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Reconstruction of Convex Bodies in the Plane from Three Non-Collinear Point Source Directed X-Rays

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by

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Abstract

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When one takes an x-ray, one learns how much material is along the line between the x-ray source and the x-ray sensor. The goal of tomography is to learn what one can about an object, by knowing how much material is on a collection of lines or rays passing through that object. Mathematically, this is a collection of line integrals of a density function of the object. In this paper, we provide and prove reconstructions for a class of convex objects of uniform density using x-rays from three point sources.

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Chapter 1

INTRODUCTION

Tomography is the study of what is determined about a function by some set of line integrals of that function. In this paper, the functions we consider are characteristic functions of bounded, convex subsets of $\mathcal{R}^2$. The line integrals we consider are integrals along rays emanating from a small number of points in the plane. Specifically, we will seek to reconstruct a convex body in the plane with x-rays from three non-collinear point sources. We provide methods for reconstructing pieces of the convex body, and prove that, for some cases, repeatedly iterating the reconstruction methods creates a sequence of sets converging to the unknown convex set.

1.1 Motivation for tomography

When one takes an x-ray of an object, there is an x-ray source and an x-ray sensor. The ratio of the x-ray intensity at the source to the intensity measured at the sensor allows one to compute the line integral of a function describing the density of the object along a line from the source to the sensor. In order to build a picture of an object from x-rays, one must try to reconstruct a function from line integrals of that function.

It has been shown in [3] that looking at all possible x-rays in n-dimensional Euclidean space will allow one to reconstruct a bounded function up to a set of measure zero. That is, one must know the line integral of the function along any line in n-dimensional Euclidean space. The same solution to the x-ray problem does not work

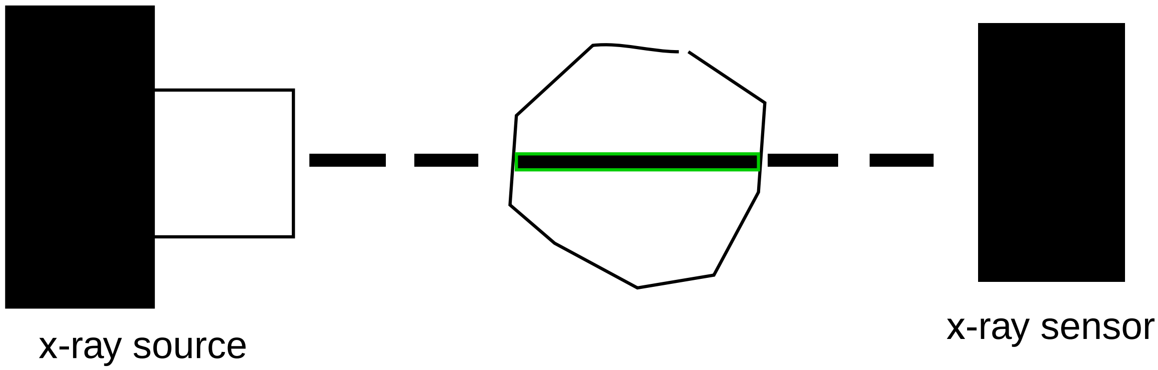

Figure 1.1: Illustration of an x-ray.

if we change the assumptions in the problem. First, it may be practical to make more assumptions about the object than just boundedness. It may be reasonable to assume, for example, that there is a maximum density within the object or that the density function of an object is zero outside a given region, not simply outside some region.

We may also restrict the known x-rays in some way, such as assuming we only know x-rays in certain directions (parallel beam tomography) or coming from certain points (fan-beam tomography). In this paper the functions are considered are characteristic functions of convex sets in the plane, emulating convex objects of a known uniform density. X-rays from three non-collinear point sources are considered.

Several previous results in reconstructing objects from a finite number of point source x-rays are presented in Chapter 2. The remainder of this thesis presents and discusses new reconstruction methods. The methods used in these reconstructions, explained in Chapter 3, will always partially reconstruct a convex body; that is we will determine some set of points must be on the interior of the unknown convex body and some other set of points must be on the exterior of the unknown body. We give complete reconstructions for several configurations of the x-ray sources and the unknown convex body being x-rayed (Chapters 4 and 5) . The first configuration is one where at least one point in the unknown object along each nonzero x-ray can be determined. The second class of shapes is polygons. When these methods succeed or fail at a complete reconstruction of an arbitrary convex body is an open question. The remainder of Chapter 1 will be spent introducing terminology.

1.2 Definitions

Definition 1.2.1 *Let $f : \mathcal{R}^2 \to \mathcal{R}^1$ be integrable along all lines, and let $\mathbf{p} \in \mathcal{R}^2$. Then the* directed x-ray *of f at $\mathbf{p}$ in direction $\mathbf{u}$, $D_{\mathbf{p}}[f](\mathbf{u})$, is*

$$D_{\mathbf{p}}[f](\mathbf{u}) = \int_0^\infty f(\mathbf{p} + t\mathbf{u})dt$$

where $\mathbf{u}$ is a unit vector.

We may also use the notation $D_p[f](\theta)$ where (θ) is an angle that indicates the direction of the unit vector with a fixed axis.

Definition 1.2.2 *A convex body K is a convex, compact subset of the plane. We will use the same symbol, K, to refer to the characteristic function of that body. That is,*

$$K(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in K \\ 0 & \text{if } \mathbf{x} \notin K \end{cases} \tag{1.1}$$

So if we refer to the x-ray of a convex body from a point $\mathbf{p}$, we refer to

$$D_{\mathbf{p}}[K](\mathbf{u}) = \int_0^\infty K(\mathbf{p} + t\mathbf{u})dt \tag{1.2}$$

Notice that $D_{\mathbf{p}}[K](\mathbf{u})$ is the length of the intersection of the ray emanating from $\mathbf{p}$ in the direction $\mathbf{u}$ with the body K. For the remainder of this paper we will pass freely between these two equivalent characterizations of the directed x-ray of a convex body.

The symbols $\triangle$ and the similar Δ will be used frequently in this paper. One will be used to indicate a triangle, and the other as a symmetric difference operator, defined below.

