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Combinatorial Proofs of Generalizations of Sperner's Lemma

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Abstract

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In this thesis, we provide constructive proofs of several generalizations of Sperner's Lemma, a combinatorial result which is equivalent to the Brouwer Fixed Point Theorem. This lemma makes a statement about the number of a certain type of simplices in the triangulation of a simplex with a special labeling. We prove generalizations for polytopes with simplicial facets, for arbitrary 3-polytopes, and for polygons. We introduce a labeled graph which we call a nerve graph to prove these results. We also suggest a possible non-constructive proof for a polytopal generalization.

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ACKNOWLEDGMENTS

Chapter 1

INTRODUCTION

In this thesis, we prove several generalizations of Sperner's Lemma, and sketch a proof of a generalization to polytopes which was conjectured by Atanassov in 1996 (see [1]). Sperner's Lemma is a combinatorial theorem dealing with triangles, or simplices in dimensions greater than 2, which are divided into many smaller triangles. It was proved by Sperner in [14] in 1928.

The key idea of Sperner's Lemma is well-represented by the planar case. Consider a triangle with vertices labeled 1, 2, and 3. Subdivide the triangle into several smaller triangles, and label the new vertices according to a special set of rules known as a *Sperner Labeling*. Sperner's Lemma claims that, in the planar case, there are an odd number of $\{1, 2, 3\}$ triangles in the subdivision:

Sperner's Lemma. *In a k-simplex with a Sperner Labeling, there are an odd number of fully-labeled elementary simplices.*

Sperner's Lemma is stated precisely in Section 2.2, and is proved in Section 4.4. The full generalization that we consider in this thesis extends the theorem to polytopes, the higher-dimensional analogue of polygons (see Section 4.1 for a precise definition):

Generalized Sperner's Lemma (polytopal conjecture). *In an arbitrary k-polytope P having n vertices, subdivided with a Sperner Labeling, there are at least $n - k$ fully-labeled elementary simplices.*

Here, "fully-labeled" means the elementary simplex has $k+1$ distinct labels. This result has been proved by Peterson and Su, for polytopes with simplicial facets, using a topological argument in [11]. Using *strictly combinatorial* arguments, we will prove this generalization for polytopes with simplicial facets (Section 4.5), and for arbitrary polytopes in 3 dimensions (Section 5.3). We will then sketch a non-constructive proof and suggest a constructive proof for the full generalization (Chapter 6).

Sperner's Lemma is an existence theorem, and it has several constructive and non-constructive proofs. Although the Lemma is very simple, it has widespread use in mathematics. For example, it is equivalent to the Brouwer Fixed Point Theorem. In the proof of this equivalence, the fixed point in Brouwer's Theorem corresponds to the $\{1, 2, 3\}$ triangle in Sperner's Lemma. This equivalence is useful not only because the proof is rather simple, but also because constructive proofs of Sperner's Lemma give an algorithm for finding the fixed point. See [7], [9], [16], and [18].

This lemma is also very useful in game-theoretic problems of *fair division*, which deals with the question of how to divide a set among several players so that all players are happy with the division. Sperner's Lemma can be used to show that a division exists in which all players feel they receive the best piece of the set. See [15].

We begin this thesis by acquainting the reader with Sperner's lemma in Chapter 2, where we describe Sperner Labeling and state Sperner's Lemma formally. In Chapter 3, we introduce *nerve graphs*, a special type of labeled graph that we will use to prove our generalizations constructively. Our main result in this chapter is the *Maximum Label Theorem*, which states that:

Maximum Label Theorem. *Given a connected k-nerve graph (N, ℓ) with $N = (B \cup C \cup D, E)$, which is not necessarily minimal, we have $L(N) - k \leq |C|$.*

Here, $L(N)$ represents the number of labels used by the nerve graph, and $|C|$ the number of "full cells" used. We begin Chapter 3 by with some motivation for nerve

graphs in Section 3.1, and the formal definition of a nerve graph in Section 3.2. We then introduce *minimalized nerve graphs* and *nerve skeletons* in Sections 3.3 and 3.4, which will help us prove the Maximum Label Theorem in Section 3.5. Finally, in Section 3.6, we introduce *subnerves*, which will help prove our generalizations.

In Chapter 4, we prove the generalization of Sperner's Lemma for a polytope with simplicial facets, which states that:

Generalized Sperner's Lemma (simplicial facets). *In a k-polytope P with simplicial faces having n vertices, subdivided with a Sperner Labeling, there are at least $n - k$ fully-labeled elementary simplices.*

Polytopes are defined in 4.1, and the Sperner Labeling is described in Section 4.2. We introduce the *simplicial graph*, a nerve graph derived from a triangulated polytope with a Sperner Labeling, in Section 4.3. In Section 4.4 we prove Sperner's Lemma itself, which we use in our proof of the generalization for simplicial facets in Section 4.5.

In Chapter 5, we give some further generalizations of the Lemma for dimensions 2 and 3. In Section 5.1, we prove a theorem generalizing the labeling scheme in the plane and a theorem making a parity claim. In Section 5.2 we conjecture a result for non-simplicial subdivisions. In Section 5.3 we prove the full generalization for 3 dimensions:

Generalized Sperner's Lemma (3-polytopes). *In an arbitrary 3-polytope P having n vertices, subdivided with a Sperner Labeling, there are at least $n - k$ fully-labeled elementary simplices.*

Finally, in Chapter 6, we sketch an alternative, non-constructive method, for proving the full generalization. This method relies on a technical conjecture regarding polytopes, and uses a simple induction argument to find the requisite $n - k$ full cells. We also suggest a constructive proof based on nerve graphs in this chapter.

Chapter 2

SPERNER'S LEMMA

In this chapter, we first describe the *Sperner Labeling* and Sperner's Lemma in the plane. We then expand this to higher dimensions and give a precise statement of Sperner's Lemma. We will prove Sperner's Lemma in Section 4.4 using nerve graphs. For alternate proofs, see [2] or [15].

2.1   *Sperner's Lemma in the Plane*

Suppose we are given a triangle in the plane, with vertices labeled 1, 2, and 3, as shown in figure (a) below. These vertices are known as *main vertices*. We will refer to a vertex labeled by i as simply *vertex i*. Now, subdivide the triangle into several smaller triangles, as in (b). This is known as a *triangulation*. The small triangles are known as *elementary triangles* or simply as *cells*.
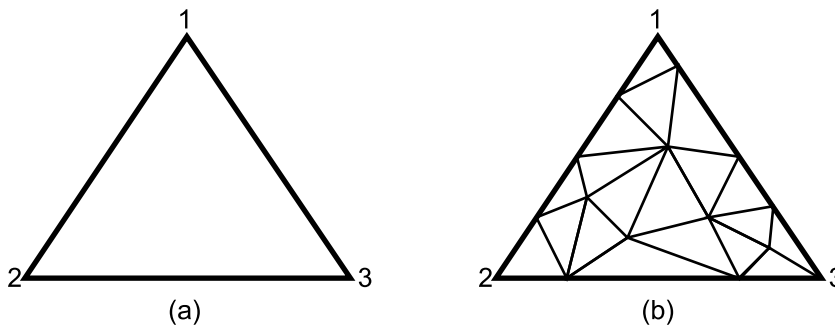
Finally, label the vertices of these elementary triangles according to the following two rules:

1. If a vertex is on the edge between the vertex i and vertex j, then the vertex must be labeled by either i or j.

2. If a vertex is in the interior of the triangle, then it may be labeled by 1, 2, or 3.

The figure below shows one possible labeling of the triangle.
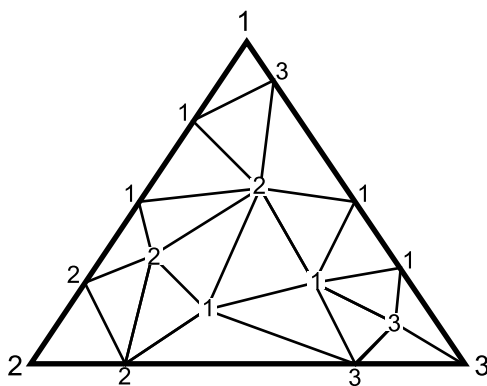
The above labeling scheme is known as a *Sperner Labeling*. We will call a triangle which is subdivided into smaller triangles and given a Sperner Labeling a *triangle with a Sperner Labeling*. We will call an elementary triangle which has all 3 possible labels a *fully-labeled elementary triangle*, or simply a *full cell*. We can now state Sperner's Lemma in the plane:

Sperner's Lemma (planar). *In a planar triangle with a Sperner Labeling, the number of fully-labeled elementary triangles (full cells) is odd.*

Note that the existence of an odd number of such simplices requires that there be at least one. In the figure above, there are 3 such full cells.

2.2 Sperner's Lemma

The extension of the *Sperner Labeling* to several dimensions is rather simple, as the rules are completely analogous. In dimensions greater than 2, Sperner's Lemma deals with *simplices* rather than triangles. A simplex is the equivalent of a triangle in an arbitrary dimension. For example, a 1-dimensional simplex (1-simplex) is just a line segment, a 2-simplex is a triangle in the plane, and a 3-simplex is a tetrahedron.

1-simplex   2-simplex   3-simplex   4-simplex
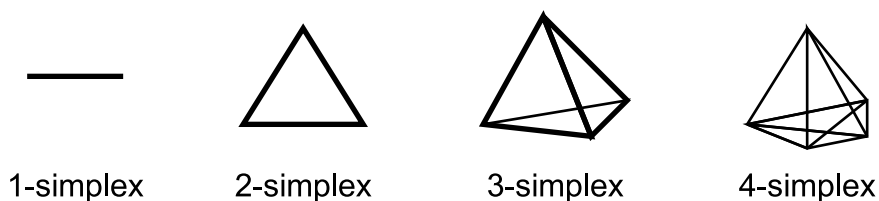
Formally, a simplex may be regarded as the convex hull of $k+1$ affinely independent points in R^n, for $n \geq k$. These points are the vertices of the simplex. A "side" of a simplex is known as a *facet*. Facets are spanned by k of the vertices of the simplex, and are thus $(k-1)$-simplices. Also, a k-simplex has $k+1$ different facets.

The Sperner Labeling can be described inductively. We have already covered the base case, so we now consider the case of a k-simplex. We first require that each of the $k+1$ vertices of a simplex be given a different label. Next, the simplex is *triangulated*, that is, subdivided into several smaller k-simplices. The smaller simplices are known as *elementary simplices* or *cells*. Now, the remaining vertices are labeled according to the following rules:

1. If a vertex is on the exterior of the simplex, and thus is in one or more facets of the simplex, then it must obey the labeling rules given by induction for each of these facets.

2. If a vertex is in the interior of the simplex, it receives an arbitrary label.

Thus, in a tetrahedron with main vertices $\{1, 2, 3, 4\}$, the vertices on the $\{1, 2, 3\}$ facet must be labeled by 1, 2, or 3, while those on the $\{1, 3, 4\}$ facet must be labeled by 1, 3, or 4. Hence, vertices on both facets, which correspond to those vertices between 1 and 3, must satisfy both of these requirements, hence be labeled by either 1 or 3. This labeling scheme is demonstrated in the following figure:
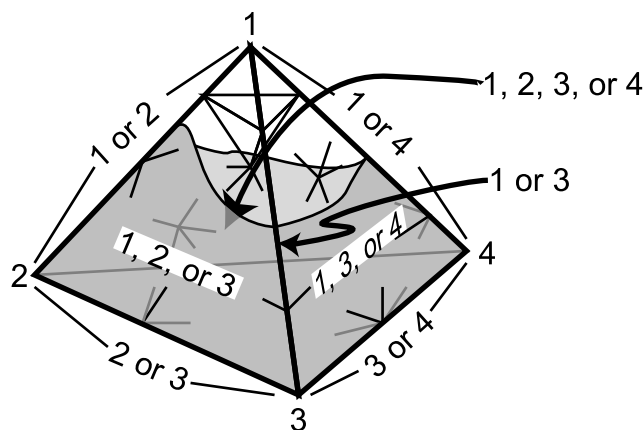
There is a shorter, non-inductive way to describe the *Sperner Labeling*. Suppose the main vertices in the k-simplex are $\{1, 2, \ldots, k+1\}$. If a vertex v is spanned by a set of main vertices $\{i_1, i_2, \ldots, i_m\}$, then its label must correspond to that of one of these main vertices. We will use this definition later to extend the Sperner Labeling to polytopes.

With the *Sperner Labeling* defined for simplices, we can now give the complete statement of Sperner's Lemma. In dimension k, a full cell will correspond to an elementary k-simplex with $k + 1$ distinct labels.

Sperner's Lemma. *In a k-simplex with a Sperner Labeling, there is an odd number of full cells.*

We will prove Sperner's Lemma in Chapter 4, when we present the generalization of Sperner's Lemma for polytopes with simplicial facets.

Chapter 3

THEORY OF NERVE GRAPHS

This chapter develops what we will call the *theory of nerve graphs*. Nerve graphs are special kinds of graphs which arise from Sperner Labelings of triangulated polytopes. Other types of labeled graphs have been used to prove generalizations of Sperner's Lemma (see [4]). Nerve graphs are useful because they give a constructive method, similar to path-following methods in [2] and [7], to find full cells.

