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Rates of Convergence to Self-Similar Solutions of Burgers' Equation

Joel Miller
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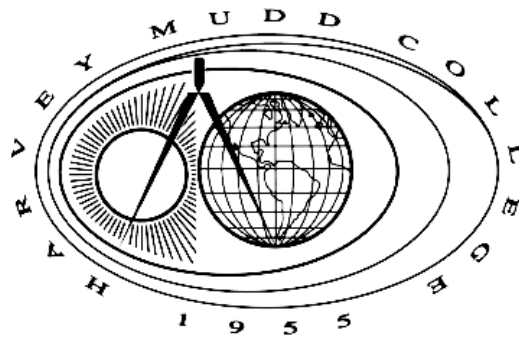
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Rates of Convergence to Self-Similar Solutions of Burgers'
Equation

by

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May 2000

Department of Mathematics

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Abstract

Rates of Convergence to Self-Similar Solutions of Burgers' Equation

by Joel Miller

May 2000

Burgers' Equation $u_t + cuu_x = \nu u_{xx}$ is a nonlinear partial differential equation which arises in models of traffic and fluid flow. It is perhaps the simplest equation describing waves under the influence of diffusion. We consider the large-time behavior of solutions with exponentially localized initial conditions, analyzing the rate of convergence to a known self-similar single-hump solution. We use the Cole-Hopf Transformation to convert the problem into a heat equation problem with exponentially localized initial conditions. The solution to this problem converges to a Gaussian. We then find an optimal Gaussian approximation which is accurate to order t^{-2} . Transforming back to Burgers' Equation yields a solution accurate to order t^{-2} .

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Chapter 1

INTRODUCTION

The general equation

$$u_t + cuu_x = \nu u_{xx} \tag{1.1}$$

with $c, \nu > 0$ is known as Burgers' Equation. It arises in applications modeling traffic flow, fluid flow in certain conditions, magneto-hydrodynamics, atmospheric behavior, and many other physical systems (cf. [6]). It is one of the simplest examples of a nonlinear partial differential equation, and it thus is useful as an example for studying their behavior. For this reason it has received considerable study.

Several people have investigated the large-time behavior of Burgers' Equation. A self-similar solution is found in Whitham [6]. Some recent work by Chern and Liu [1], [2], [3] and Escobedo and Zuazua [4], [8] shows rates of convergence to self-similar solutions. Chern and Liu found a self-similar asymptotic approximation which differs from the true solution by a uniform error of order $1/t$. Zuazua and Escobedo studied a generalization in higher dimension, and found a self-similar asymptotic state with a similar error.

In this thesis, we improve on these estimates, making the assumption that the initial conditions are exponentially localized. We find a self-similar asymptotic state whose uniform error is of order $1/t^2$.

To simplify our analysis of the problem, we can scale time and space to yield the equation

$$u_t + uu_x = u_{xx} \tag{1.2}$$

To see this, let $\tau = \alpha t$ and $\xi = \beta x$. Then Equation (1.1) becomes

$$\alpha u_\tau + c\beta uu_\xi = \nu\beta^2 u_{\xi\xi}$$

Setting $\alpha = c^2/\nu$ and $\beta = c/\nu$ leaves (1.2).

Chapter 2

TRANSFORMATION TO HEAT EQUATION

We are interested in the large-time asymptotic behavior of Burgers' Equation:

$$u_t + uu_x = u_{xx} \quad (2.1)$$

given initial conditions $u(x, 0) = f(x)$ which are exponentially localized, that is,

$$|f(x)| \leq ce^{-a|x|} \quad c, a > 0. \quad (2.2)$$

We can conclude from the exponential localization condition (2.2) that the moments,

$$M_j(f) = \int_{-\infty}^{\infty} x^j f(x) dx \quad (2.3)$$

are bounded, $|M_j(x)| \leq 2cj!/a^{j+1}$. In particular, the *mass*, $M_0(f) = \int_{-\infty}^{\infty} f(x) dx$ is finite. We will make the additional assumption that $f(x) \geq 0$ and that it is nonzero for some region with positive measure. This restriction can be relaxed, but it makes the analysis cleaner.