Definition 1.2.3 *The triangle with vertices* $\mathbf{p_1}$, $\mathbf{p_2}$, *and* $\mathbf{p_3}$ *will be denoted by* $\triangle\mathbf{p_1p_2p_3}$.

Definition 1.2.4 *Let* K *and* K' *be sets.* $K\Delta K'$ *denotes the symmetric difference of* K *and* K', *defined by*

$$K\Delta K' = (K \cup K') \setminus (K \cap K').$$

When K *and* K' *are thought of as characteristic functions of sets,* Δ *is equivalent to function addition modulo 2.*

Two more important concepts are convex hull and a star-shaped set, a slight relaxation of convex set.

Definition 1.2.5 *If* E *is a collection of points in the plane, the* convex hull *of* E, *denoted* $CH(E)$, *is the smallest convex set containing* E.

Definition 1.2.6 *A set* K *is* star-shaped *from a point* $\mathbf{p}$ *if every ray emanating from* $\mathbf{p}$ *that intersects* K *intersects* K *in a connected set. We call a set star-shaped if it is star-shaped from all x-ray sources being considered.*

Note that a set that is star-shaped from every point if and only if it is convex; thus anything shown true for star-shaped bodies holds true for convex bodies as well.

Chapter 2

PREVIOUS RESULTS

In this chapter we give several previous results that are important to the development of a reconstruction of a convex body from three non-collinear points. One shows that a convex body in the plane is uniquely determined by directed x-rays from three non-collinear sources [3]. The unique determination is not, however, reconstructive. A second, due to Falconer [1], is a reconstruction of a convex body in the plane from two sources, provided the convex body intersects the line containing both point sources. Finally, we prove a result about uniqueness of convex polygons from a single point source x-ray, first published in [4].

2.1 Uniqueness of a convex body from three non-collinear directed x-rays.

Directed x-rays from three non-collinear points uniquely determine a convex body in the plane. The method of proof depends on where the body lies in relation to the three source points. Here we present the case when the body lies completely inside the triangle formed by the three point x-ray sources, since this is the case for which certain objects will be reconstructed. The other cases, when the body intersects the triangle or is outside the triangle, may be found in [3].

Theorem 2.1.1 *Given directed x-rays from three non-collinear points* $\mathbf{p_1}$*,* $\mathbf{p_2}$*, and* $\mathbf{p_3}$ *in the plane, a convex body* K *in the interior of the triangle with vertices at* $\mathbf{p_1}$*,* $\mathbf{p_2}$*, and* $\mathbf{p_3}$ *is uniquely determined by these three directed x-rays.*

Proof. For purposes of contradiction, assume there are two different convex bodies K and K' such that $D_{\mathbf{p_i}}[K] = D_{\mathbf{p_i}}[K']$, for $i = 1, 2, 3$. Let C be a connected component of the interior of $K \Delta K'$ of maximal area. Consider the line l containing the two points of $\partial K \cap \partial K' \cap \partial C$. Some $\mathbf{p_i}$ must be on the same side of this line l as C. Without loss of generality, assume this point is $\mathbf{p_1}$.

We are given that $D_{\mathbf{p_1}}[K] = D_{\mathbf{p_1}}[K']$. Thus there must be a connected component C' of $K \Delta K'$ on the other side of l with the same directed x-ray from $\mathbf{p_1}$ as C. Call this component C'. The area of C' must be greater than that of C, since it is farther from $\mathbf{p_1}$. This contradicts the assumption that C was maximal. $\square$

The same method can be used to show that any star-shaped body on the interior of $\triangle \boldsymbol{p_1 p_2 p_3}$ is uniquely determined from the x-rays from $\boldsymbol{p_1}$, $\boldsymbol{p_2}$, $\boldsymbol{p_3}$.

2.2 Two related reconstructions

Theorem 2.2.1 *Let K be a convex body in the plane, let $\mathbf{x}$ and $\mathbf{y}$ be points in the plane and let l be the line through $\mathbf{x}$ and $\mathbf{y}$. If l intersects K, we may reconstruct K from directed x-rays at $\mathbf{x}$ and $\mathbf{y}$.*

The proof is given in [1]. The most important consequence of this theorem is that the problem of reconstructing a convex body from three point source x-rays has been solved in the case where the convex body intersects a line containing two of the point sources.

We now show that a wedge, a region of a plane known to be bounded by two lines and two rays which intersect a single x-ray source, $\mathbf{p}$, having no parallel sides is uniquely determined by an x-ray from $\mathbf{p}$.

Theorem 2.2.2 *Let K be a compact, convex subset of the plane known to be bounded by two rays emanating from some point, $\mathbf{p}$, and two non-parallel lines. Then K is uniquely determined by $D_{\mathbf{p}}[K]$.*