We begin in Section 3.1 by giving the motivation for these graphs. We define them formally in Section 3.2 and prove some elementary results. In Sections 3.3 and 3.4 we introduce *minimalized nerves* and *nerve skeletons*, two types of graphs which can be derived from nerve graphs. They are both used to prove the *Maximum Label Theorem* in Section 3.5, our main result for this chapter. This is the most important step in obtaining the $n - k$ result for the main theorem. Finally, in Section 3.6 we describe *subnerves*, which will be necessary for proving our generalizations later.

3.1 Motivation

The constructive proof of the generalization of Sperner's Lemma for simplicial facets presented in this paper makes extensive use of graph theory. Specifically, it requires several theorems about a special kind of graph which we will call a *nerve graph*.

Nerve graphs are basically graphs in which the vertices are labeled, and in which all vertices have degree 1, 2, or $(k + 1)$ for some k, with some restrictions on the labeling of the vertices. This will give us a purely combinatorial description of triangulated polytopes with Sperner Labelings.

Let us take the planar example to see how Sperner's Lemma is related to graph theory. Recall (see Section 2.2) that Sperner's Lemma asserts the existence of a full cell in any triangulation of a triangle T labeled according to the following rules, indicated in the figure below:

1. The three main vertices are labeled by 1, 2, and 3.

2. A vertex on the edge between vertices i and j is labeled by either i or j.

3. A vertex in the interior is labeled by 1, 2, or 3.
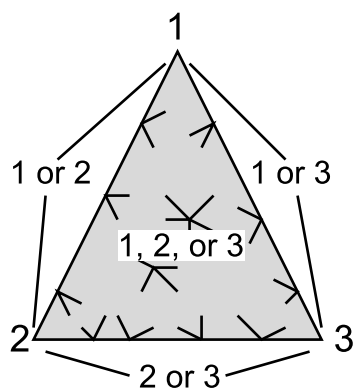
With any triangulation, we can divide the main triangle T into several distinct regions such that the vertices in each region have the same label, and two adjacent regions have different labels, as in the figure below.
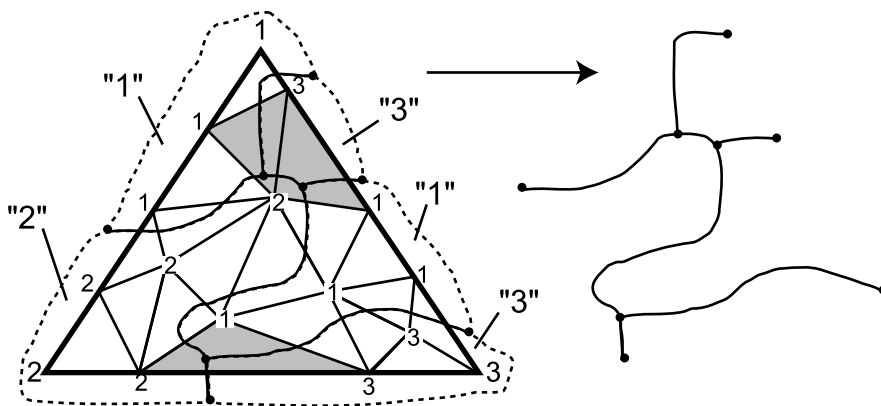
Note that the boundaries of these areas form a graph. The vertices of this graph consist of two types: those points adjacent to three regions, and those points adjacent to two regions and the exterior. The edges are the boundaries between two regions.

Removing the edges that border the outer area, we can obtain a nice graph from the subdivided triangle. Then, the first type of vertex (connected to three areas) now has degree 3, while the second type has degree 1. The figure above shows the graph obtained from a specific subdivision of a triangle. Note that the degree 3 vertices correspond to the triangles in the subdivision which have all three labels (since they border three areas).

Recall the degree-edge relationship from graph theory, which states that the sum of the degrees of all vertices in a graph must be even. Let the number of degree 1 vertices be $|V_1|$, and the number of degree 3 vertices be $|V_3|$. Then, we know that $|V_1| \equiv |V_3| \mod 2$. We can show by induction that $|V_1|$ is odd, meaning that $|V_3|$ must also be odd, proving the result.

Note the use of graph theory in this argument. Although simpler counting methods can be used to obtain the same result, this 'nerve graph' method is more easily generalized.

3.2   Definition of a Nerve Graph and Elementary Propositions

Our proof of the generalized Sperner's Lemma for simplicial facets uses the subdivision of a polytope to form several graphs, which we call *nerve graphs*, in a similar manner to that described above. The details of this will not be given until the next chapter. We choose to present the theory of nerve graphs in this chapter, as it does not require any knowledge about polytopes or their triangulation.

There are two types of restrictions which must be placed on nerve graphs so they can represent the triangulation of polytopes. First, the *degree restriction* requires vertices in the graph to have degree 1, 2 or $(k + 1)$, for some $k > 2$. The different

types of vertices in a nerve graphs correspond to different simplices in the triangulation. Note that the degree 3 vertices are in triangles (2-simplices) with three labels. Also, although they are outside the triangle in the above figure, the degree 1 vertices correspond to edges (1-simplices) with two labels. Finally, the degree 2 vertices which we have added below are in triangles with two labels.
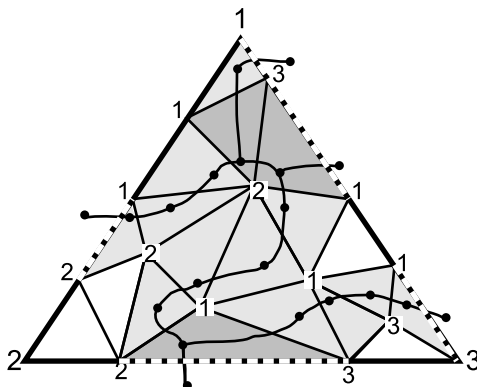
The second restriction is the *labeling restriction*, which requires adjacent vertices to have a certain number of labels in common. In the planar case, adjacent vertices should share two labels, because adjacent triangles in the plane share two vertices.

Of course, the form of a nerve graph in higher dimensions is slightly different. When the subdivided polytope lies in dimension k, we will have vertices of degree 1, 2, and $(k+1)$, which receive k, k, and $(k+1)$ labels, respectively.

3.2.1 Formal Definition of a Nerve Graph

Given an (n,k)-*polytope,* that is, a polytope with n vertices in k dimensions, its corresponding nerve graph will be an (n,k)-*nerve graph*, or simply a k-*nerve graph*. The definition of a nerve graph will depend on both n and k. Note that we must have $n \geq k+1$, because a k-dimensional polytope must have at least $k+1$ vertices.

We now define the label sets used by nerve graphs. Given n and j, where $0 \le j \le n$, we define I^n and I^n_j as follows:

$$I^n = \{1, \ldots, n\}, \tag{3.1}$$

$$I^n_j = \{I^* \subset I^n : |I^*| = j\}. \tag{3.2}$$

Thus, I^n_j is the set of j element subsets of I^n. As an example, the set I^4_2 consists of all 2 element subsets of $I^4 = \{1, 2, 3, 4\}$:

$$I^4_2 = \{12, 13, 14, 23, 24, 34\}.$$

In the theory of nerve graphs, we will be concerned primarily with I^n_k and I^n_{k+1}.

We now give the definition of an (n, k)-nerve graph.

Definition 3.1. Let $k > 1$ and $n \ge k + 1$. An (n, k)-*nerve graph* is a pair (N, ℓ), where $N = (V, E)$ is a graph and ℓ is a labeling function

$$\ell : V \to I^n_k \cup I^n_{k+1}$$

such that each vertex $v \in V$ satisfies the following rules:

(a) *Degree Range Rule*: $\deg(v) \in \{1, 2, k + 1\}$,

(b) *Degree $k + 1$ Rule*: $\deg(v) = k + 1$ if and only if $\ell(v) \in I^n_{k+1}$.

(c) *Edge Limit Rule*: if $\ell(v) \in I^n_{k+1}$ and $\ell(w) \in I^n_{k+1}$, then $(v, w) \notin E$.

(d) *Label Subset Rule*: if $(v, w) \in E$, and $\deg(v) \le \deg(w)$, then $\ell(v) \subset \ell(w)$.

(e) *Three Vertex Rule*: if $\ell(v) \in I^n_{k+1}$ and $(v, w_1) \in E$, $(v, w_2) \in E$ with $w_1 \ne w_2$, then $\ell(w_1) \ne \ell(w_2)$.

We will often refer to the five rules (a) through (e) according to the names given. Note that *this definition will be used very often throughout the remainder of the thesis, and the reader should study it carefully.* We often omit the label n or the label k and write simply *k-nerve graph* or *nerve graph* in place of "(n, k)-nerve graph."

Consider the definition of the label function ℓ above. Although we have only defined it here for a single vertex, we will "overload" ℓ by using it to mean several different things. Speaking of a set of vertices $W = \{w_1, w_2, \ldots, w_j\} \subset V$, we let

$$\ell(W) = \bigcup_{i=1}^{j} \ell(w_i).$$

For a nerve graph $N = (V, E)$, we let $\ell(N) = \ell(V)$. Often we are concerned with the number of labels in a given set. We will denote this number by $L(x) = |\ell(x)|$.

The following figure shows an example of a nerve graph, complete with labelings. The reader may check to see that the five nerve rules hold for all vertices. If we call the nerve graph (N, ℓ), then $\ell(N) = \{1, 2, 3, 4, 5\}$ and $L(N) = 5$.
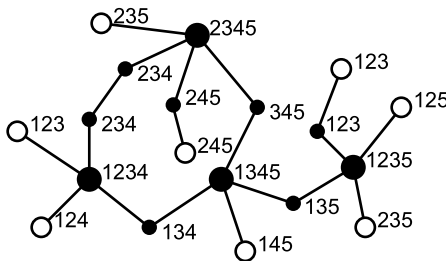
The next few sections will explain the five conditions above. They will also give some simple combinatorial results based on the definition of a nerve graph.

3.2.2   The degree of the vertices

The degree range rule (a) states that the degree of any vertex has three possible values: 1, 2, or $k + 1$, for some positive k. We can classify the vertices according to their degree, as in the following definition.

Definition 3.2. Given a nerve graph (N, ℓ) with $N = (V, E)$, we define the sets B, C, and D as follows:

$$B = \{v \in V : \deg(v) = 1\} \tag{3.3}$$

$$C = \{v \in V : \deg(v) = k + 1\} \tag{3.4}$$

$$D = \{v \in V : \deg(v) = 2\}. \tag{3.5}$$

Thus, B consists of all degree 1 vertices, which we call *boundary vertices*; C consists of all degree $k + 1$ vertices, which we call *connecting vertices* or *cell vertices*; and D consists of all degree 2 vertices, which we call *door vertices*. This is demonstrated in the figure below.

connecting
(cell) vertices

boundary
vertices
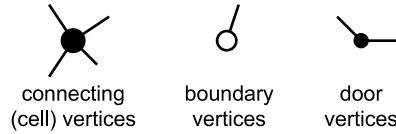
door
vertices

The labels for these sets are drawn from the corresponding simplices in the triangulation of a polytope, in which the *boundary* simplices correspond to vertices in B, the full *cells* to vertices in C, and the *door* simplices to vertices in D. These labels also make some sense in terms of the graph itself. The outer (boundary) vertices are those with degree 1, and one might pass to the main (cell) vertices by passing through degree 2 vertices (doors).

Are there some combinatorial conclusions we can draw from the form of such a graph? Certainly! Suppose we have a connected nerve graph. Recall from graph theory that, for a graph $G = (V, E)$,

$$\sum_{v \in V} \deg(v) = 2|E|. \tag{3.6}$$

In the case of nerve graphs, this can be written as

$$\sum_{b \in B} \deg(b) + \sum_{c \in C} \deg(c) + \sum_{d \in D} \deg(d) = \sum_{b \in B} 1 + \sum_{c \in C} (k + 1) + \sum_{d \in D} 2.$$

It is thus clear that

$$|B| + 2|D| + (k+1)|C| = 2|E|, \tag{3.7}$$

and hence

$$|B| \equiv (k+1)|C| \mod 2. \tag{3.8}$$

What does this tell us? In the case where k is even, $|B|$ and $|C|$ must have the same parity, and in the case where k is odd, $|B|$ must have even parity! Thus, we are already starting to see some results regarding the possible numbers of the different types of vertices. We give this result formally in the following observation:

Observation 3.1. *In a connected k-nerve graph with vertices $V = B \cup C \cup D$, we know that $|B| \equiv |C| \mod 2$ if $k \equiv 0 \mod 2$, and $|B| \equiv 0 \mod 2$ if $k \equiv 1 \mod 2$.*

3.2.3   The Labeling Rules: Edge Perspective

We will now examine the labeling rules for nerve graphs, from both an *edge perspective* and a *neighbor perspective*. The former involves looking at the relationship between the labelings of two connected vertices. The latter, covered in the next section, involves looking at how the labeling of a vertex is related to the labelings of its neighbors.

Observe that the degree $k+1$ rule (b) of nerves, which states that

$$\deg(v) = k+1 \text{ if and only if } \ell(v) \in I_{k+1}^n,$$

just indicates how many labels each type of vertex receives.

