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Solution to Problem 1751, A Combinatorial Identity

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A combinatorial identity. **October 2006**

1751. *Proposed by Iliya Bluskov, University of Northern British Columbia, Prince George, BC, Canada*

Let k_1, $k_2, \ldots, k_n$ be integers with $k_i \geq 2$, $i = 1, 2, \ldots, n$, and let $N = \sum_{i=1}^{n} \binom{k_i}{2}$. Prove that

$$\sum_{1 \leq i < j \leq n} \binom{k_i}{2}\binom{k_j}{2} + 3\sum_{i=1}^{n} \binom{k_i + 1}{4} = \binom{N}{2}. \tag{1}$$

II. *Solution by Arthur Benjamin and Andrew Carman (student) Harvey Mudd College, Claremont, CA.*

We give a combinatorial proof.

A city has n districts and, for $i = 1 \ldots, n$, district i has k_i representatives on the city council. The number of ways to select a *ticket*, an unordered pair of representatives from the same district is $N = \sum_{i=1}^{n} \binom{k_i}{2}$.

On the right side of (1), $\binom{N}{2}$ counts the ways to pick two different tickets. We claim the left side also gives this count.

The number of ticket pairs of tickets from two different districts is $\sum_{1 \leq i < j \leq n} \binom{k_i}{2}\binom{k_j}{2}$. The number of ticket pairs using four different people from district i is $3\binom{k_i}{4}$, because once we have chosen ticket members a, b, c, d, there are three ways to select the running mate of a. The number of ticket pairs involving three different people from district i is $3\binom{k_i}{3}$, since once we have chosen ticket members a, b, c, there are three ways to pick which representative appears on both tickets. Since $\binom{k_i}{4} + \binom{k_i}{3} = \binom{k_i+1}{4}$, the total number of ways to pick a pair of tickets from the same district is $3\sum_{i=1}^{n} \binom{k_i+1}{4}$. The result now follows.

Also solved by JPV Abad, Steve Abbot, Armstrong Problem Solvers, Ovidiu Bagdasar (Romania), Michel Bataille (France), J. C. Binz (Switzerland), Robert Calcaterra, Johann Chen, Haiwen Chu, Con Amore Problem Group (Denmark), Chip Curtis, Knut Dale (Norway), Joe DeMaio, M. N. Deshpande (India), Fejéntaláltuka Szeged Problem Solving Group (Hungary), G.R.A.20 Problem Solving Group (Italy), Ralph P. Grimaldi, Enkel Hysnelaj (Australia) and Elton Bojaxhiu (Albania), Ronald A. Kopas, Harris Kwong, Elias Lampakis (Greece), Kathleen E. Lewis, McDaniel College Problem Group, Kim McInturff, William Moser (Canada), Michael Natanson, Rob Pratt, Nicholas C. Singer, Albert Stadler, Marian Tetiva (Romania), Thomas R. Wilkerson, Stuart V. Witt, Paul Zwier, and the proposer.