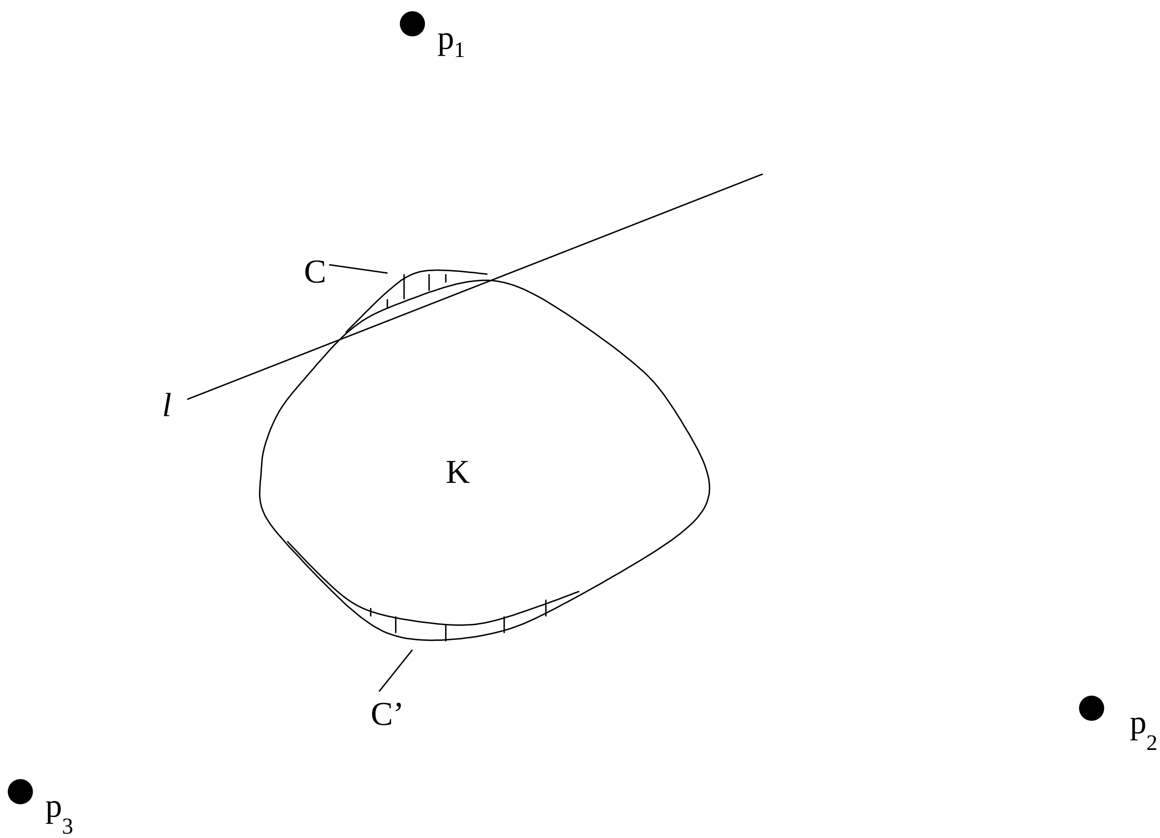

Figure 2.1: Uniqueness of a convex body form three non-collinear x-rays.

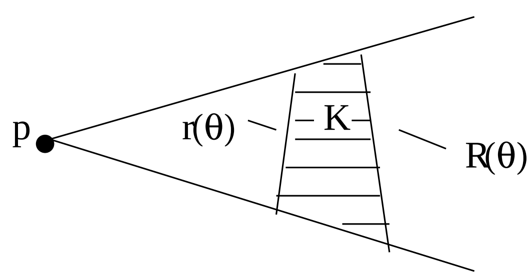

Figure 2.2: Parameterizing a wedge.

Proof. First we assume that the x-ray source is at the origin, and $D_{\mathbf{p}}[K](\frac{\pi}{2}) \neq 0$, since we can impose a change of coordinates to make this true. Then we parameterize the near edge of K by $r(\theta)$ and the far edge of K by $R(\theta)$.

Let the farthest edge have slope B and y-intercept A in rectangular coordinates, and the bottom edge have slope b and y-intercept a in rectangular coordinates, and parameterize $R(\theta)$ and $r(\theta)$ as follows:

$$\begin{aligned} R(\theta) &= \frac{A}{\sin(\theta) + B\cos(\theta)} \\ r(\theta) &= \frac{a}{\sin(\theta) + b\cos(\theta)}. \end{aligned}$$

Now we may take three polar derivatives and evaluate at $(\frac{\pi}{2})$, obtaining 4 equations and 4 unknowns. The equations yield:

$$\begin{aligned} D_{\mathbf{p}}[K](\pi/2) &= A - a \\ D_{\mathbf{p}}[K]'(\pi/2) &= AB - ab \\ D_{\mathbf{p}}[K]''(\pi/2) &= 2(AB^2 - ab^2) + A - a \\ D_{\mathbf{p}}[K]^{(3)}(\pi/2) &= 6(AB^3 - ab^3) + AB - ab. \end{aligned}$$

Taking linear combinations to cancel all but the highest order term in each equation yields constants C_0, C_1, C_2, C_3 such that

$$\begin{aligned} C_0 &= A - a \\ C_1 &= AB - ab \\ C_2 &= AB^2 - ab^2 \\ C_3 &= AB^3 - ab^3. \end{aligned}$$

Algebraic manipulation of these equations yields:

$$C_2 = C_1(B+b) - C_0Bb$$

$$C_0C_3 - C_1C_2 = (C_0C_2 - C_1^2)(B+b).$$

These yield two solutions for (b, B) each of which will determine A and a when one looks at the equations for C_0 and C_1. Thus there will be two solutions to the system. Assume that $(A = A_0, a = a_0, B = B_0, b = b_0)$ solves the system. We note that $(A = -a_0, a = -A_0, B = b_0, b = B_0)$ also solves the system presented. Only one of these solutions will have $A > 0$, which is necessary by our assumptions, so the wedge is uniquely determined by the x-ray data from $\mathbf{p}$. $\square$

Chapter 3

METHODS FOR RECONSTRUCTING PIECES OF A CONVEX BODY FROM THE X-RAY DATA.

In this chapter, we present several original methods for piecing together K from x-ray data.

3.1 The general method of reconstruction

Definition 3.1.1 *An* inner guess *of K is any set E such that $E \subset K$. We say that a set E is a better inner guess than a set F if $E \neq F$ and $F \subset E \subset K$.*

Definition 3.1.2 *An* outer guess *of K is any set E such that $E \supset K$. We say that a set E is a better outer guess than a set F if $E \neq F$ and $F \supset E \supset K$.*

That is, we desire to make outer guesses smaller and inner guesses larger.

The methods that follow show ways to create an inner guess and outer guess for K. After initial guesses are completed, different methods can be used to get better inner guesses and better outer guesses. The next chapter will show how, in certain cases, these methods can be used to create a sequence of sets that converges to K.

Method 3.1.3 (Initial outer guess) *Let*

$$\Psi = \{\boldsymbol{x} \mid \boldsymbol{x} \text{ is in some ray } \psi \text{ based at } \boldsymbol{p_1}, \boldsymbol{p_2}, \text{ or } \boldsymbol{p_3} \text{ such that } \psi \cap K = \emptyset\}.$$

Let $A = \Psi^c$, the complement of Ψ. Then A is an outer guess for K.

That is, start with the plane and discard all points that are along a ray where the x-ray is 0. What is left of the plane must contain the undetermined convex set.

There is also a method for establishing an initial inner guess, but it is more complicated.