$I^5 = \{1,2,3,4,5\}$
$I_4^5 = \{1234,1235,1245,1345,2345\} \longrightarrow \bullet C$
$I_3^5 = \{123,124,125,134,135,145, \searrow \rightarrow OD$
$\qquad 234,235,245,345\} \qquad \searrow \bullet B$

Vertices in B (degree 1) and D (degree 2) receive k labels out of I^n, while vertices in C (degree $k+1$) receive $k+1$ labels. In the figure above, in which $n = 5$ and $k = 3$, the vertices in B and D receive 3 labels, while the vertices in C receive 4 labels.

Now, consider the label subset rule (d), which states that

$$\text{if } (v, w) \in E, \text{ and } \deg(v) \leq \deg(w), \text{ then } \ell(v) \subset \ell(w).$$

Given $\beta_1, \beta_2 \subset \{B, C, D\}$, let

$$(\beta_1, \beta_2) = \{(v, w) : v \in \beta_1, w \in \beta_2\}.$$

Note that $(C, C) = \emptyset$. We make the following observations:

Observation 3.2. *If $(v, w) \in (B \cup D, B \cup D)$, then $\ell(v) = \ell(w)$.*

Proof. Recall that $L(v) = |\ell(v)|$. We know that $L(v) = L(w)$ and also that either $\ell(v) \subset \ell(w)$ or $\ell(w) \subset \ell(w)$. Hence, $\ell(v) = \ell(w)$. $\qquad\square$

Observation 3.3. *If $(v, w) \in (B \cup D, C)$, then $\ell(v) \subset \ell(w)$.*

Proof. Consequence of label subset rule (d). $\qquad\square$

What can we say about the figure below, where $v \in C$, $w_1, w_2 \in D$, and $w_3 \in B$?
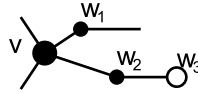
The above observations tell us that $\ell(w_3) = \ell(w_2) \subset \ell(v)$, and $\ell(w_1) \subset \ell(v)$.

Note that these labeling rules are very restrictive, as adjacent vertices will always share at least k labels. This makes sense, of course, when thinking about two adjacent simplices; in k dimensions, they have $k+1$ vertices, and if they are adjacent they will share a face, which has k vertices.

These observations allow us to prove the following proposition, which indicates that a connected k-nerve graph with no vertices in C has only k labels, while a connected k-nerve graph with at least one vertex in C has the same label set as C.

Proposition 3.4. *(a) Let (N, ℓ) be a connected k-nerve graph with $|C| = 0$. Then*

$$L(N) = |\ell(N)| = k. \tag{3.9}$$

If $|B| > 0$, then $\ell(N) = \ell(B)$. If $|D| > 0$, then $\ell(N) = \ell(D)$.

  (b) Let (N, ℓ) be a connected k-nerve graph with $|C| > 0$. Then,

$$\ell(N) = \ell(C). \tag{3.10}$$

Equivalently, $\ell(B) \subset \ell(C)$ and $\ell(D) \subset \ell(C)$.

Proof. (a) Suppose $|C| = 0$, and let $v, v' \in N$. By connectedness, there is a path

$$\left(v_0, v_1, \ldots, v_j\right): \quad v_0 = v, \quad v_j = v'$$

between v and v'. Note that $v_i \in B \cup D$ for all i, since $|C| = 0$. Observation 3.2 tells us that $\ell(v_i) = \ell(v_{i+1})$ for $0 \le i \le j - 1$ and so $\ell(v) = \ell(v')$. Since v and v' were arbitrary, all vertices in N must have the same label set. Thus, $\ell(N) = \ell(v)$ for any vertex v, proving the three claims. In the figure below, since $\ell(v_0) = \{2, 3, 4\}$, we must have $\ell(v_1) = \{2, 3, 4\}$, and we eventually obtain $\ell(v_j) = \{2, 3, 4\}$.
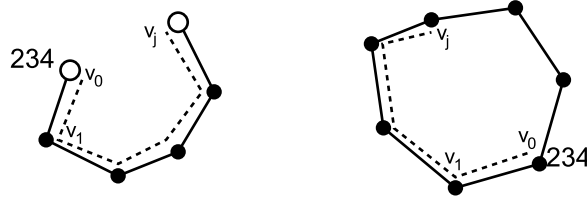
(b) Now, suppose $|C| > 0$, and let $c \in C$. Then, by connectedness, for any vertex $v \in B \cup D$, there is a path

$$\left(v_0, v_1, \ldots, v_j\right): \quad v_0 = v, \quad v_j = c$$

between v and c. Choose m such that $v_i \in B \cup D$ for $i < m$, but $v_m \in C$. Then,

$$\ell(v) = \ell(v_0) = \ell(v_{m-1}) \subset \ell(v_m) \subset \ell(C).$$

Hence, $\ell(v) \subset \ell(C)$ for all $v \in B \cup D$. Thus, $\ell(N) = \ell(B) \cup \ell(C) \cup \ell(D) = \ell(C)$. In the figure below, we have $i = 2$. In this case, we know that $\ell(v_0) = \ell(v_1) \subset \ell(v_2)$. $\quad\Box$
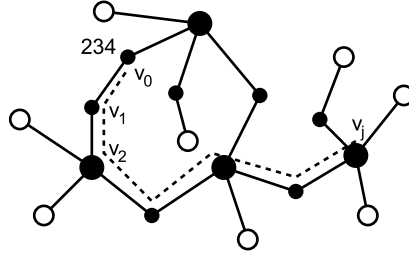
3.2.4   The Labeling Rules: Neighbor Perspective

The restrictions imposed by the label subset rule (d) can be alternately expressed in terms of neighbors. We can classify the neighbors of a given vertex by degree.

Definition 3.3. Given a vertex $v \in V$, we define $b(v)$, $c(v)$, and $d(v)$ by

$$b(v) = \{w \in B : (v, w) \in E\} \tag{3.11}$$

$$c(v) = \{w \in C : (v, w) \in E\} \tag{3.12}$$

$$d(v) = \{w \in D : (v, w) \in E\}. \tag{3.13}$$

These just represent the set of vertices adjacent to the vertex v in one of the sets B, C, or D. For the vertex v above, the sets $b(v)$ and $d(v)$ are indicated. Of course, since $v \in C$, $c(v)$ is empty.

We make the following observations regarding a vertex and its neighbors:

Observation 3.5. *Let $v \in B \cup D$, and let $w \in b(v) \cup c(v) \cup d(v)$. Then, $\ell(v) \subset \ell(w)$.*
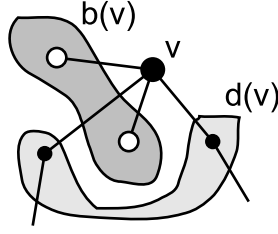
Proof. Consequence of the label subset rule (d). □

Observation 3.6. *Let $v \in C$. If $W = b(v) \cup d(v)$, then*

$$\ell(v) = \bigcup_{w \in W} \ell(w). \tag{3.14}$$

Proof. Let $w_1, w_2 \in b(v) \cup d(v)$ with $w_1 \neq w_2$. Then, by the three vertex rule (e) of nerves, $\ell(w_1) \neq \ell(w_2)$, and so $|\ell(w_1) \cup \ell(w_2)| \geq k + 1$. Note that $\ell(w_1) \cup \ell(w_2) \subset \ell(v)$ as well. Looking at the corresponding sizes of these label sets, we see that

$$k + 1 \leq |\ell(w_1) \cup \ell(w_2)| \leq |\ell(v)| = L(v) = k + 1.$$

Thus, $L(v) = |\ell(w_1) \cup \ell(w_2)| = k + 1$, and so

$$\ell(v) = \ell(w_1) \cup \ell(w_2) \subset \bigcup_{w \in W} \ell(w) \subset \ell(v),$$

since $\ell(w) \subset \ell(v)$ for all $w \in W$. This proves our claim. □

This last observation is probably the most non-intuitive, but important, basic result in this theory. It allows us to follow a certain label set $\ell(v)$ for a vertex $v \in D$ along a path to another vertex $v_2 \in D$, where the path consists of vertices all of which contain $\ell(v)$ as part of their label set. Returning to the applications of this theory to Sperner's Lemma, remember that graphs really represent paths of simplices, and we are essentially trying to figure out how to find all the vertices in C (the full cells), while starting at a vertex in D (full facets). This path-following method will be useful for finding full cells.

3.3 Minimal Nerve Graphs

In the above section, we have seen that the labelings of vertices in B and those in D are essentially the same. Also, adjacent vertices in $B \cup D$ are labeled identically. This is indicative of some redundancy in the nerve graph. It seems as though we could combine adjacent vertices in $B \cup D$ without really affecting the graph. Minimal nerve graphs eliminate this redundancy. They will also be useful for the Maximum Label Theorem, in the next section.

We will first define nerve reductions.

Definition 3.4. Let (v, d, w) be a path in a nerve graph (N, ℓ) with $v \in B \cup C \cup D$, $d \in D$, and $w \in B \cup D$. Then, the *nerve reduction of N with respect to (d, w)* is the nerve graph defined by $(N^-(d, w), \ell^-)$ where $N^-(d, w) = (V^-, E^-)$ with $V^- = V \setminus \{d\}$ and $E^- = E \setminus \{(v, d), (d, w)\} \cup \{(v, w)\}$. We let $\ell^-(v) = \ell(v)$ for all $v \in V'$, and we will refer to ℓ^- as simply ℓ.

Thus, the nerve reduction removes one of the extra vertices in D and connects its two neighbors together, as shown below.
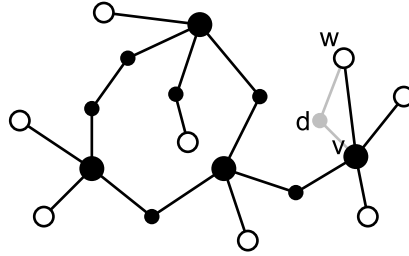
Observation 3.7. *A nerve reduction of (N, ℓ) is still a nerve graph.*

Proof. All properties of nerves except the label subset rule (d) follow because neither the degrees of the vertices v and w nor their labelings change. In a nerve reduction, the label subset rule is valid because if $\deg(v) = k + 1$ then $\ell(w) = \ell(d) \subset \ell(v)$, and if $\deg(v) \in \{1, 2\}$ then $\ell(w) = \ell(d) = \ell(v)$. $\square$

The following observation notes that, despite taking away vertices, the label set of a nerve reduction and the number of its vertices in C is the same as that of the original nerve graph.

**Observation 3.8.** *Let $(N^-(d, w), \ell)$ be a nerve reduction of a nerve graph (N, ℓ). Then, $\ell(N) = \ell(N^-)$ and $|C| = |C^-|$, the number of degree $k + 1$ vertices in N^-.*

Proof. By Observation 3.2, $\ell(d) = \ell(w) \subset \ell(N^-)$. Thus, $\ell(N) = \ell(N^-) \cup \ell(d) = \ell(N^-)$. Also, $|C| = |C^-|$ holds since the vertex d removed is in D. □

If there does not exist a nerve reduction of some nerve graph (N, ℓ), we will call (N, ℓ) a *minimal nerve graph*. Note that there are three types of minimal nerve graphs. The first type consists of two connected vertices in B. The second type consists of a single vertex in D, connected to itself. The last type consists of nerve graphs having vertices in C. Examples of these three types are shown below.
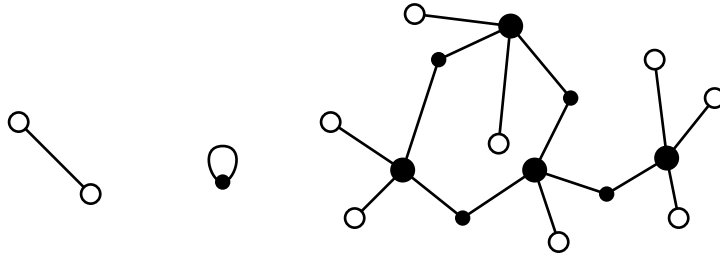
We make the following observation regarding minimal nerve graphs:

Observation 3.9. *Let (N, ℓ) be a minimal nerve graph, with $(v, w) \in E$. If $\deg(v) = 2$, then $\deg(w) = k + 1$; if $\deg(v) = 1$ and $(v, w) \in E$, then $\deg(w) \in \{1, k + 1\}$.*

Proof. If one of these statements did not hold, then it would be possible to find a nerve reduction. □

By this observation, we see that no vertex in D is adjacent to a vertex in either B or D. Hence, all vertices in D must be adjacent to exactly two degree $(k + 1)$

vertices (in C), and all vertices in B must be either adjacent to another vertex in B, or adjacent to a single vertex in C. This excludes, of course, the case when a vertex in D is connected to itself. These observations are reflected in the figure below.
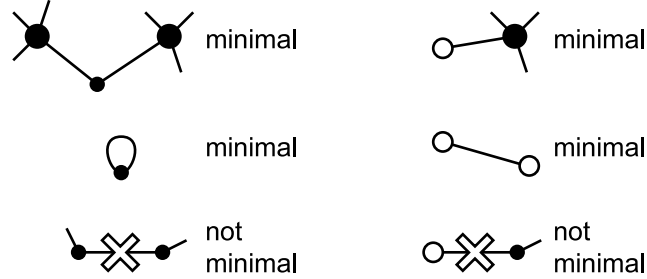
Suppose we have a nerve graph (N, ℓ). Consider any nerve reduction (N^-, ℓ) of (N, ℓ). This *new* nerve graph might also have a reduction, call it (N^{-2}, ℓ). Because (N, ℓ) is finite, we must eventually obtain a minimal nerve graph (N^{-m}, ℓ) for some m. We will call a minimal nerve graph obtained in this way $(\bar{N}, \ell)$. Note that this is not a well-defined process; there might be several different minimal nerve graphs which can be obtained in this way. This will not matter, so we will simply use $(\bar{N}, \ell)$ to mean one of the possible minimal nerve graphs.