To begin our analysis, we turn to the Cole-Hopf transformation¹ [6].

$$u = -2\phi_x/\phi \quad (2.4)$$

which reduces Burgers' Equation to the heat equation: $\phi_t = \phi_{xx}$.

We can determine ϕ explicitly in terms of u . We get the ordinary differential equation

$$\phi_x + \frac{u}{2}\phi = 0$$

¹See Appendix A for a derivation

which solves to

$$\phi(x, t) = \exp \left[-\frac{1}{2} \int_{-\infty}^x u(s, t) ds \right] \quad (2.5)$$

In particular, the initial conditions become

$$\phi(x, 0) = \exp \left[-\frac{1}{2} \int_{-\infty}^x u(s, 0) ds \right]$$

Note that the integral in the exponent approaches $M_0(f)$ as $x \rightarrow \infty$ and 0 as $x \rightarrow -\infty$. In between, it is bounded because the integral is monotonically increasing. By the maximum principle of the heat equation [5] we can conclude that for all $t > 0$

$$0 < e^{-M_0(f)}/2 \leq \phi(x, t) \leq 1 < \infty. \quad (2.6)$$

Since $\phi(x, t)$ is not localized, we cannot use tools such as the Fourier Transform.

We can modify the problem to create a localized initial condition by using the following observation: if $\phi_t = \phi_{xx}$, then $\phi_{tx} = \phi_{xxx}$. Defining $\psi = -\phi_x$, we achieve $\psi_t = \psi_{xx}$. Analyzing the initial condition for ψ , we get

$$\psi(x, 0) = \frac{1}{2} u(x, 0) \exp \left[-\frac{1}{2} \int_{-\infty}^x u(s, 0) ds \right] \equiv h(x). \quad (2.7)$$

Because $u(x, 0)$ is exponentially localized and the exponential term is bounded, we conclude $h(x)$ is exponentially localized. We can determine ϕ in terms of ψ as

$$\phi(x, t) = 1 - \int_{-\infty}^x \psi(s, t) ds. \quad (2.8)$$

In particular, since ϕ is strictly positive, we know that for all x

$$\int_{-\infty}^x \psi(x, t) dx < 1. \quad (2.9)$$

We will be able to recover u from ψ and ϕ by modifying Equation (2.4):

$$u = 2\psi/\phi. \quad (2.10)$$

Chapter 3

ANALYSIS OF THE HEAT EQUATION

We have now transformed the original problem into solving

$$\begin{aligned}\psi_t &= \psi_{xx} \\ \psi(x, 0) &= h(x)\end{aligned}\tag{3.1}$$

where h is exponentially localized.

Because h is exponentially localized, we know that its Fourier Transform¹ exists. Applying the Fourier Transform we get

$$\begin{aligned}\widehat{\psi}_t(k, t) &= -k^2 \widehat{\psi}(k, t) \\ \widehat{\psi}(k, 0) &= \widehat{h}(k)\end{aligned}$$

which can be solved as $\widehat{\psi}(k, t) = \widehat{h}(k)e^{-k^2 t}$.

We expand $\widehat{h}(k)$ for $k > 0$ using the Taylor remainder theorem.

$$\widehat{h}(k) = \left(\sum_{j=0}^2 \frac{\widehat{h}^{(j)}(0)}{j!} k^j \right) + \frac{\widehat{h}^{(3)}(c)}{3!} k^3 \quad c \in (0, k)$$

Note that $c = c(k)$ depends on k . Using identities (B.5) and (B.4)

$$\begin{aligned}\widehat{h}(k) &= \left(\sum_{j=0}^2 \frac{(-i)^j M_j(h)}{j!} k^j \right) + \frac{\mathbb{F}[(-ix)^3 h(x)](c)}{3!} k^3 \\ &= \left(\sum_{j=0}^2 \frac{(-i)^j M_j(h)}{j!} k^j \right) + \frac{\int_{-\infty}^{\infty} (-ix)^j h(x) e^{-icx} dx}{3!} k^3\end{aligned}$$

where $M_j(h) = \int_{-\infty}^{\infty} x^j h(x) dx$ is the j -th moment of h .