Method 3.1.4 (Initial inner guess) *Let A be as in 3.1.3. A will be a hexagon with sides along the six rays where $D_{\mathbf{p_i}}[K]$ goes to zero. Some sides may be degenerate. Label the vertices of the hexagon $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{f}$ moving clockwise. Let T_1 be $\triangle\boldsymbol{ace}$. Let T_2 be $\triangle\boldsymbol{bdf}$. Then $T_1 \cap T_2$ is an inner guess of K.*

Proof. At least one point on each side of A will also be in ∂K, the boundary of K. Label these points $\boldsymbol{p_{ab}}$, $\boldsymbol{p_{bc}}$, $\boldsymbol{p_{cd}}$, $\boldsymbol{p_{de}}$, $\boldsymbol{p_{ef}}$, and $\boldsymbol{p_{fa}}$, where the subscripts indicate which side of the hexagon the point lies on. Let $H = CH(\boldsymbol{p_{ab}}, \boldsymbol{p_{bc}}, \boldsymbol{p_{cd}}, \boldsymbol{p_{de}}, \boldsymbol{p_{ef}}, \boldsymbol{p_{fa}})$. Because K is convex, $K^c \subset H^c$. Thus, $A \cap K^c \subset A \cap H^c$. All points in H^c must be contained in some triangle formed by consecutive corners of A. We conclude H, and therefore K, will necessarily have $T_1 \cap T_2$ on its interior. Thus $T_1 \cap T_2$ is an inner guess. $\square$

3.2 Specific methods of reconstruction

In this section specific methods of creating better inner and outer guesses are discussed. The next chapter will deal with analyzing how successful these methods are at reconstructing K.

3.2.1 The bar-in-trough and kid-in-blanket methods

This is where we use the the fact that certain points must be in K in order for the x-ray data to fit inside the outer guess. We can also improve the outer guess by pinning the x-ray down to the inner guess and seeing if there is any part of the current outer guess that cannot be reached by the given the current inner guess.

To understand the bar-in-trough method picture the following scenario. There is a bar of length l and a trough of length d. We place the bar into the trough. Now if

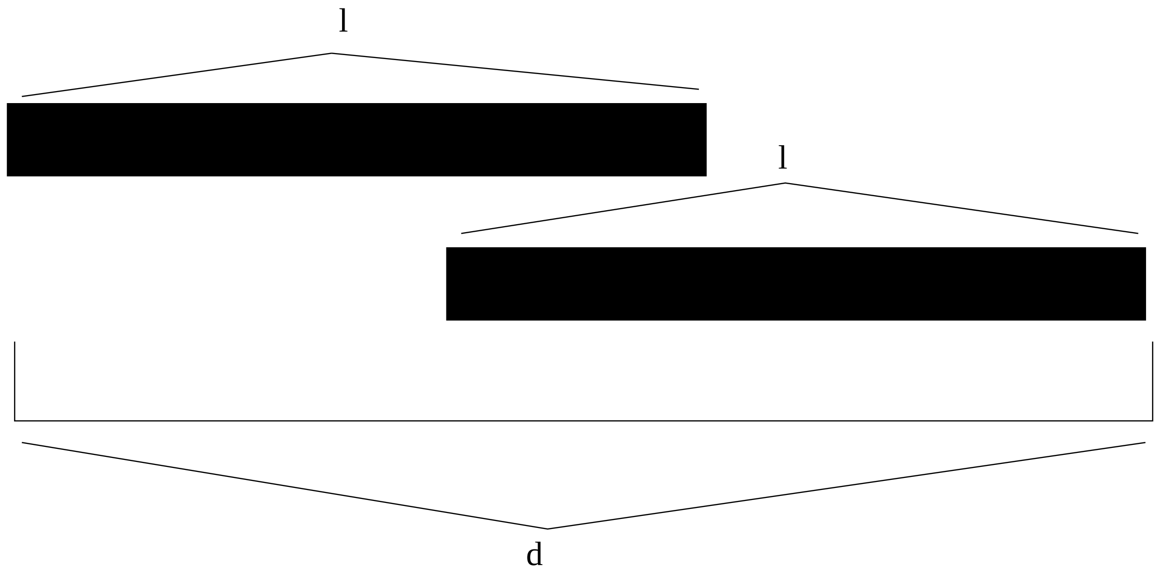

Figure 3.1: The bar-in-trough method.

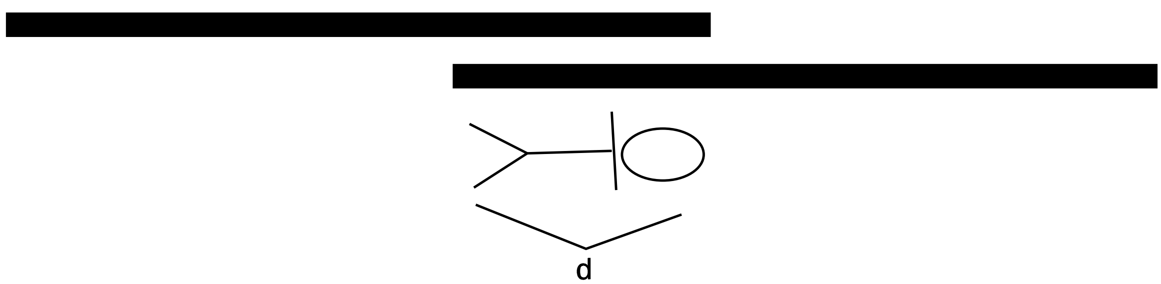

Figure 3.2: The kid-in-blanket method.

$2l > d$, we can conclude that the a middle section of the trough of length $2l - d$ must contain the bar, regardless of how the bar was placed in the trough.

We use similar reasoning to reconstruct pieces of the inside of K by thinking of the x-ray of K along a given ray as a bar, and the section of that same ray intersecting the best outer guess of K as a trough.