Definition 3.5. Given a nerve graph (N, ℓ), a *minimalized nerve graph of* (N, ℓ), denoted by $(\bar{N}, \ell)$, is a minimal nerve graph in the sequence (N, ℓ), (N^-, ℓ), (N^{-2}, ℓ), (N^{-3}, ℓ), ..., obtained by iterating nerve reductions as long as possible.
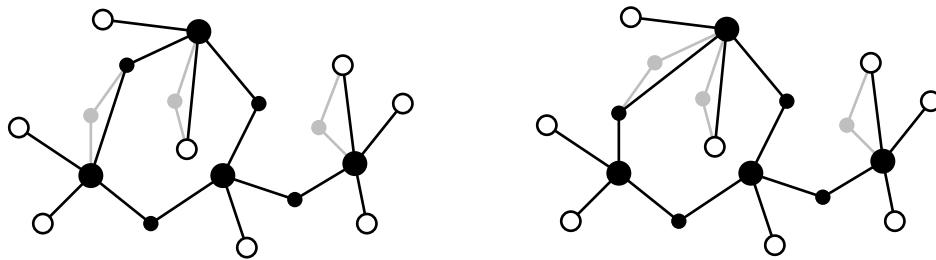
As we have stated, the minimalized nerve graph is not necessarily unique. The figure above shows two possibilities for the minimalized nerve graph for the nerve graph we have been using in this section.

The following lemma, which will be used in Section 3.5, follows directly from Observation 3.8.

Lemma 3.10. *Let* $(\bar{N}, \ell)$ *be a minimalized nerve graph of a connected nerve graph* (N, ℓ). *If* $|C| > 0$, *then* $|C|_N = |C|_{\bar{N}}$, $\ell(N) = \ell(\bar{N})$, *and* $L(N) = L(\bar{N})$.

3.4   Nerve Skeletons

In this section, we study *nerves skeletons*, which are labeled graphs derived from nerve graphs. Nerve skeletons are easy to work with, but have the same number of labels, and the same number of vertices in C, as their parent nerve graphs. The *Skeleton Lemma*, which we prove later in this section, verifies these properties. We can derive a nerve skeleton from any minimal nerve graph:

Definition 3.6. The *nerve skeleton* (N_S, ℓ_S) of a connected minimal nerve graph (N, ℓ) is defined by $N_S = (C, E_S)$, where $E_S = \{(v, w) : (v, d), (d, w) \in E \text{ for some } d \in D\}$. We let $\ell_S(v) = \ell(v)$ for $v \in C$, and will refer to ℓ_S by simply ℓ.

minimalized nerve graph   nerve skeleton
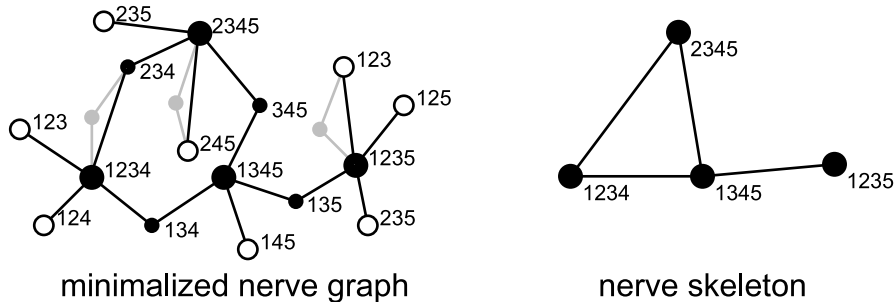
The nerve skeleton consists only of vertices in C, and basically serves as a description of paths between vertices. An example of a nerve skeleton is shown above. The

following proposition gives some important properties of nerve skeletons:

Proposition 3.11. *Given a nerve skeleton (N_S, ℓ), with $N_S = (V_S, E_S)$, of a connected minimal (n, k)-nerve graph (N, ℓ), then N_S is connected and $\ell : V_S \to I_{k+1}^n$. Moreover, if $(v, w) \in E_S$, then $|\ell(v) \cap \ell(w)| \geq k$.*

Proof. Note first that all vertices in N_S had degree $k + 1$ in the original nerve graph (N, ℓ), and so $\ell : V_S \to I_{k+1}^n$. Now, if $(v, w) \in E_S$, then $(v, d) \in E$ and $(w, d) \in E$ for some vertex d. By the label subset rule (d) of nerves, $\ell(d) \subset \ell(v) \cap \ell(w)$, and so

$$|\ell(v) \cap \ell(w)| \geq |\ell(d)| = k.$$

Finally, we verify that N_S is connected. If c and c' are two vertices in N_S, then there is a path

$$(c, d_0, c_1, d_1, \ldots, c_j, d_j, c')$$

in N connecting them. Observation 3.9 shows that, because N is minimal, we must have $d_i \in D$ and $c_i \in C$. But this means that $(c, c_1, c_2, \ldots, c_j, c')$ is a path in N_S connecting c and c'. Hence, N_S is connected. $\qquad\square$

We define a *skeleton graph* to be a pair (N_S, ℓ) with $N_S = (V_S, E_S)$ satisfying the three properties above. Thus, N_S is connected, $\ell : V_S \to I_{k+1}^n$, and if $(v, w) \in E_S$ then $|\ell(v) \cap \ell(w)| \geq k$. Clearly, a nerve skeleton is a skeleton graph.

We now give the *Skeleton Lemma*, which shows that the number of cell vertices $|C|_N$ in a nerve graph is the same as the number of vertices $|N_S|$ in its nerve skeleton. Also, the label set of the nerve graph is the same as the label set of its nerve skeleton.

Skeleton Lemma. *Let (N_S, ℓ) be the nerve skeleton of a connected minimal nerve graph (N, ℓ). If $|C| > 0$, then $|C|_N = |N_S|$, $\ell(N) = \ell(N_S)$, and $L(N) = L(N_S)$.*

Proof. First, $|C|_N = |N_S|$ by definition. Second, since $|C| > 0$, (3.10) in Proposition 3.4 gives $\ell(N) = \ell(C) = \ell(N_S)$. But this means that we must have $|\ell(N)| = |\ell(N_S)|$ or $L(N) = L(N_S)$. $\qquad\square$

3.5 The Maximum Label Theorem

This section contains the most important theorem for nerve graphs, which we call the *Maximum Label Theorem*. This theorem gives an inequality which is used directly to demonstrate the minimum number of full cells in our generalizations of Sperner's Lemma. The main result of the theorem is that

$$|C| \geq L(N) - k. \tag{3.15}$$

That is, the number of vertices in C is at least the total number of labels in the nerve graph, minus k. From the perspective of Sperner's Lemma, this means that the total number of fully-labeled simplices is greater than or equal to the number of vertices n on the polytope, minus the dimension k. Proving this theorem will bring us most of the way to proving our generalizations of Sperner's Lemma.

In this section, we will first review maximal paths. Then, we will prove the Maximum Label Theorem for skeleton graphs, which will imply the Maximum Label Theorem itself.

3.5.1 Maximal Paths

We first review some results from graph theory regarding maximal paths. Recall that a *path* is a sequence of vertices $(v_0, v_1, \ldots, v_j)$ in a graph such that no vertices are repeated, and two successive vertices v_i and v_{i+1} are connected.

Definition 3.7. A path $P = (v_0, v_1, \ldots, v_j)$ in a finite graph (or skeleton graph) C is said to be a *maximal path* if it is not strictly contained in any other path in the graph (skeleton graph).

This means that if a vertex $v \in C$ is not in a maximal path $P = (v_0, v_1, \ldots, v_j)$, then $(v, v_0) \notin E$ and $(v, v_j) \notin E$. Simply put, no more vertices can be tacked on to the beginning or the end of P.

The following lemma gives a fundamental result for maximal paths.
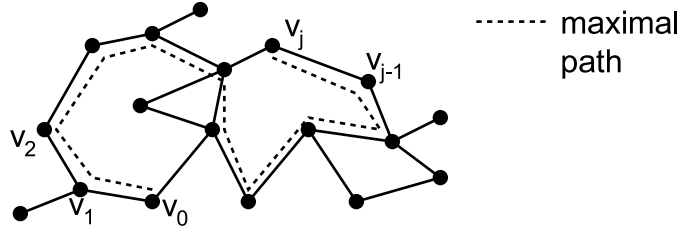
Lemma 3.12. *If $P = (v_0, \ldots, v_N)$ is a maximal path in a connected graph G, then $G - v_N$ is connected.*

Proof. We proceed by contradiction. Suppose that $G - v_N$ were not connected. Then, there exist two vertices w_1 and w_1' which do not have a path connecting them. However, since G is connected, they must both have a path to v_N: let these paths be $(w_1, w_2, \ldots, w_j, v_N)$ and $(w_1', w_2', \ldots, w_k', v_N)$.

Now, we cannot have both $w_j \in P$ and $w_k' \in P$, because this would give a path between w_j and w_k' in $G - v_N$, and hence between w_1 and w_1'. Let $w \in \{w_j, w_k'\}$ be the vertex which is not in P. Then, $(v_0, \ldots, v_N, w)$ is a path which contains our maximal path P, a contradiction. Hence, $G - v_N$ must be connected. $\qquad\square$

3.5.2 The Maximum Label Theorem

We will first prove the Maximum Label Theorem for skeleton graphs.

Maximum Label Theorem (skeleton graphs). *Let (N_S, ℓ) be a skeleton graph with $N_S = (C_S, E_S)$. Then, $L(N_S) - k \leq |C_S|$, where $L(N_S)$ is the number of labels used in the skeleton graph and $|C_S|$ is the number of vertices in the skeleton graph.*

Proof. We proceed by induction on $|C_S|$. For the base case, $|C_S| = 1$, we have

$$L(N_S) - k = L(C_S) - k = k + 1 - k = 1 = |C_S|.$$

For the induction step, note that, by Lemma 3.12 above, there is a vertex $c \in C_S$

such that $N_S - c$ is connected. But since $N_S - c$ is also a skeleton graph, we have

$$L(N_S - c) - k \le |C_S - c| = |C_S| - 1.$$

Now, $(c, d) \in E_S$ for some $d \in C_S$, since N_S is connected and $|C_S| > 1$. Note that

$$\ell(N_S) \;=\; \ell(N_S - c) \;\cup\; \ell(c) \;=\; \ell(N_S - c) \;\cup\; \big(\ell(c) \cap \big(\ell(d)\big)^c\big),$$

where $\ell(c) \cap (\ell(d))^c$ consists of the vertices in c but not in d. Because c and d share k labels, we know that $|\ell(c) \cap (\ell(d))^c| \le 1$. Hence,

$$L(N_S) \;=\; |\ell(N_S - c) \cup \big(\ell(c) \cap \big(\ell(d)\big)^c\big)| \;\le\; L(N_S - c) + 1 \le |C_S| + k.$$

$\square$

We can now prove the Maximum Label Theorem.

Maximum Label Theorem. *Given a connected k-nerve graph (N, ℓ) with $N = (B \cup C \cup D, E)$, which is not necessarily minimal, we have $L(N) - k \le |C|$.*

Proof. Note that for $|C| = 0$, we have shown in Proposition 3.4 that $L(N) = |\ell(N)| = k$, and so $L(N) - k = 0 \le 0 = |C|$.

Now, if $|C| > 0$, let $(\bar{N}, \ell)$ be a minimalized nerve graph of (N, ℓ), and let $(\bar{N}_S, \ell)$ be the nerve skeleton of this minimalized nerve graph. According to the Skeleton Lemma, $|C|_{\bar{N}_S} = |C|_{\bar{N}}$, and $L(\bar{N}_S) = L(\bar{N})$. Also, by Lemma 3.10 above, $|C|_N = |C|_{\bar{N}}$ and $L(N) = L(\bar{N})$. We have already proved that, since (N_S, ℓ) is a skeleton graph, $L(\bar{N}_S) - k \le |C|_{\bar{N}_S}$. By the equalities above, this gives us $L(N) - k \le |C|$. $\square$

3.6   Subnerves

We now consider *subnerves*, which consist of those vertices of a nerve graph which include all the labels in a certain label set. Formally, we make the following definition:

Definition 3.8. Given a label set $I' \subset I^n$ and a nerve graph (N, ℓ) with $N = (V, E)$, the *subnerve induced by* I' is the induced graph $(N(I'), \ell')$ with $N(I') = (V', E')$, where

$$V' = \{v \in V : I' \subset \ell(v)\}.$$

We let $B' = B \cap V'$, $C' = C \cap V'$, and $D' = D \cap V'$. Also, the edge set is defined by $E' = \{(v, w) : v, w \in V'\}$, and the labeling function by $\ell'(v) = \ell(v)$ for all $v \in V'$. We will refer to ℓ' by simply ℓ.