¹See Appendix B for facts about the Fourier Transform.

At this point we want to find an self-similar asymptotic approximation for ψ . We will do this by finding a self-similar solution to the heat equation whose zeroth, first, and second moments (equivalent to mass, mean, and variance) match those of ψ .

We start by observing that given the heat equation with initial conditions $C\delta(x)$, we get the solution [5]

$$\frac{Ce^{-x^2/4t}}{\sqrt{4\pi t}}. \quad (3.2)$$

Under the change of variables $x \mapsto x - x_*$ and $t \mapsto t + t_*$ the heat equation is unchanged, that is, the heat equation commutes with translations in space and time, we conclude that the heat equation with initial conditions $C\delta(x - x_*)$ at $t = -t_*$ is solved by the self-similar Gaussian

$$G(x, t) = \frac{Ce^{-(x-x_*)^2/4(t+t_*)}}{\sqrt{4\pi(t+t_*)}}.$$

It is this observation that allows us to improve on previous results. Previous asymptotic estimates were found by choosing the optimal value for C . We are able to also choose the optimal values for x_* and t_* .

We will see later that $t_* \geq 0$. At $t = 0$, G will solve the initial condition

$$g(x) \equiv G(x, 0) = Ce^{-(x-x_*)^2/4t_*}/\sqrt{4\pi t_*}$$

We can expand \hat{g} in the same manner we expanded \hat{h} above. We want to find values for C , x_* , and t_* which make g match the first moments of h .

We need to solve the system [7]

$$M_0(h) = \int_{-\infty}^{\infty} g(x) dx \quad (3.3)$$

$$M_1(h) = \int_{-\infty}^{\infty} xg(x) dx \quad (3.4)$$

$$M_2(h) = \int_{-\infty}^{\infty} x^2g(x) dx \quad (3.5)$$

by proper choice of C , x_* , and t_* . To solve (3.3) we use the fact that the integral on the right side is simply C , so we choose $C = M_0$.

We now solve (3.4)

$$\begin{aligned}
M_1 &= \int_{-\infty}^{\infty} xg(x) \\
&= M_0 \int_{-\infty}^{\infty} \frac{e^{(-x-x_*)^2/4(t+t_*)}}{\sqrt{4\pi(t+t_*)}} x dx \\
&= M_0 \left(\int_{-\infty}^{\infty} \frac{e^{-\xi^2/4(t+t_*)}}{\sqrt{4\pi(t+t_*)}} \xi d\xi + x_* \int_{-\infty}^{\infty} \frac{e^{-\xi^2/4(t+t_*)}}{\sqrt{4\pi(t+t_*)}} d\xi \right)
\end{aligned}$$

where $\xi = x - x_*$. Using the fact that the first integrand has odd symmetry, we get the first integral goes to 0. The second integral evaluates to 1. So we conclude that $M_1 = M_0x_*$, and so $x_* = M_1/M_0$.

Finally we solve (3.5)

$$\begin{aligned}
M_2 &= \int_{-\infty}^{\infty} x^2g(x) \\
&= \int_{-\infty}^{\infty} x^2G(x, t) dx \Big|_{t=0}
\end{aligned}$$

Let $S(t) = \int_{-\infty}^{\infty} x^2G(x, t) dx$. We want $S(0) = M_2$. We know that $S(-t_*) = \int_{-\infty}^{\infty} x^2M_0\delta(x_*) dx = M_0x_*^2$. We also have

$$\begin{aligned}
\frac{d}{dt}S(t) &= \int_{-\infty}^{\infty} x^2G_t dx \\
&= \int_{-\infty}^{\infty} x^2G_{xx} dx \\
&= x^2G_x - 2xG \Big|_{-\infty}^{\infty} + 2 \int_{-\infty}^{\infty} G dx \\
&= 2M_0
\end{aligned}$$