Method 3.2.1 (The bar-in-trough method) *Let O be an outer guess of K. Let $\boldsymbol{p}$ be an x-ray source. Assume for some angle θ, $2D_{\boldsymbol{p}}[K](\theta) > D_{\boldsymbol{p}}[O](\theta)$. Let L be a line segment of length $2D_{\boldsymbol{p}}[K](\theta) - D_{\boldsymbol{p}}[O](\theta)$ centered at the center of the intersection* *of the ray emanating from* $\boldsymbol{p}$ *at angle* θ *with* O*.* L *must be in the interior of* K*.*

Proof. We apply the bar-in-trough argument to K and O along the ray emanating from $\boldsymbol{p}$ at angle θ. $\square$

Similar reasoning is applied in the kid-in-blanket method. Assume a child of length d desires to hide under a blanket of length l. The blanket must be contained in a segment of length $2l - d$ centered on the kid if it is going to cover the child. We now think of the blanket as x-ray data and the child as an inner guess. Using the kid-in-blanket method will provide a bound on the outer guess.

Method 3.2.2 (The kid-in-blanket method) *Let* E *be an inner guess of* K*. Let* $\boldsymbol{p}$ *be an x-ray source. We conclude that if for some angle* θ*,* $D_{\boldsymbol{p}}[E](\theta) \neq 0$*, then a line segment of length* $2D_{\boldsymbol{p}}[K](\theta) - D_{\boldsymbol{p}}[I](\theta)$ *centered at the center of the intersection of the ray emanating from* $\boldsymbol{p}$ *at angle* θ *with* E *completely contains* K *along the ray emanating from* $\boldsymbol{p}$ *at angle* θ*.*

Proof. Apply the kid-in-blanket method argument along the ray emanating from $\boldsymbol{p}$ at angle θ. $\square$

3.2.2 The convexity-line method

Knowing that a boundary point must lie on a certain line and knowing an inner guess forces us to conclude, by convexity arguments, that certain other points must also be in the interior of K.

Method 3.2.3 (The convexity-line method) *Let* E *be an inner guess of* K*. Let* F *be a subset of the plane known to contain a point of* K*. Then* $\bigcap_{x \in F} CH(E \cup x)$ *is an inner guess of* K*.*

Proof. Since we know that for some $x' \in F$, $CH(E \cup x') \subset K$, and $\bigcap_{x \in F} CH(E \cup x) \subset CH(E \cup x')$, we can conclude that $\bigcap_{x \in F} CH(E \cup x) \subset K$. $\square$

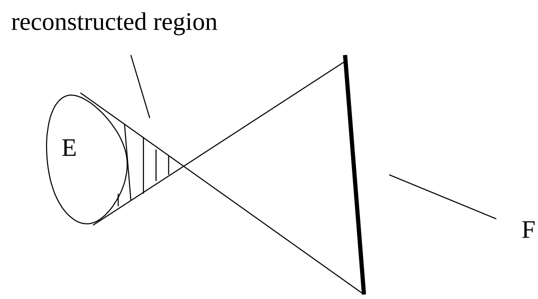

Figure 3.3: The convexity-line method.

These methods allow us to attain better inner and outer guesses. In the next chapter we find a set of convex bodies that can be completely reconstructed. That is, repeatedly applying these methods will create a sequence of inner guesses that approach K.

Chapter 4

A RECONSTRUCTION IN A SPECIAL CASE

The previously discussed methods can assist in partially reconstructing a convex body in the plane. The question arises, "When do these methods completely reconstruct the convex body?" Ideally the answer would be always, but we first show that reconstruction is possible in a special case. To do this we define a metric on the set of convex bodies in the plane. We then show that we can construct a sequence of guesses that converge to the unknown convex body being x-rayed.

4.1 The x-ray metric

Here we introduce a metric that measures the largest difference in x-rays between two sets. We also show that sets that are close together in the x-ray metric have a small symmetric difference.

Definition 4.1.1 *Given three x-ray sources* $\mathbf{p_1}$*,* $\mathbf{p_2}$*, and* $\mathbf{p_3}$*, and two convex bodies* K_1 *and* K_2 *in the interior of* $\triangle\mathbf{p_1p_2p_3}$*, the distance between the two bodies in the* x-ray metric *is given by*

$$d(K_1, K_2) = \left|\sup_{i,\theta} D_{\mathbf{p_i}}[K_1](\theta) - D_{\mathbf{p_i}}[K_2](\theta)\right|$$

To see that d is a metric, first notice that since x-rays from the three points uniquely determine an object, $d(K_1, K_2) = 0$ if and only if $D_{\mathbf{p_i}}[K_1] = D_{\mathbf{p_i}}[K_2]$ if and only if $K_1 = K_2$. Now we demonstrate the triangle inequality. Without loss of generality, assume $d(K_1, K_2) = \sup_\theta |D_{\mathbf{p_1}}[K_1](\theta) - D_{\mathbf{p_1}}[K_2](\theta)|$. Then

$$
\begin{aligned}
d(K_1, K_3) + & d(K_2, K_3) \\
& \geq \sup_\theta |D_{\mathbf{p_1}}[K_1](\theta) - D_{\mathbf{p_1}}[K_3](\theta)| + \sup_\theta |D_{\mathbf{p_1}}[K_2](\theta) - D_{\mathbf{p_1}}[K_3](\theta)| \\
& \geq \sup_\theta |D_{\mathbf{p_1}}[K_1](\theta) - D_{\mathbf{p_1}}[K_2](\theta)| \\
& = d(K_1, K_2).
\end{aligned}
$$

The last equality is true by assumption. Therefore, the triangle inequality is satisfied, establishing d as a metric.

Theorem 4.1.2 *Let K_i be a sequence of either increasing or decreasing convex bodies in the triangle $\triangle \mathbf{p_1 p_2 p_3}$. If $d(K_i, K) \to 0$ then $\lambda_2(K \triangle K_i) \to 0$, where λ_2 is the two dimensional Lebesgue measure.*

Proof. Let R be the maximum distance between any two points in $\triangle \mathbf{p_1 p_2 p_3}$. Then $\lambda_2(K \triangle K_i) \leq 2\pi R d(K_i, K)$. $\square$

4.2 An example of reconstruction

Now we can turn our attention to actual examples of reconstruction. Reconstruction can be accomplished by creating a sequence of outer or inner guesses that converge to the actual convex body. We use the generic term *progress* to indicate an outer guess smaller than the smallest outer guess attained, or a inner guess larger than the largest inner guess attained.