An example of a nerve graph and two of its subnerves is shown below:

nerve graph

{1,2,3} subnerve   {1,3} subnerve
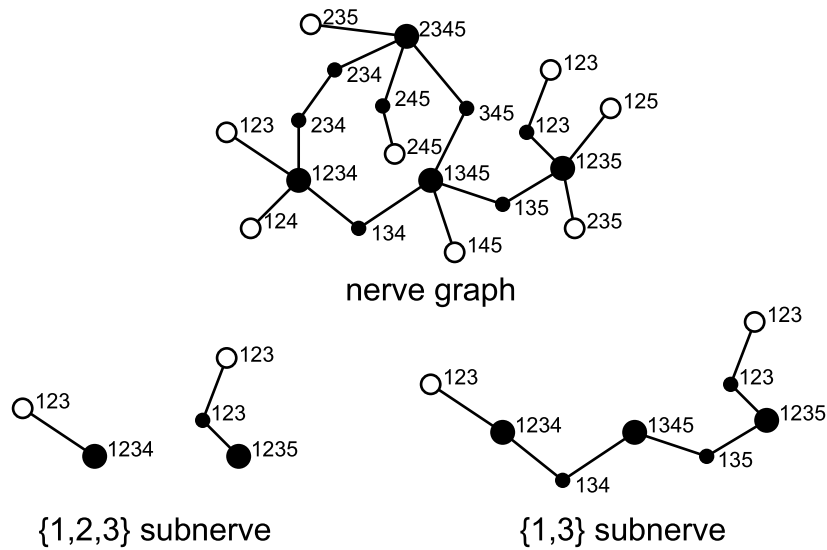
Note that subnerves of connected nerve graphs may be disconnected. Also, note that the form of the subnerves depends on the size of the label set. The one with k labels (the $\{1, 2, 3\}$ subnerve in the figure), for example, consists of a vertex in C connected with a string of vertices in D, which ends up at a vertex in B. However, the subnerve with $(k-1)$ labels (the $\{1, 3\}$ subnerve in the figure), consists of two vertices with degree 1, and several with degree 2. This last property is why subnerves are useful. We can use them to define a path through the nerve graph, or, in terms of the

polytopal problem, a path through the interior triangles of the subdivided polytope. These paths will be very useful in proving the main result.

We begin the study of subnerves with a fundamental lemma, which states that for any k element subset of the label set of a vertex $c \in C$, there is a vertex adjacent to c which has that exact label set.

Lemma 3.13. *Given a nerve graph* (N, ℓ) *with* $N = (V, E)$, *if* $v \in C$, *and* $\ell(v) = \{a_1, \ldots, a_{k+1}\}$, *then for all* $i, 1 \leq i \leq k+1$, *there is a vertex* $w \in b(v) \cup d(v)$ *such that*

$$\ell(w) = \ell(v) \setminus \{a_i\}. \tag{3.16}$$

Proof. Let $w_1, \ldots, w_{k+1}$ be the vertices connected to v. For a given i, we know that $|\ell(w_i)| = k$, $|\ell(v)| = k + 1$, and $\ell(w_i) \subset \ell(v)$. Thus, $\ell(w_i) = \ell(v) \setminus \{b_i\}$ for some label $b_i \in \ell(v)$. However, property (e) of nerves requires that $\ell(w_i) \neq \ell(w_j)$ if $i \neq j$, and thus $b_i \neq b_j$ if $i \neq j$.

However, since there are $k+1$ neighbors of v, $\{b_1, \ldots, b_{k+1}\}$ must be a permutation of $\ell(v) = \{a_1, \ldots, a_{k+1}\}$. This demonstrates (3.16). $\square$

The following proposition describes the form of a subnerve in terms of the possible degrees of its vertices.

Proposition 3.14. *Given an* (n, k)-*nerve graph* (N, ℓ) *with* $N = (V, E)$, *consider a subnerve* $(N(I'), \ell)$ *with* $N(I') = (V', E')$ *such that* $I' \in I_j^n$ *for some* j, $1 \leq j \leq k$. *If* $v \in B'$, *then* $\deg'(v) = 1$; *if* $v \in D'$, *then* $\deg'(v) = 2$; *and if* $v \in C'$, *then*

$$\deg'(v) = k + 1 - j. \tag{3.17}$$

Proof. First, note that if $v \in B' \cup D'$, and $(v, w) \in E$, then $I' \subset \ell(v) \subset \ell(w)$ by the label subset rule (d) of nerves, and so $w \in V'$. Hence, $(v, w) \in E'$. This demonstrates that $\deg'(v) = \deg(v)$ in this case.

Otherwise, let $v \in C'$. Let $a_i \in I'$ be a label in the label set I'. Then, by (3.16) in Lemma 3.13 above, there is some vertex w such that $(v, w) \in E$ and $\ell(w) = \ell(v) \setminus \{a_i\}$.

Hence, $w \notin V'$. However, if $a_i \in \ell(c)$ but $a_i \notin I'$, then there is some vertex w such that $(v, w) \in E$ and $\ell(w) = \ell(v) \setminus \{a_i\} \supset I'$. Hence, $w \in V'$. Thus, of the $k + 1$ neighbors connected to v in N, $|I'|$ are not connected to it in N', while $|\ell(v)| - |I'| = k + 1 - j$ are connected to it in N'. $\qquad\square$

Consider the subnerve $N(\{1, 2, 3\})$ induced from a $(5, 3)$-nerve graph in the figure below. Since $\{1, 2, 3\} \in I_3^5$, for $v \in C'$ we have $\deg'(v) = k + 1 - j = 3 + 1 - 3 = 1$. For the $N(\{1, 3\})$ subnerve, on the other hand, we have $\deg'(v) = 3 + 1 - 2 = 2$ for $v \in C'$. Note that the degrees of vertices in B' and D' does not change.

{1,2,3} subnerve   {1,3} subnerve
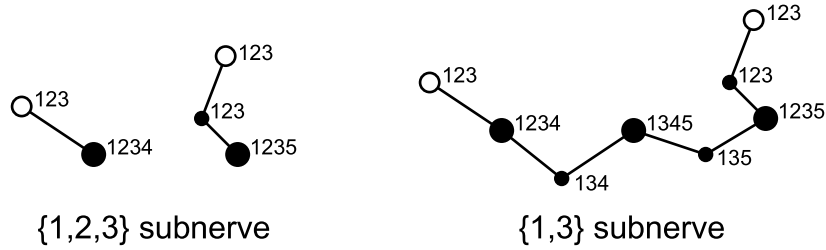
This proposition gives the following corollary, which is similar to Observation 3.1 and follows from the degree-edge relationship of graphs (3.6).

Corollary. *Given an (n, k)-nerve graph (N, ℓ) with $N = (V, E)$, consider a subnerve $(N(I'), \ell)$ with $N(I') = (V', E')$ such that $I' \in I_j^n$ for some j, $1 \le j \le k$. If $k - j \equiv 0$ mod 2, then $|B'| \equiv |C'|$ mod 2. Otherwise, if $k - j \equiv 1$ mod 2, then $|B'| \equiv 0$ mod 2.*

This completes our discussion of the theory of nerve graphs. Our results here are based entirely on graph theory, and form the major part of this thesis. The extensive development of this theory in this chapter greatly simplifies our main result, which we will discuss in the next chapter.

Chapter 4

GENERALIZATION OF SPERNER'S LEMMA FOR SIMPLICIAL FACETS

This chapter will demonstrate the proof of the Generalized Sperner's Lemma for polytopes with simplicial facets, which states that:

Generalized Sperner's Lemma (simplicial facets). *In a k-polytope P with simplicial faces having n vertices, subdivided with a Sperner Labeling, there are at least $n - k$ fully-labeled elementary simplices.*

In [1], Atanassov proved this result for $k = 2$. The case where $n = k + 1$ is just Sperner's Lemma itself.

We will first define a polytope and a *Sperner Labeling* for a polytope, expanding on the earlier discussion in Chapter 2. We will then demonstrate how nerve graphs can be introduced into the problem. Finally, using the results in the previous chapter, we will prove the above generalization.

4.1 Definition of a Polytope

We can define a polytope formally:

Definition 4.1. A *polytope* is a bounded set which is the intersection of finitely many closed halfspaces in $\mathbf{R}^k$.

Although polytopes are not always defined with the boundedness condition, we will only be concerned with bounded polytopes in this thesis, so this definition is

sufficient. A polytope can alternately be defined as the convex hull of a finite set of points in $\mathbf{R}^k$. For example, a k-simplex Δ_k can be defined in $\mathbf{R}^{k+1}$ by

$$\Delta_k = \{\mathbf{x} \in \mathbf{R}^{k+1} : \sum_1^{k+1} x_i = 1, x_i \geq 0\} = \text{conv}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d+1}\}.$$

Now, a *face* of a convex k-polytope P is any set $F = P \cap \{\mathbf{x} \in \mathbf{R}^k : \mathbf{c} \cdot \mathbf{x} = c_0\}$ where $\mathbf{c} \cdot \mathbf{x} \leq c_0$ for all $\mathbf{x} \in P$. A *facet* of a polytope is a face of the polytope of dimension $k - 1$, while an *edge* of a polytope is a face of dimension 1. Thus, for a 3-polytope, facets correspond to the "sides" or polygons on the boundary of the given polytope.

In this chapter, we will rely on our intuition regarding a polytope and its faces rather than its formal definition. For a complete discussion of polytopes, see [19]. When we write "polytope" without referring to a dimension, we will assume the dimension is k.

4.2   The Sperner Labeling

A *Sperner Labeling* is defined for polytopes which are subdivided into small simplices. This labeling is a simple extension of the labeling used for a triangle in the plane:
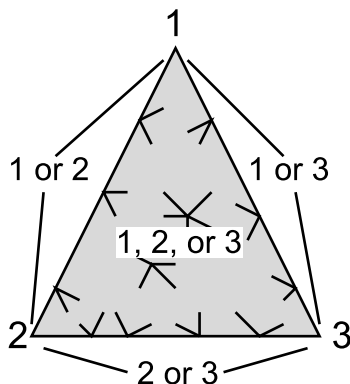
4.2.1   Labeling Simplices

We can easily generalize this labeling to simplices (that is, higher dimensional tri-angles). Suppose we are given the standard k-simplex $\Delta_k = (E_0, E_1, \ldots, E_k)$. This simplex can be subdivided into several smaller simplices, giving a finite vertex set $V = \{v_n\}$. A Sperner Labeling will assign one of $k+1$ possible labels to each of the vertices, with a few conditions, as explained below.

Definition 4.2. Given a simplex $\Delta_k = (E_0, E_1, \ldots, E_k)$ which is subdivided, so that it has a finite vertex set $V = \{v_n\}$, then a *Sperner Labeling* of that simplex is a map $\ell : V \to I^{k+1}$ such that:

1. $\ell(\{E_0, \ldots, E_k\}) = I^{k+1}$.

2. Given any set $\{E_{i_1}, E_{i_2}, \ldots, E_{i_m}\}$ such that $v \in \text{span}(E_{i_1}, E_{i_2}, \ldots, E_{i_m})$, we must have $\ell(v) \in \{\ell(E_{i_1}), \ldots, \ell(E_{i_m})\}$.
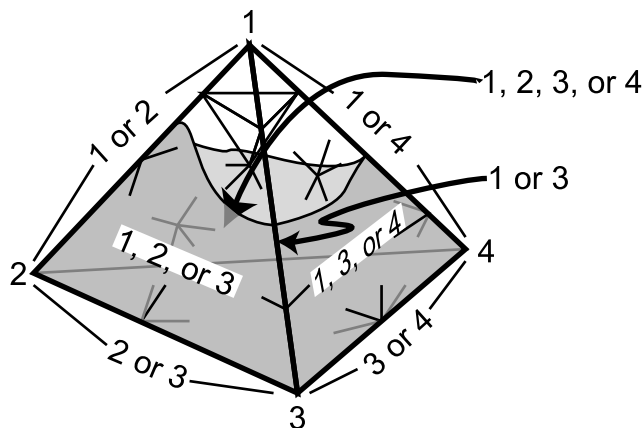
Here, the first condition requires that each of the main vertices receives a different label. For the second condition, note that a vertex may be spanned by more than one set of vertices. For example, if $v \in \text{span}(E_1, E_2)$, then $v \in \text{span}(E_1, E_2, E_3)$ as well. Because *both* of these statements are true, we must have *both* $\ell(v) \in \{\ell(E_1), \ell(E_2)\}$

and $\ell(v) \in \{\ell(E_1), \ell(E_2), \ell(E_3)\}$. Thus, we cannot have $\ell(v) = \ell(E_3)$. Because the rule must work for *any* spanning set, it must work for the *smallest* spanning set. The figure above shows the labeling scheme in 3 dimensions.