So we have $S(-t_*) = M_0x_*^2$ and $S' = 2M_0$. From this it follows that $M_2 = S(0) = M_0x_*^2 + 2M_0t_*$. Since we know $x_* = M_1/M_0$, we solve this to obtain

$$t_* = (M_2M_0 - M_1^2)/M_0^2.$$

Some calculus shows that an equivalent way of expressing t_* is

$$t_* = \frac{1}{M_0} \int_{-\infty}^{\infty} (x - x_*)^2h(x) dx.$$

It turns out that t_* is positive because of our assumption that $u(x, 0) \geq 0$. This assumption guarantees that $h \geq 0$. From this we have $M_0 > 0$. Since $\int_{-\infty}^{\infty} (x - x_*)^2 h(x) dx > 0$, we have $t_* > 0$. Dropping the assumption $u(x, 0) \geq 0$ is possible as long as $M_0 \neq 0$, but it may allow $t_* < 0$, corresponding to a gaussian whose initial conditions are at some positive time.

We now have $\widehat{\psi} = \widehat{h}e^{-k^2t}$ and $\widehat{G} = \widehat{g}e^{-k^2t}$ where the first three terms of the expansion of \widehat{h} and \widehat{g} are identical. Define E to be the error between ψ and G . That is

$$E(x, t) = \psi(x, t) - G(x, t) \quad (3.6)$$

Then using the Taylor Remainder Theorem, $\widehat{E}(k) = \widehat{\psi} - \widehat{G}$ will be given by $(\widehat{h} - \widehat{g})'''(c) = k^3 e^{-k^2t} \int_{-\infty}^{\infty} x^3 (h - g)(x) e^{-icx} dx$ where $g = G(x, 0)$ and $c = c(k) \in (0, k)$.

For future reference, it will be useful to have a uniform bound in the x -variable on $|E(x, t)|$ and on $|\int_{-\infty}^x E(s, t) ds|$. To bound $|E|$ we first find a bound on $|\widehat{E}|$.

$$\begin{aligned} |\widehat{E}| &= \left| k^3 e^{-k^2t} \int_{-\infty}^{\infty} x^3 (h - g) e^{-icx} dx \right| \\ &\leq |k^3 e^{-k^2t}| \int_{-\infty}^{\infty} |x^3 (h - g)| |e^{-icx}| dx \\ &\leq |k^3 e^{-k^2t}| \int_{-\infty}^{\infty} |x^3 (h - g)| dx \\ &\leq C e^{-k^2t} |k^3| \end{aligned} \quad (3.7)$$

Where we have used the fact that $h - g$ is exponentially localized so that the integral must be finite.

We can now find the bounds we need.

$$\begin{aligned} |E| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{E} dk \right| \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{E}| dk \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} C e^{-k^2t} |k^3| dk \\ &\leq \frac{2C}{2\pi} \int_0^{\infty} e^{-k^2t} k^3 dk \end{aligned}$$

$$\leq \frac{C}{2\pi t^2} \quad (3.8)$$

Using $\mathbb{F}[\int_{-\infty}^x E(\xi, t) d\xi] = \widehat{E}(k, t)/k$ and similar techniques, we can find

$$\left| \int_{-\infty}^x E(s, t) ds \right| \leq \frac{C}{4t^{3/2}\sqrt{\pi}}. \quad (3.9)$$

We now have bounds on $|E(x, t)|$ and $\int_{-\infty}^x |E(x, t)|$, and we can return to Burgers' Equation.