Definition 4.2.1 *A method* allows progress *when applying the method will yield a larger inner guess or a smaller outer guess than the current inner guess or outer guess, respectively.*

Theorem 4.2.2 *After an initial use of the convexity-line method, the bar-in-trough or kid-in-blanket method will allow progress.*

Proof. Almost every ray emanating from one of the x-ray sources, $\mathbf{p_1}$, $\mathbf{p_2}$, and $\mathbf{p_3}$ intersecting the inner guess created by the convexity-line method intersects the boundary of the inner guess and boundary of the outer guess at two lines. That is most rays intersecting the inner guess intersect no corners of the inner guess or outer guess. By Theorem 2.2.2, no two non-parallel wedge polygons may have the same x-ray from a single point. By the way the convexity-line method works, the rays intersecting the inner guess will not intersect four parallel boundary segments of the inner and outer guesses. (See Figure 4.2.) Thus in some neighborhood in each of these x-rays, the bar-in-trough method or kid-in-blanket method will allow progress. □

This is the most progress that has been made to date for reconstructing a generic convex body. The initial inner guess and outer guess can always be applied, and then a little bit more progress may be made by applying Theorem 4.2.2. For certain shapes of convex bodies, complete reconstruction from the given techniques will be shown. First we define the shapes.

Definition 4.2.3 *Let α be an angle, and $\mathbf{p}$ a point such that α is on the boundary of the support of $D_{\mathbf{p}}[K](\theta)$. Then we call a ray emanating from $\mathbf{p}$ at angle α a* boundary ray *of $\boldsymbol{p}$.*

We are now going to prove that if there are two points in the plane where three boundary rays, one from each x-ray source, intersect, the bar and trough method will reconstruct K. First we show that progress can be made at every stage. We then show that the inner guesses converge to K.

Theorem 4.2.4 *If K and $\mathbf{p_1}$, $\mathbf{p_2}$, and $\mathbf{p_3}$ are such that there are two points where three boundary rays intersect, then progress may be made for any inner guess, E and outer guess O, unless $E = O$, that is unless K has already been reconstructed.*

Proof. Call the two points where the three boundary rays intersect q_1 and q_2. The initial inner guess will be the line segment connecting q_1 and q_2. This means that

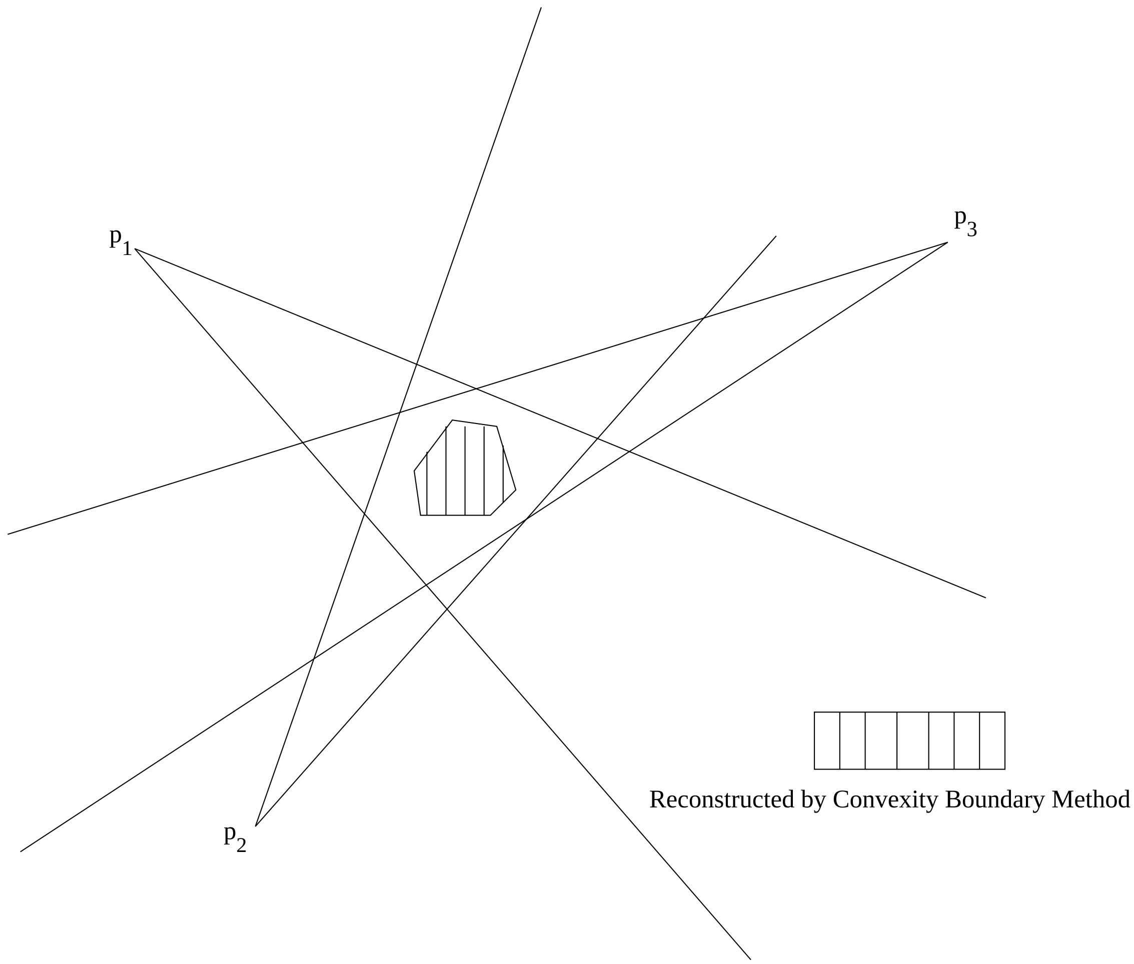

Figure 4.1: After the initial application of the convexity-line method.

every ray along which the x-ray is not zero will intersect the inner guess. Now let F_1^{max} be the region of maximal area containing the current inner guess such that $D_{p_1}[K] = D_{p_1}[F_1^{max}]$. Similarly let F_1^{min} be the region of minimal area containing the current inner guess such that $D_{p_1}[K] = D_{p_1}[F_1^{min}]$.