Thus, vertices on the $\{i, j, k\}$ facet receive either i, j, or k and vertices on the $\{i, j\}$ edge receive either i or j. Vertices in the interior may receive any of the possible labels. An example of a subdivided 3-simplex follows:
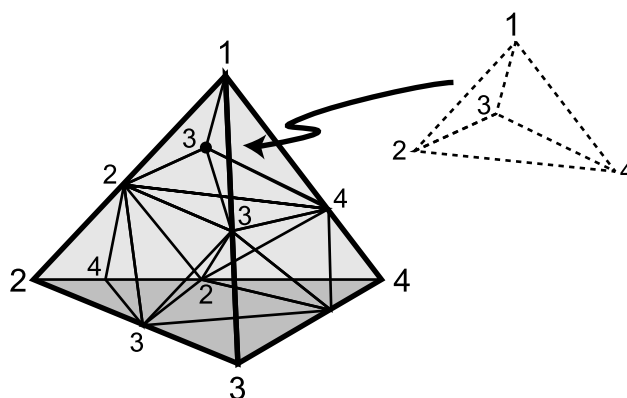
Note an interesting feature of this labeling: if we consider a single facet of a subdivided k-simplex with a Sperner Labeling, it corresponds to a $(k-1)$-simplex with a Sperner Labeling. This property suggests using induction in the proof of Sperner's Lemma, and it is indeed used in many proofs of the Lemma.

4.2.2 Labeling Polytopes

The rules for labeling a k-polytope with n vertices are similar; the only difference is that the n main vertices receive n different labels. The second condition in Definition 4.2 above does not change for a polytope. Note that a vertex in a polytope may have more than $k + 1$ options for its label. In the cube below, for example, the vertices on each facet have 4 possible labels, while the interior vertices have 8 possible labels, even though $k + 1 = 4$.
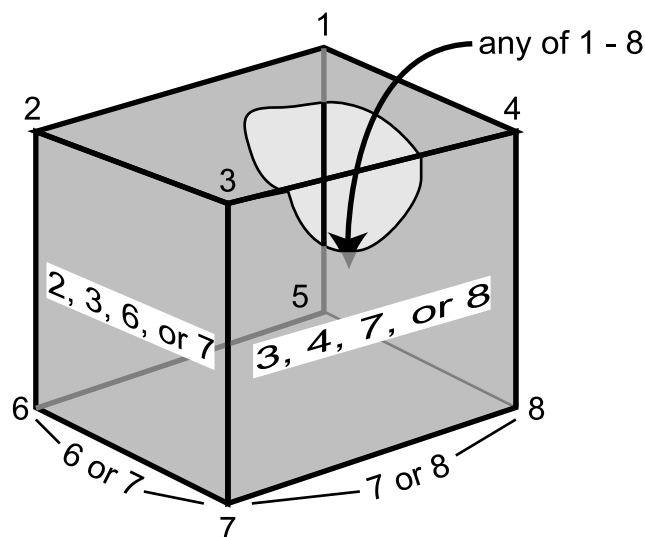
4.3 The Simplicial Graph

We will now describe how to obtain a nerve graph from a polytope with a Sperner Labeling. We can create a graph based on the polytope, which we call the *simplicial graph* of the polytope. This graph can be shown to be a nerve graph, allowing us to use the results from the previous chapter.

4.3.1 Preliminaries and Vertex Categories

Suppose we are given a k-polytope P with n vertices, which is subdivided, creating a vertex set V, and given a Sperner labeling $\ell : V \to I^n$. Given an arbitrary elementary simplex σ, we define $\ell(\sigma)$ to be the set of labels of the vertices of that simplex. Here, we allow σ to have any dimension, from 1 up to k. Thus, it could have anywhere from 2 to $k + 1$ labels.

We start by defining some special vertices to be used in our graph. These vertices correspond to simplices in our subdivided polytope (although they are not necessarily k-dimensional).

Definition 4.3. Given a triangulated polytope P, we let S be the set of elementary k-simplices, we let F be the set of elementary facets on the boundary of the polytope, and we let F_i be the set of elementary facets on the interior of the polytope.

This means elements of F are adjacent to exactly one k-simplex (which is in S), while elements of F_i are adjacent to exactly two k-simplices (which are in S). Note that $f \in F$ can be viewed as the *facet* of a simplex in S, and that $f_i \in F_i$ can be viewed as the facet of one of two simplices in S. The following figure shows the classification of simplices and facets in the triangulation of a 2-simplex:
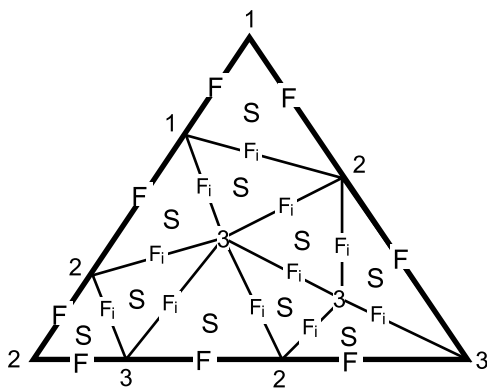
Every simplex in S has exactly $k + 1$ facets, each of which borders either another simplex in S or an elementary facet. As well, each elementary facet borders exactly one simplex in S. We will now use these adjacencies to create a graph out of the vertices in S, F, and F_i.

4.3.2   The Simplicial Graph

In this section, we define the simplicial graph. First, we will need to classify the elements in S, F, and F_i to be used. We will first split the vertices in S, F, and F_i up into several categories, corresponding to the vertex sets B, C, and D in the previous chapter, with the following definitions:

Definition 4.4. We define the sets B, C, D_c, and D_f as follows:

1. We will call a simplex $\sigma_f \in F$ with $l(\sigma_f) = k$ a *full facet*. We denote the set of full facets by B, for boundary.

2. We will call a simplex $\sigma \in S$ with $l(\sigma) = k + 1$ a *full cell*. We denote the set of full cells by C, for cell.

3. We will call a simplex $\sigma \in S$ with $l(\sigma) = k$ a *semi-full cell*. We denote the set of semi-full cells by D_c, for "cell" door.

4. We will call a facet $\sigma_f \in F_i$ with $l(\sigma_f) = k$ an *interior door*, or simply a *door*. We denote the set of interior doors by D_f, for "facet" door.

The following figure indicates each of these types of vertices for a sample subdivided polytope (in this case, $k = 2$, so the facets are just edges, and the cells just triangles).
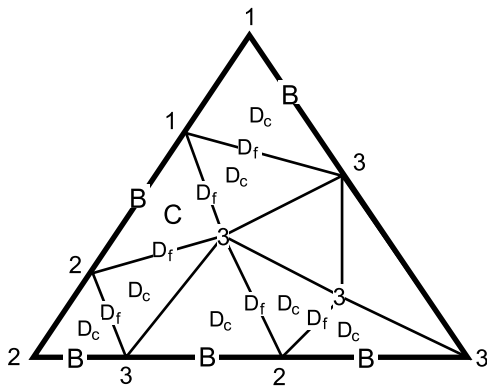
Note how closely these sets of simplices correspond to the sets B, C, and D in the previous chapter. As before, B and D have k labels, while C has $k+1$ labels. We have two separate groups of simplices for D because there are two types of such vertices: fully-labeled facets bordering two cells, and cells having only k different labels. The

elements of B and D_f are facets, or $(k-1)$ simplices, while the elements of C and D_c are cells, or k simplices.

We can use these vertices to define the simplicial graph, but we still need to describe the edges to be used. Given two simplices σ and τ, we define their intersection to be $\sigma \cap \tau$. Note that if σ and τ may or may not be the same dimension. A few examples of simplex intersections are shown below.

$$\sigma \cap \tau = 234$$
$$\sigma \cap \upsilon = 4$$
$$\sigma \cap \omega = 34$$
$$\tau \cap \upsilon = 45 = \upsilon \cap \omega$$
$$\tau \cap \omega = 345 = \omega$$
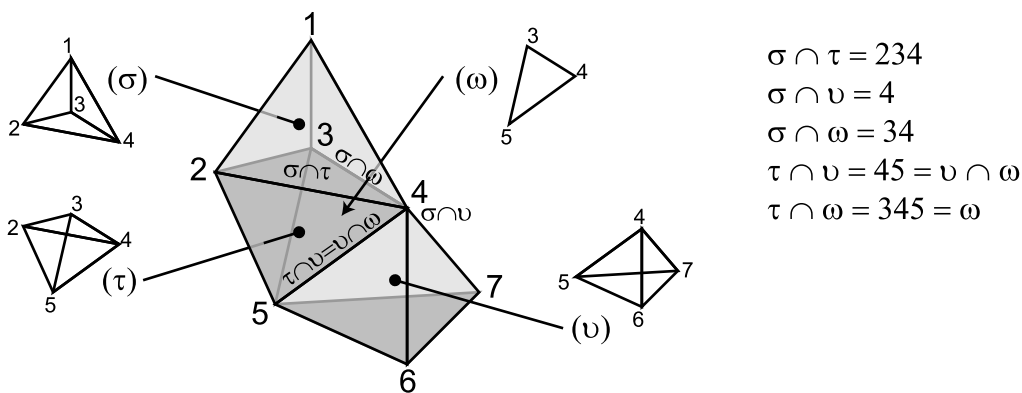
Note that it is possible to intersect a 2-simplex with a 3-simplex, and in general a k_1-simplex with a k_2-simplex. It is also possible to have empty intersections. For example, $\sigma \cap (567) = \emptyset$ in the above figure. This intersection is the basis for our definition of an edge in the simplicial graph:

Definition 4.5. The *edge set* E in the simplicial graph of a polytope, with vertex sets B, C, D_c, and D_f, is defined by

$$E = \{(\sigma, \tau) : \sigma \in B \cup D_f, \tau \in C \cup D_c, \sigma \cap \tau = \sigma\}.$$

Thus, we put an edge between a $(k+1)$-simplex τ and a k-simplex σ if σ is one of the facets of τ and both σ and τ are in our vertex set. This allows us to explicitly define the simplicial graph:

Definition 4.6. The *simplicial graph* of a polytope P is the graph $G(P) = (A, E)$, where $A = B \cup C \cup D_c \cup D_f$ and E is the edge set defined above.

The following figure shows the simplicial graph of the 2-polytope given previously.
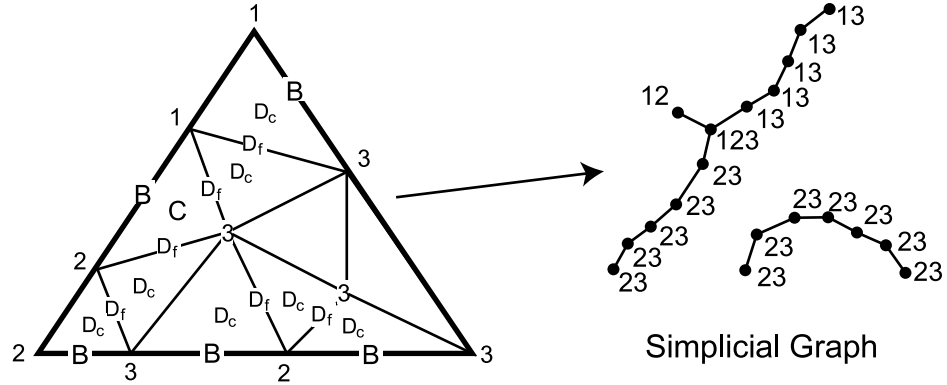
Simplicial Graph

Note that each vertex in $G(P)$ has a natural labeling in $I_k^n \cup I_{k+1}^n$: if $\sigma \in A$, we let the labeling be $\ell(\sigma)$, as defined for simplices. Our next task is to demonstrate that the simplicial graph, together with the labeling function ℓ, is actually a nerve graph.

4.3.3 Simplicial Graph and Nerve Components

We have already created a graph using the polytope P, and now we need to show that it is a nerve graph. Note that $G(P)$ need not be connected; of course, nerve graphs do not have to be connected either.

Theorem 4.1. *The labeled simplicial graph $(G(P), \ell)$ of a polytope P is actually a nerve graph.*

Proof. We need to verify each of the five conditions for a nerve graph.

We must first verify the degree range rule (a), which states that

$$\deg(v) \in \{1, 2, k+1\}.$$

First, if $v \in B$, then $v \in F$ as well, so v borders some simplex $w \in S$. Thus, $|\ell(w)| \geq |\ell(v)| = k$, and so $w \in D_c \cup C$, giving $(v, w) \in E$. No other edges are possible because v is the facet of only one simplex. Hence, $\deg(v) = 1$.

Similarly, if $v \in D_f$, then v borders two simplices $w_1, w_2 \in D_c \cup C$, so that $(v, w_1) \in E$ and $(v, w_2) \in E$. Hence, $\deg(v) = 2$.

Now, if $v \in C$, then v has $k + 1$ facets. Since $L(v) = |\ell(v)| = k + 1$, if w_i is a facet of v, then $L(w_i) = k$. Hence, $(w_i, v) \in E$. Since this holds for all facets w_i, we must have $\deg(v) = k + 1$. Again, no other edges are possible since any vertex connected to v must be one of its facets.

Finally, if $v \in D_c$, then v has $k + 1$ facets. However, since $L(v) = k$, only two of these facets have k labels. This is because there are two vertices α_i and α_j with the same label, and only the facets labeled by $\ell(v) \setminus \ell(\alpha_i)$ and $\ell(v) \setminus \ell(\alpha_j)$ have k different labels. Thus, v will be connected to these two facets only, so that $\deg(v) = 2$. This completes the proof of the degree range rule (a).

The degree $k + 1$ rule (b) is trivial, because the only vertices of degree $k + 1$ are those in C, with $k + 1$ labels. Also, there are only vertices between $(B \cup D_f)$ and $(C \cup D_c)$ by definition, precluding edges between two vertices in C. This demonstrates the edge limit rule (c).