Chapter 4

RETURN TO BURGERS' EQUATION

Using Transformation (2.10) and Equation (3.6) we return to Burgers' Equation, getting

$$\begin{aligned} u(x, t) &= \frac{2\psi(x, t)}{\phi(x, t)} \\ &= \frac{2[G(x, t) + E(x, t)]}{\phi(x, t)} \end{aligned}$$

We expect the solution corresponding to G to be an asymptotic approximation for u . We use (2.8) and (2.10) to give

$$\theta(x, t) = \frac{2G(x, t)}{1 - \int_{-\infty}^x G(s, t) ds}$$

as the Burgers' Equation solution corresponding to the Heat Equation solution G .

We consider the difference between u and θ .

$$\begin{aligned} u(x, t) - \theta(x, t) &= \frac{2[G + E] \left[1 - \int_{-\infty}^x G ds \right] - 2G\phi}{\phi(x, t) \left[1 - \int_{-\infty}^x G ds \right]} \\ &= \frac{2 \left(G \int_{-\infty}^x E ds + E \left[1 - \int_{-\infty}^x G + E ds \right] \right)}{\phi(x, t) \left[1 - \int_{-\infty}^x G ds \right]} \end{aligned} \quad (4.1)$$

where in the second step we substituted for ϕ in the numerator using Equation (2.8) and performed some algebra.

To get a uniform bound on this error, we need to determine bounds on the individual terms of (4.1). We will start off by bounding the first term of the numerator

$$\left| G \int_{-\infty}^x E \right| ds = \left| M_0 \frac{e^{-(x-x_*)^2/4(t+t_*)}}{\sqrt{4\pi(t+t_*)}} \int_{-\infty}^x E ds \right|$$

$$\begin{aligned}
&\leq \frac{M_0}{\sqrt{4\pi(t+t_*)}} \frac{C}{4t^{3/2}\sqrt{\pi}} \\
&\leq \frac{CM_0}{8\pi t^2}
\end{aligned} \tag{4.2}$$

Before we can bound the second term of the numerator, we need some more information about $\int_{-\infty}^x G ds$. We know that it is strictly positive because G is positive, and we know that its magnitude is strictly increasing. We further know that as x goes to infinity, this goes to $1 - \phi(\infty, 0)$. Since $\phi(\infty, 0) = \exp[-M_0(f)/2] > 0$, we know that $M_0(G) = M_0(h) < 1$.

$$\left| E \left[1 - \int_{-\infty}^x G ds \right] \right| \leq \frac{C(1 - M_0)}{2\pi t^2} \tag{4.3}$$

We now seek a bound on the denominator. We know from Equation (2.6) that $\phi \geq a$ for $a = e^{-M_0(f)}/2$ and that $1 - \int_{-\infty}^x G > 1 - M_0$. From this it follows that

$$\begin{aligned}
|u(x, t) - \theta(x, t)| &< \frac{2 \left(\frac{CM_0}{8\pi t^2} + \frac{C(1-M_0)}{2\pi t^2} \right)}{a(4 - M_0)} \\
&< \frac{C(4 - 3M_0)}{4a(1 - M_0)\pi t^2}
\end{aligned} \tag{4.4}$$

Where $C = \int_{-\infty}^{\infty} |x^3(g - h)| dx$, $M_0 = \int_{-\infty}^{\infty} h dx$, and $a = \inf \phi(x, 0) = e^{-M_0(f)}/2$. In particular, this is $\mathcal{O}(t^{-2})$

Chapter 5

EXAMPLE

We consider Burgers' Equation

$$u_t + uu_x = u_{xx} \tag{5.1}$$

with the tophat initial condition

$$u(x, 0) = \begin{cases} 0 & |x| > 1 \\ 1 & |x| \leq 1 \end{cases}$$

This initial condition transforms to

$$\psi(x, 0) = \begin{cases} 0 & |x| > 1 \\ e^{-2(x+1)} & |x| \leq 1 \end{cases}$$

where

$$\psi_t = \psi_{xx} \tag{5.2}$$

In Figure 5.1 we show how the true solution to (5.2) converges to the optimal gaussian solution. Notice that at time $t = 10$, this is a very good approximation, and