For purposes of contradiction, assume progress is not possible. Then for $i = 1, 2, 3$, $D_{p_i}[F_1^{max}] = D_{p_1}[F_i^{min}]$, since otherwise the bar and trough method would allow progress. But by the unique determination of convex bodies from three point x-rays, we would have to conclude $F_1^{max} = F_1^{min}$. Thus K has been reconstructed. □

A series of inner guesses, E_j, can now be constructed by repeatedly applying Theorem 4.2.4. We may assume that the corresponding outer guesses O_j created by applying the bar-in-trough method from point 1 are star-shaped from all x-ray sources. If O_j is not, consider a ray that intersects it in more than one connected component. Only one of these connected components contains a piece of the inner guess, so the other pieces may be discarded, allowing us to force the outer guess to be star-shaped, and improving the outer guess.

The series of inner guesses converges, since it is increasing and bounded above by K. We now show that this series of guesses converges to K.

Theorem 4.2.5 *Let E_j be a sequence of inner guesses as in Theorem 4.2.4. Then $\lim_{j\to\infty} E_j = K$, with convergence in the x-ray metric.*

Proof. Let $F_{1_j}^{max}$ and $F_{1_j}^{min}$ be as above, with the sub index indicating which inner guess, E_j generated $F_{1_j}^{max}$ and $F_{1_j}^{min}$. Let $\boldsymbol{p_i}, i = 1, 2, 3$ be the three point sources, then, since this is a limit of inner guesses attained by the bar and trough method,

$$\lim_{j\to\infty} D_{\boldsymbol{p_i}}[F_{1_j}^{max} - F_{1_j}^{min}](\theta) = 0.$$

However all the functions being considered are dominated by $D_{\boldsymbol{p_j}}[\mathrm{CH}(\triangle p_1 p_2 p_3)]$, so we may apply the dominated convergence theorem to exchange limits and integrals (x-rays). Thus

$$D_{p_i} \lim_{j\to\infty} [F_{1_j}^{max} - F_{1_j}^{min}](\theta) = 0$$

Therefore,

$$D_{p_i} \lim_{j\to\infty} F_{1_j}^{max} = D_{p_i} \lim_{j\to\infty} F_{1_j}^{min}$$

But since three point x-rays will uniquely determine any body, we must conclude:

$$\lim_{j\to\infty} F_{1_j}^{max} = \lim_{i\to\infty} F_{1_j}^{min}.$$

But this will only happen when $\lim_{j\to\infty} E_j = K$. $\square$

Chapter 5

A FEW NOTES ABOUT NON-SMOOTH BODIES

In this chapter we explore ways in which non-smooth boundary points make locating boundary points easier. The idea is that any non-smooth point will cause a discontinuity in the first derivative of the x-ray. By seeing where the rays where the first derivative of the x-ray is discontinuous intersect, we can locate the non-smooth boundary points.

5.1 Locating the rays containing non-smooth boundary points

First we show that convex bodies with non-smooth boundary points have a discontinuity in the first derivative of their x-rays.

Theorem 5.1.1 *Let K be a convex body with a non-smooth point $\mathbf{q}$ in its boundary. Let $\mathbf{p}$ be an x-ray source positioned such that the line containing $\mathbf{p}$ and $\mathbf{q}$ intersects the interior of K. Then the derivative of the x-ray with respect to θ, $D_{\mathbf{p}}[K]'(\theta)$, will have a discontinuity at θ_0 where the ray emanating from $\mathbf{p}$ in the direction θ_0 contains $\mathbf{q}$.*

Proof. Consider polar coordinates centered at $\mathbf{p}$. Let $r(\theta)$ parameterize the near boundary of K and let $R(\theta)$ parameterize the far boundary of K as in Figure 5.1. Then $D_{\mathbf{p}}[K](\theta) = R(\theta) - r(\theta)$. therefore $D_{\mathbf{p}}[K]'(\theta) = R'(\theta) - r'(\theta)$. But by assumption, either R' or r' has a discontinuity, and by the convexity of K they cannot have discontinuities that cancel to make $R'(\theta) - r'(\theta)$ continuous. $\square$

Not every discontinuity will be counted in this way, as along the boundary rays, $D_{\mathbf{p}}[K]'$ will always have a discontinuity, regardless of whether there is a non-smooth

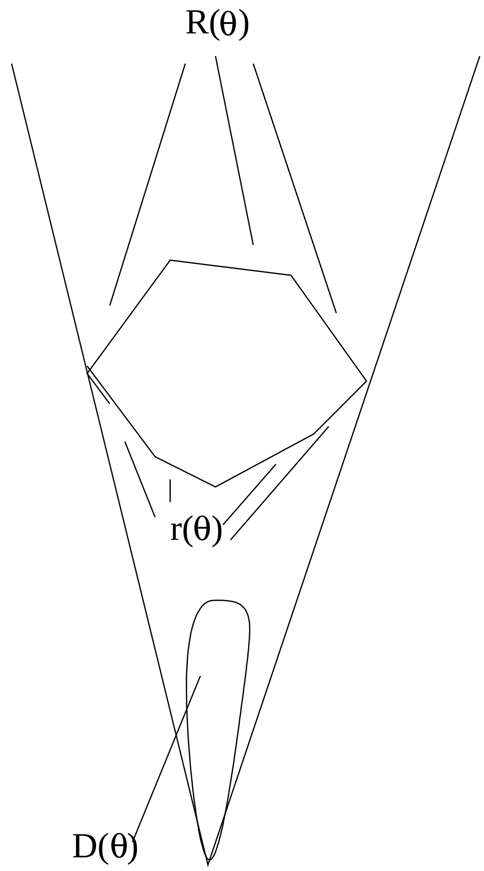

Figure 5.1: Parameterization of the near and far edges.

point along the ray.