For the label subset rule (d), we must show that

$$\text{if } (v, w) \in E, \text{ and } \deg(v) \leq \deg(w), \text{ then } \ell(v) \subset \ell(w).$$

By definition of the edge set, if $(v, w) \in E$ then we can assume $v \in B \cup D_f$ and $w \in C \cup D_c$. In this case, $\deg(v) \leq \deg(w)$ and since v is a facet of w, $\ell(v) \subset \ell(w)$.

Finally, we must verify the three vertex rule (e), which states that

$$\text{if } \ell(v) \in I_{k+1}^n, (v, w_1), (v, w_2) \in E \text{ with } w_1 \neq w_2, \text{ then } \ell(w_1) \neq \ell(w_2).$$

Note that for these requirements, w_1 and w_2 must be two distinct facets of a full cell $v \in C$. These facets must correspond to *distinct* k-element subsets of $\ell(v)$. Since $|\ell(v)| = k + 1$, w_1 and w_2 cannot have the same label set.

This completes the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following corollary allows us to apply the Maximum Label Theorem to the connected components of $G(P)$:

Corollary. *If N_i is a connected component of $G(P)$, the simplicial graph of an (n,k)-polytope P, then (N_i, ℓ) is a connected (n,k)-nerve graph.*

Before we move on to a proof of Sperner's Lemma, there are a few details of notation which should be noted. In general N_i will correspond to a connected component of the simplicial graph $G(P)$. The *set* of connected components is given by $N_1, N_2, \ldots, N_m$. The *label set* of a component N_i is thus given by

$$\ell(N_i) = \bigcup_{\sigma \in A_i} \ell(\sigma),$$

which corresponds to the definition of labels given for nerve graphs. We will refer to a connected component N_i of the simplicial graph of a polytope as a *nerve component* of that polytope. When we write N_i, we will assume the existence of a labeling function ℓ, as in the corollary above. Nerve components which give us the requisite number $n - k$ of full cells are called *full nerve components*, and their precise definition follows:

Definition 4.7. A *full nerve component* of a polytope P is a connected component N_i of the polytope's simplicial graph $G(P)$ which uses all n possible labels. That is, $L(N_i) = n$ or $\ell(N_i) = \ell(P)$.

In the figure below, we have $n = 3$ and $k = 2$. Thus, a full nerve component must use all of the labels, $\{1, 2, 3\}$. The nerve component on the left uses all, making it a full nerve component, while the one on the right only uses 2 and 3.

full nerve component   not full
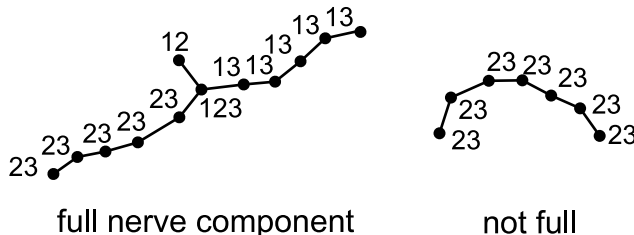
We can use the Maximum Label Theorem to show that these full component graphs have $n - k$ full cells. We will see in the following sections that, in order

to prove Sperner's Lemma and its generalizations, it will be sufficient to prove the existence of a full nerve component.

4.4 Proof of Sperner's Lemma

Although the theory of nerve graphs is most useful in proving the main theorem, it can also aid in the proof of Sperner's Lemma itself. Since it is used in the proof of the generalized Sperner's Lemma for simplicial facets, we will now prove Sperner's Lemma:

Sperner's Lemma. *In a k-simplex with a Sperner Labeling, there are an odd number of fully-labeled elementary simplices (full cells).*

Proof. We proceed by induction. First, the $k = 1$ case just represents the division of a line segment into several shorter segments, as shown below.
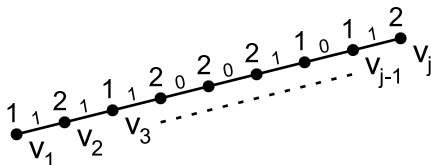
By the labeling assumption in the theorem, these vertices are mapped by ℓ to the vertex set I^2. Let us call these vertices $v_1, v_2, \ldots, v_j$, as shown above, with $\ell(v_1) = 1$ and $\ell(v_j) = 2$. Then,

$$\sum_{i=1}^{j} \big(\ell(v_i) + \ell(v_{i+1})\big) \equiv \ell(v_1) + \ell(v_j) \equiv 1 \mod 2.$$

The first equality follows since each of the vertices $v_2, v_3, \ldots, v_{j-1}$ is counted twice in the sum. Thus, there must be an odd number of terms which satisfy

$$\ell(v_i) + \ell(v_{i+1}) \equiv 1 \mod 2.$$

Of course, each of these terms corresponds to a fully-labeled edge, so we have proved the base case.

Now, assume the theorem holds for a $(k-1)$ simplex, and suppose we are given a k-simplex P_k with a Sperner Labeling from I^{k+1}.

Consider the simplicial graph N of the simplex P_k, and let (N', ℓ) with $N' = N(I'), I' = I^k$ be the *subnerve* induced by the label set $I^k = \{1, 2, \ldots, k\}$. Recall that this subnerve will include only those vertices in N which have all of the labels in I^k. Then, by the corollary to Proposition 3.14, we must have $|B'| \equiv |C'| \mod 2$. Since B' consists of all full facets on the $(12 \cdots k)$ facet of the simplex, $|B'|$ is odd by induction. Also, $|C| = |C'|$, since $I^k \subset \ell(v)$ for all $v \in C$. Thus, $|C|$ is odd. $\qquad \square$

The idea for this proof is shown in the figure below. We look at the induced subnerve for the simplicial graph, which might not be connected. Then, a proposition proved in our discussion on subnerves indicates that the parity of $|B'|$ (that is, the number of full facets on the $\{1, 2\}$ edge) must be the same as the parity of $|C|$.

triangulated simplex

simplicial graph

induced subnerve
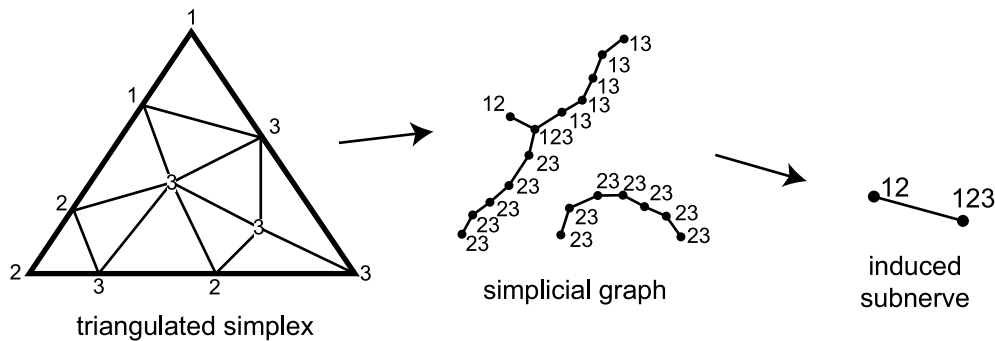
4.5 Generalized Sperner's Lemma for Simplicial Facets

The primary difficulty remaining in proving the generalized version of Sperner's Lemma is finding a full nerve component in the polytope.

The following lemma will show, in the case where the facets of the polytope are all simplicial, that there is at least one nerve component which uses an odd number of full facets on each of its faces. This is, of course, a full nerve component.

Note that the *simplicial facet* requirement prohibits polytopes such as cubes, as shown below; each facet of such a k-polytope must be a $(k-1) - simplex$, having exactly k labels. It turns out to be easier to prove the theorem in this case.

simplicial facets   non-simplicial facets
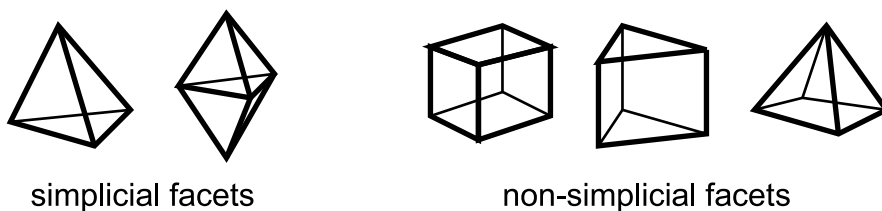
We will now show that we can assign a single parity value to each nerve component N_i of a k-polytope. Note the following general observation on subnerves

Observation 4.2. *Let N_i be a nerve component and suppose $I' \subset \ell(N_i)$ with $|I'| = k - 1$. Then, for the subnerve $(N_i(I'), \ell)$, we have $|B_i'| \equiv 0 \mod 2$. That is, the number of vertices in B_i having all of the labels in I' is even.*

Proof. By the corollary to Proposition 3.14, since $k - |I'| = 1$ is odd, $|B'| \equiv 0 \mod 2$. $\qquad\square$

Now, we show that the parity of the number of full facets used by a nerve component for two adjacent facets of a polytope is the same. First, we will introduce some new notation.

Definition 4.8. Given a nerve component N_i and a face F of a k-polytope P, we let

$$f_i(F) = |\{b \in B_i : \ell(F) \subset \ell(b)\}|.$$

Note that because F is a *face*, and not necessarily a *facet* of the polytope, it may have any dimension less than k. Hence, this definition applies for an edge of a 3-polytope, as well as one of its facets.

Lemma 4.3. *Let N_i be a nerve component of a k-polytope with simplicial facets, and let F_1, F_2 be two faces of the polytope sharing a common $(k-2)$-dimensional face E. Then, $f_i(F_1) \equiv f_i(F_2) \mod 2$.*

Proof. Let $I' = \ell(E)$ be the label set of the edge E, and note that E borders only these two facets of the polytope. Consider the subnerve $(N_i(I'), \ell)$, the vertices of which all contain the label set I'. Because the polytope has simplicial facets, $|I'| = k - 1$, and so $|B_i'| \equiv 0 \mod 2$ by Observation 4.2 above.

This means that $|B_i'| = f_i(F_1) + f_i(F_2)$, because if $v \in B_i' = B_i \cap N_i(I')$, then v must occur on either F_1 or F_2. As well, all vertices in B_i on one of the facets F_1 or F_2 must be contained in $N_i(I')$. Thus, we have $f_i(F_1) \equiv f_i(F_2) \mod 2$. $\qquad\square$

The figure below shows the consequence of the above lemma. The number of full facets used on the $\{1, 2, 3\}$ facet and the number on the $\{1, 3, 4\}$ facet must have the same parity. Here, the shared $(k-2)$-facet is $\{1, 3\}$.
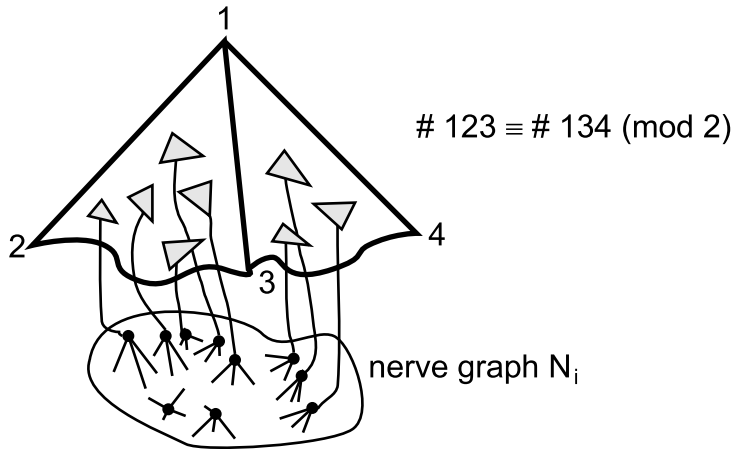
What the above lemma means is that each pair of adjacent facets of the polytope has some parity associated it for every nerve component. By extension, the parity must be the same for all facets of the polytope, that is, the parity of $f_i(F)$ depends

only on the nerve component N_i chosen and not on the facet F. Thus, if we define $\rho(N_i) = f_i(F) \mod 2$ for some facet F, then $\rho(N_i) \equiv f_i(F_j) \mod 2$ for *all* facets F_j.

If we can show the sum of the parities of all nerve components is odd, then there must be some nerve component with odd parity. In the following theorem, we show the existence of an odd nerve component, and show that it must be full.

Full Nerve Component Theorem. *Given the simplicial graph $G(P)$ of a polytope P with nerve components $N_1, \ldots, N_m$, there is some full nerve component N_i.*

Proof. According to Sperner's Lemma, a facet F of P must have an odd number of full facets, since it is a simplex with a Sperner labeling, and so $\sum_{i=1}^{m} f_i(F) \equiv \mod 2$. In terms of the nerve components, this gives

$$\sum_{i=1}^{m} \rho(N_i) \equiv \sum_{i=1}^{m} f_i(F) \equiv 1 \mod 2.$$

Hence, there is at least one nerve component N_i with $\rho(N_i) = 1$. This nerve component must then use at least one full facet from each facet of the polytope and therefore all possible labels. Thus, N_i is full. $\qquad\square$

Of course, a full nerve component will give us the desired number of full cells. We can now present the generalization in the case of simplicial facets.