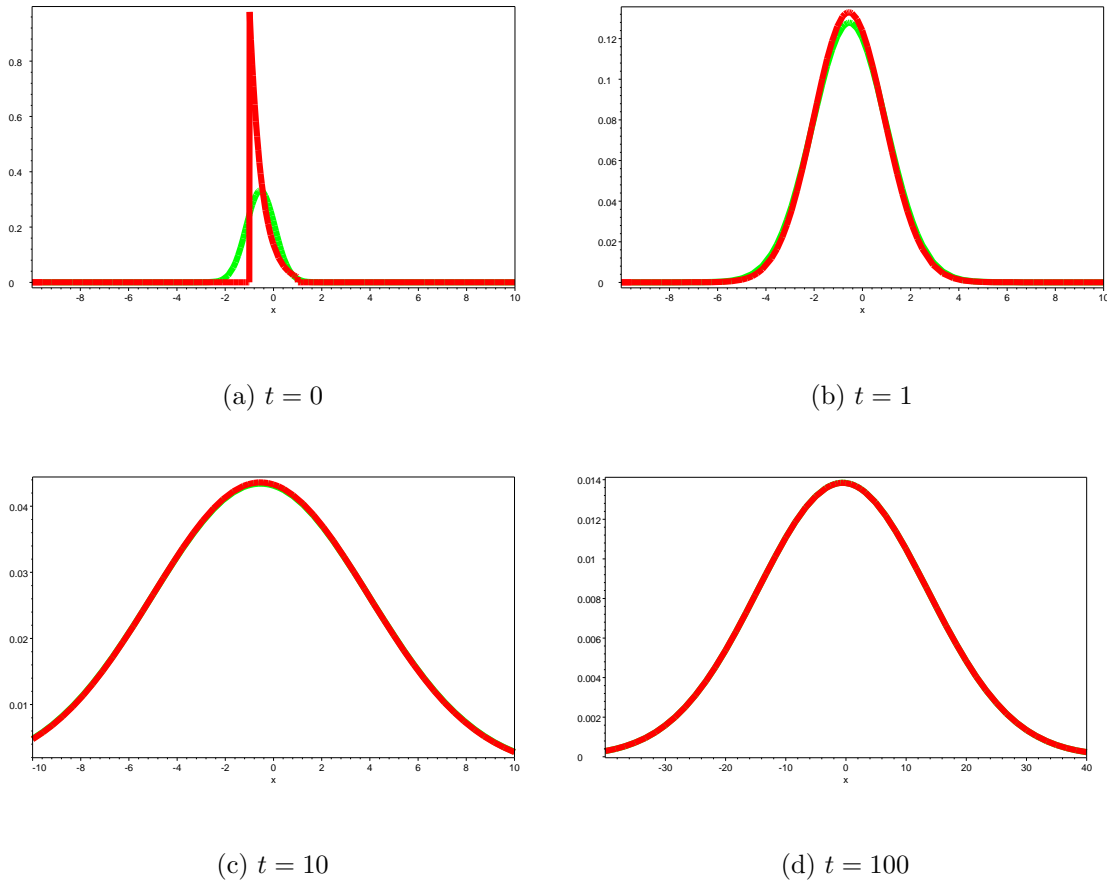


Figure 5.1: Heat Equation Convergence [red = true solution; green = approximation]

at time $t = 100$, there is no detectable difference between the true solution and the asymptotic approximation.

We now consider the true solution to Burgers' Equation u compared with the solution to which the optimal gaussian corresponds. This is shown in Figure 5.2. Again, note that at $t = 10$ this is a very good approximation, and at $t = 100$, there is no detectable error.