5.2 Counting the discontinuities

When one looks at x-ray data, there are two possible sources for non-smooth points. The first is discontinuities in the derivative of the x-ray and the second is where the x-ray goes to zero. For each discontinuity in the derivative there could be one or two non-smooth points. At each boundary ray there could be zero, one, or two non-smooth points. Thus we attain the following bound:

$$\begin{aligned}&\text{number of discontinuities in the x-ray's derivative}\\&\qquad\le \text{number of non-smooth points}\\&\qquad\le (2 * \text{ number of discontinuities in the x-ray's derivative }) + 4.\end{aligned}$$

5.3 When can the non-smooth boundary points be located?

Only the rays containing non-smooth points can be located if x-ray data from one point is used. However, if data from more than one point is used, all non-smooth boundary points must lie on the intersections of rays containing discontinuities from all the sources. See Figure 5.2.

Once data from two x-ray sources is taken into account, there will be a finite number of points on the intersection of rays containing non-smooth boundary points. Call these points *candidates.* If the third x-ray source is not on a line containing two candidates, then the data from the third x-ray will determine which points are non-smooth boundary points; these points will be all the points on rays from the third x-ray source potentially containing non-smooth points that are also candidate points from the first two x-ray sources.

Method 5.3.1 (Finding Corners) *We observe that for any non-smooth region,*

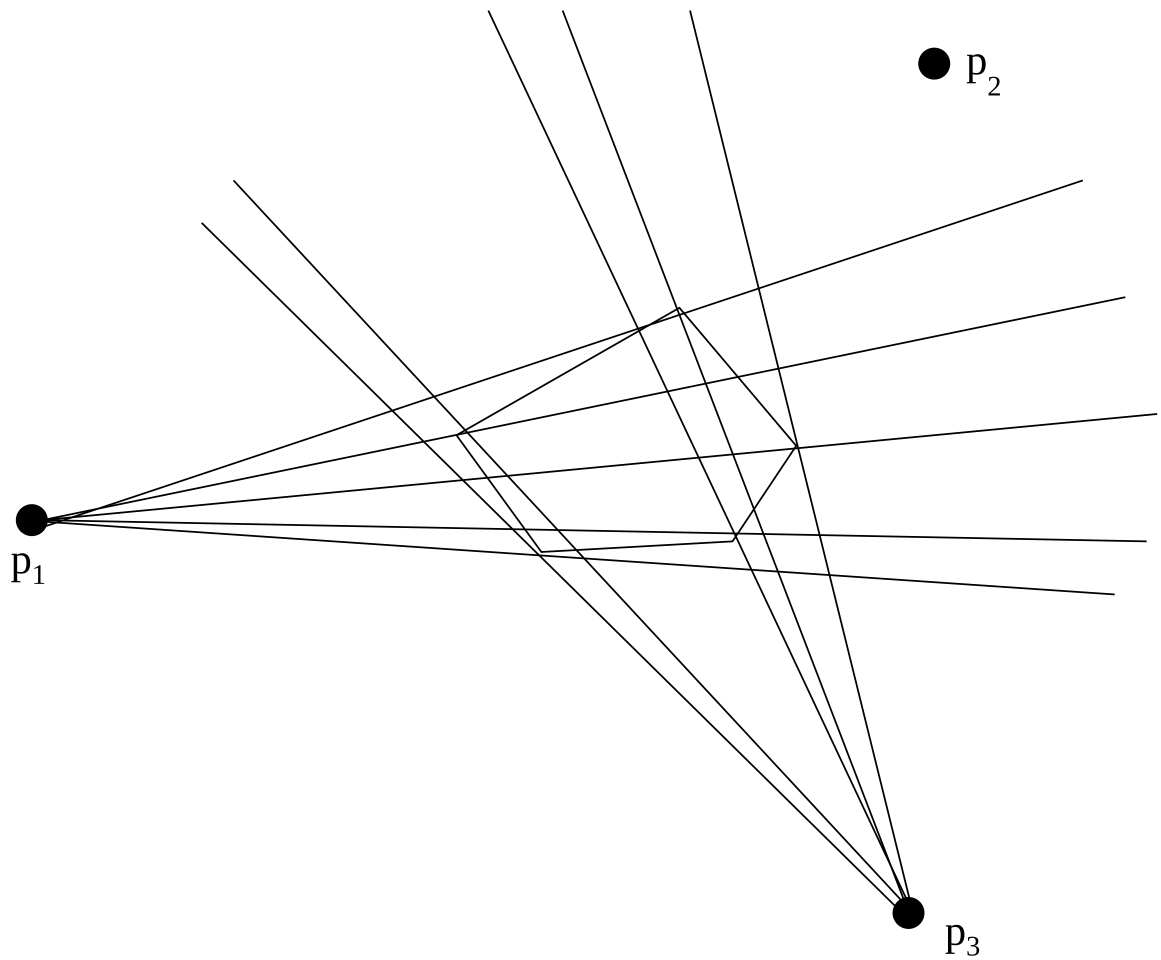

Figure 5.2: Intersecting rays from two points yield a finite collection of candidate points.

taking the convex hull of the region's non-smooth boundary points will yield an inner guess.

We note that for most polygons, those where all the non-smooth boundary points can be located, the finding corners method will yield the polygon itself.

Chapter 6

CONCLUSION

While many reconstruction methods have been posed, and proofs of complete reconstruction have been demonstrated in several special cases. We have shown the bar in trough method reconstructs a class of bodies where rays where the x-rays go to zero intersect in two points. We can use discontinuities in the derivatives of the x-ray data to reconstruct most polygons and locate non-smooth points on the boundary of any convex set.

The general reconstruction problem is still open. The methods introduced here provide some information about any convex set, but weather they lead to a complete reconstruction in the general case is unknown. If there is a class of objects that are not reconstructed using these methods, can additional methods be found to reconstruct an unknown convex object from x-ray data from three points?

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Appendix A

SYMBOLS USED

$D_{\mathbf{p}}[f]$	the directed x-ray from $\mathbf{p}$ of f.
K	a compact, convex subset of the plane.
$K(\mathbf{x})$	the characteristic function of K.
$\triangle\mathbf{p_1p_2p_3}$	a triangle with vertices $\mathbf{p_1p_2p_3}$
Δ	the symmetric difference operator (Definition 1.2.4).
$CH(E)$	the convex hull of the set E (Definition 1.2.5).