class of approximating domains, circle domains, we outline a method for constructing bounded domains from possible $h$-functions $f(r)$. We prove some results about these domains, and we finish with a possible for a proof of the convergence of the sequence of domains constructed.