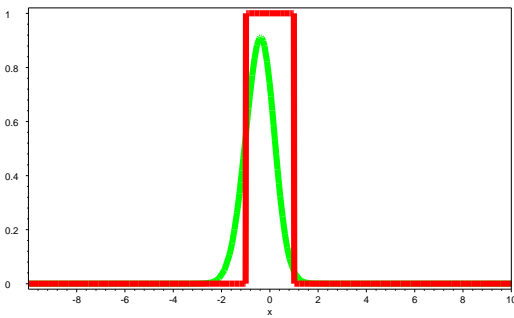
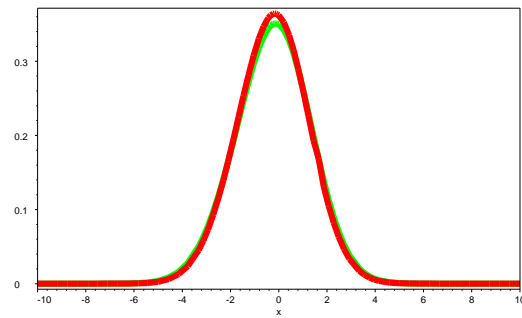
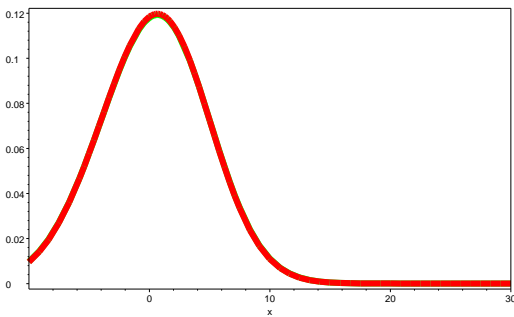
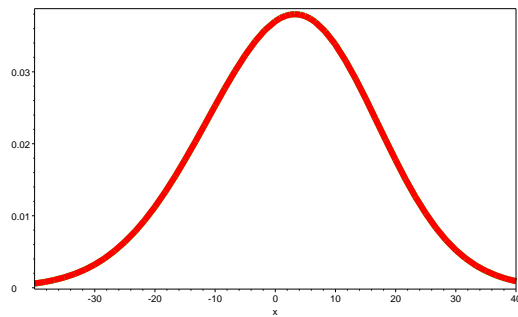
(a) $t = 0$ (b) $t = 1$ (c) $t = 10$ (d) $t = 100$

Figure 5.2: Burgers' Equation Convergence [red = true solution; green = approximation]

The convergence in our solution is very good in comparison to the asymptotic approximation from Chern [1] shown in Figure 5.3.

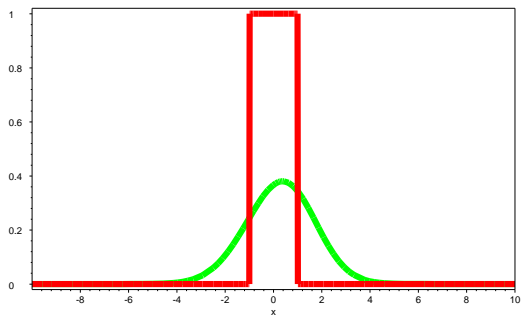
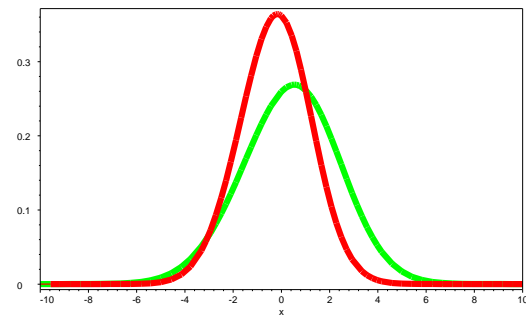
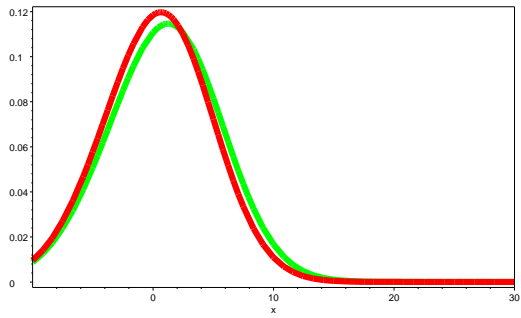
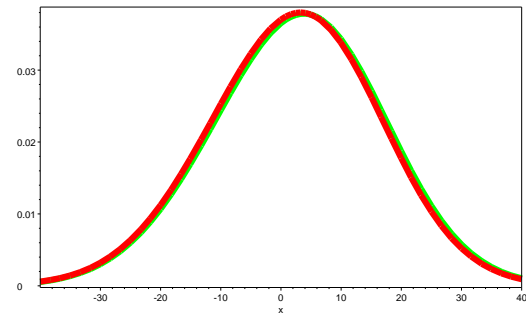
(a) $t = 0$ (b) $t = 1$ (c) $t = 10$ (d) $t = 100$

Figure 5.3: Chern's Convergence [red = true solution; green = approximation]

Chapter 6

CONCLUSIONS AND FUTURE WORK

Burgers' Equation with exponentially localized initial conditions has self-similar behavior as t grows. We have found an asymptotic approximation which matches the true solution with an error term of order t^{-2} . This is an improvement on previous work of Chern and others which found asymptotic approximations with errors of order t^{-1} .

In future work, we would like to find L^p norms on the error terms. The L^p norms of the error found by Chern for his approximation are of order $t^{-1+1/2p}$. We suspect that we can achieve $t^{-2+1/2p}$. We also hope to extend this sort of analysis to other nonlinear PDEs, in particular the lubrication model of a thin fluid film

$$u_t + u^m u_x = -(u^n u_{xxx})_x.$$

Appendix A

THE COLE-HOPF TRANSFORMATION

The Cole-Hopf Transformation was discovered independently by Cole and Hopf around 1950. It changes Burgers' Equation $u_t + uu_x = u_{xx}$ into the Heat Equation $\phi_t = \phi_{xx}$ as shown in [6]

To derive the transform, we let $u = \gamma_x$. Then Burgers' Equation can be integrated yielding $\gamma_t + \gamma_x^2/2 = \gamma_{xx}$. Let $\gamma = -2 \log \phi$. Then we get

$$-2\frac{\phi_t}{\phi} + 2\frac{\phi_x^2}{\phi^2} = -2\frac{\phi\phi_{xx} - \phi_x^2}{\phi^2}$$

Applying some algebra to this yields $\phi_t = \phi_{xx}$

Appendix B

FOURIER IDENTITIES

We define the Fourier Transform by

$$\mathbb{F}[g(x)] = \widehat{g}(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx} dx$$

The transform is inverted by

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g}(k)e^{ikx} dk$$

We have the following identities (cf. [5]):

$$\mathbb{F}[\alpha f(x) + \beta g(x)] = \alpha \widehat{f} + \beta \widehat{g} \tag{B.1}$$

$$\mathbb{F}[g'(x)] = ik\widehat{g} \tag{B.2}$$

$$\mathbb{F}[g_t(x, t)] = \widehat{g}_t \tag{B.3}$$

$$\mathbb{F}[(ix)^j g(x)] = \widehat{g}^{(j)}(k) \tag{B.4}$$

Note that from Equation (B.4)

$$\begin{aligned} \widehat{g}^{(j)}(0) &= \int_{-\infty}^{\infty} (ix)^j g(x)e^0 dx \\ &= i^j M_j(g) \end{aligned} \tag{B.5}$$

We have the transform for a gaussian $g(x, t) = e^{-x^2/4\pi t}/\sqrt{4\pi t}$ is

$$\widehat{g} = e^{-k^2 t} \tag{B.6}$$

We finally develop the transform of shifts in x and t :

$$\mathbb{F}[g(x - x_*)] = e^{-ikx_*} \widehat{g} \tag{B.7}$$

$$\mathbb{F}[g(x, t + t_*)] = \widehat{g}(k, t + t_*) \tag{B.8}$$

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