Generalized Sperner's Lemma (simplicial facets). *In a k-polytope P with simplicial faces having n vertices, subdivided with a Sperner Labeling, there are at least $n - k$ fully-labeled elementary simplices.*

Proof. Let N_i be the full nerve component in P's reduced simplicial graph. Then, $\ell(N_i) = k$, and so $|C_i| \geq n - k$, according to the Maximum Label Theorem. $\qquad\square$

Given the machinery we developed for nerve graphs, it is easier to prove this theorem. Note that this proof is constructive. In order to find all full cells in the polytope, it is sufficient to find all the nerve components, and this is possible since, by induction, we can find all full facets of the polytope.

Chapter 5

GENERALIZATIONS FOR DIMENSIONS 2 AND 3

This chapter describes how Sperner's Lemma can be further generalized in dimensions 2 and 3. Specifically, in Section 5.1 we can show that in a polygon with n vertices but only $m \leq n$ labels, there are still at least $m - 2$ full cells. We also prove that the parity of full cells is the same as that of m in the planar case. In Section 5.2, we make a conjecture regarding the subdivision of a polygon into n-gons rather than triangles. In Section 5.3, we prove the existence of at least $n - 3$ full cells in the triangulation of an arbitrary 3-polytope.

5.1 Parity and Arbitrary Labeling in 2 Dimensions

There are a number of planar generalizations of Sperner's Lemma which can be proved using the theory of nerve graphs. In this section, we extend Sperner's Lemma in the plane to allow for more general labelings of the main vertices, and we also prove the parity claim.

We begin with some notation. When referring to a vertex v_j with $j \notin \{0, \ldots, m\}$ in a set of vertices $\{v_0, v_1, \ldots, v_m\}$, we will mean $v_{j'}$, where $j' = j \mod (m + 1)$. This notation allows us to denote a 2-polytope, or polygon, by $(v_0, v_1, \ldots, v_m)$, with the condition that $E_j = (v_j, v_{j+1})$ is an edge of the polygon for all i. Note that this includes the case $E_m = (v_m, v_0) = E_{-1}$.

Now, given a polygon $P = (v_0, v_1, \ldots, v_{n-1})$, we will say that the main vertices of P are *properly m-labeled* if they are labeled by $\ell : \{v_j\}_{j=0}^{n-1} \to I^m$ for $3 \leq m \leq n$ such that $\ell(v_0) = 1$, $\ell(v_{n-1}) = m$, and if $0 \leq j < l \leq n - 1$ then $\ell(v_j) \leq \ell(v_l)$. This

restriction makes sure that main vertices having the same label are adjacent. For example, figure (a) below has an illegal labeling because the vertices labeled by 2 are separated, while the octagon in figure (b) is properly 5-labeled.

(a) not properly 5-labeled   (b) properly 5-labeled
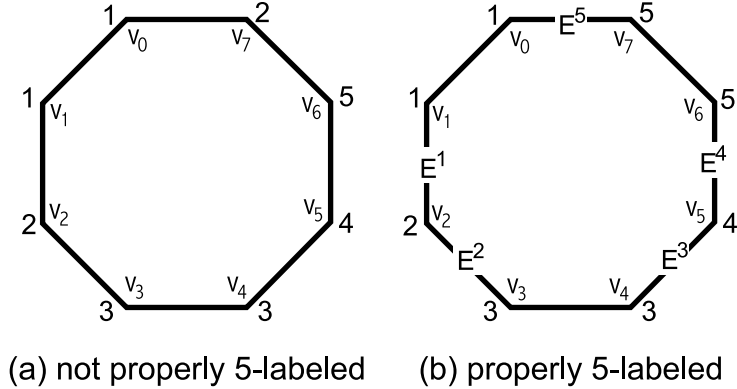
Based on the labeling rules for the main vertices of a properly m-labeled polygon, we make the following observation:

Observation 5.1. *Given a properly m-labeled polygon $P = (v_0, \ldots, v_{m-1})$, if $0 \leq j \leq m - 1$, then there is a unique edge $E_{j'} = (v_{j'}, v_{j'+1})$ with $\ell(E_{j'}) = \{j, j + 1\}$.*

Given $0 \leq j \leq m - 1$, we let $E^j = E_{j'}$ for the edge $E_{j'}$ defined as in the above observation. Note that these edges correspond to the edges of the above octagon having 2 different labels. With the above restriction on the labels of the main vertices, the Sperner Labeling is completely analogous. For example, on a (23) edge, a vertex may be labeled by either 2 or 3, while on a (33) edge, it must be labeled by 3.

Recall that in Definition 4.8 we defined $f_i(F)$ for a nerve component N_i and a face F of a polytope by

$$f_i(F) = |\{b \in B_i : \ell(F) \subset \ell(b)\}|,$$

the number of full facets in the nerve component N_i containing all the labels of F.

We now present our first theorem:

Theorem 5.2. *Let* $P = (v_0, v_1, \ldots, v_{n-1})$ *be a properly* m*-labeled polygon with edges* E^j *defined as above. If* P *is triangulated and given a Sperner Labeling, and there are* $|C|$ *full cells in the polygon, then* $|C| \geq m - 2$.

This generalization extends the proof to an arbitrary number of labels (not necessarily equal to the number of vertices). The case where $n = m$ was proved by Atanassov in [1]. In the figure below, we have a hexagon with 4 different labels, which must have at least 2 full cells.
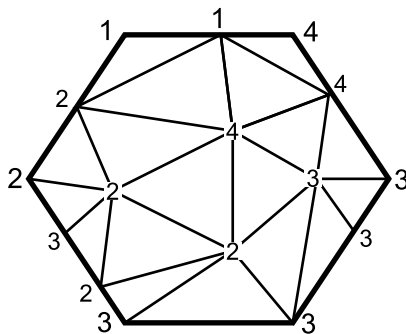
For the theorem above, it suffices to prove the existence of a full nerve component. Note that with the conditions given, the simplicial graph will be an $(m, 2)$-nerve graph rather than an $(n, 2)$-nerve graph, because there are only m labels.

Full Nerve Component Theorem (2 dimensions). *There is some full nerve component in any properly* m*-labeled polygon* P *which is triangulated and given a Sperner Labeling.*

Proof. Let $j \in I^m$. Suppose that, as in Observation 5.1, $\ell(E^{j-1}) = \{j-1, j\}$ and $\ell(E^j) = \{j, j+1\}$. By the corollary to Proposition 3.14, for the subnerve (N', ℓ) with $N' = N(I'), I' = \{j\}$ we must have $|B'| \equiv 0 \mod 2$. Since facets in B containing the vertex j can occur only along the E^{j-1} or E^j edge, for any nerve component N_i we must have $f_i(E^{j-1}) \equiv f_i(E^j) \mod 2$.

By extension, we see that $f(E^j) \equiv f(E^l) \mod 2$ for any j, l. Thus, if we define $\rho(N_i) = f_i(E^0) \mod 2$ for a nerve component N_i, then we have $\rho(N_i) \equiv f_i(E^j) \mod 2$ for all j.

Now, suppose the simplicial graph N has m components $N_1, N_2, \ldots, N_m$. Then, since $\sum_{i=1}^{m} f_i(E^0)$ is odd by induction, we have

$$\sum_{i=1}^{m} \rho(N_i) \equiv \sum_{i=1}^{m} f_i(E^0) \equiv 1 \mod 2.$$

Hence, there is some odd nerve component N_i. Since this uses a full facet from every full edge of the polygon, it must be a full nerve component. □

The figure below shows the simplicial graph for a properly 4-labeled hexagon. Note that it has two nerve components, one of which must be full.
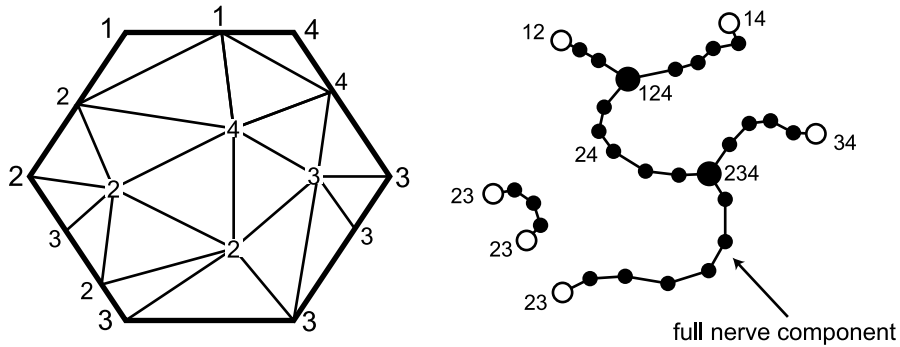
Of course, the existence of this full nerve component of the $(m, 2)$-nerve graph N gives us $m - 2$ full cells, by the Maximum Label Theorem, proving Theorem 5.2.

We now prove a theorem regarding the parity of the number of full cells.

Theorem 5.3. *Let $P = (v_0, v_1, \ldots, v_{n-1})$ be a properly m-labeled polygon with edges E^j defined as above. If P is triangulated and given a Sperner Labeling, and there are $|C|$ full cells in the polygon, then $|C| \equiv m \mod 2$.*

Proof. Note that, as Observation 3.1 states, $|B| \equiv |C| \mod 2$. Since the number of full facets in B on any full edge E^j of the polygon is odd, we must have $|B| \equiv m \mod 2$. Hence, $|C| \equiv m \mod 2$. $\qquad\qquad\qquad\qquad\square$

5.2 Arbitrary Subdivisions in 2 Dimensions

It is possible to consider the subdivision of a polygon in the plane by n-gons rather than triangles, as in the following hexagon subdivided into small 4-gons.
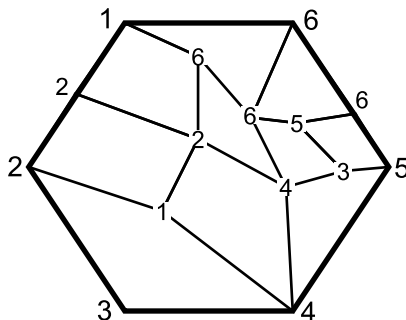
The simplicial graphs created from such figures are not nerve graphs, because vertices may have any degree from 1 up to q, and any number of labels from 2 up to q. It would be worthwhile to study the kind of graph which arises from such subdivisions. By extending the results of Chapter 3 to this new type of graph, we believe the following conjecture can be proved:

Conjecture. *Let $P = (v_1, v_2, \ldots, v_n)$ be a properly m-labeled polygon which is subdivided into several smaller q-gons and given a Sperner Labeling. Then, there are at least $m - q + 1$ full cells in the subdivision, and the number of full cells has the same parity as $m - q + 1$.*

The existence of a cubical full cell in cubical subdivisions of hypercubes is well-known. Cubical subdivisions are studied in [4] and [5].

5.3 Generalized Sperner's Lemma in 3 Dimensions

In this section, we will demonstrate the existence of a full nerve component in *any* 3-polytope, giving us a constructive proof for the full generalization of Sperner's Lemma to polytopes in dimension $k = 3$.

We must come up with a new notion of nerve component parity $\rho(N_i)$, and thus a new notion of parity for a given facet (possibly non-simplicial) of a 3-polytope.

It will be necessary to expand the definition of f_i. For a nerve component N_i, a face or set of vertices E, and a facet F, we define

$$f_i(E)[F] = |\{b \in B_i : \ell(E) \subset \ell(b), b \text{ occurs on } F\}|.$$

Thus, $f_i(E)[F]$ is the number of full facets occurring on F with the label set E.

Our presentation of the main proof of this section is similar to that in Section 4.5. We begin with some lemmas establishing the notion of parity for a nerve component of a 3-polytope.

Lemma 5.4. *Let* $F = (v_0, v_1, \ldots, v_m)$ *be a facet of a 3-polytope. Given a nerve component* N_i, *if* $|j - l| > 1$, *so that* (v_j, v_l) *is not an edge of the polytope, then* $f_i(v_j, v_l)[F] \equiv 0 \mod 2$. *Also,* $f_i(E_{j-1})[F] \equiv f_i(E_j)[F] \mod 2$ *for all* j.

Proof. First, suppose that $|j - l| > 1$, and thus (v_j, v_l) is not an edge of the polytope. Let $I' = \ell(v_j) \cup \ell(v_l)$. Since $|I'| = 2 = k - 1$, Observation 4.2 shows that $|B_i'|$ is even in the subnerve (N', ℓ) with $N' = N_i(I')$. But since (v_j, v_l) is not an edge of the polytope, the only full facets in the 3-polytope using both of these vertices must occur on the facet F. Hence, $f_i(v_j, v_l)[F] = |B_i'| \equiv 0 \mod 2$.

Now, consider the case with edges E_i and E_{i+1}. Note that any full facet (v_j, v_{j_2}, v_{j_3}) with the label v_j is counted in both the sum $f_i(v_j, v_{j_2})[F]$ and the sum $f_i(v_j, v_{j_3})[F]$. Thus, the sum of all full facets with the label v_j, which is just $f_i(v_j)[F]$, can be computed by

$$2f_i(v_j)[F] = \sum_{v \in F, v \neq v_j} f_i(v_j, v)[F].$$

Thus, the sum on the right is even. Since many of the terms $f_i(v_j, v)[F]$ will also be even, we see that

$$0 \equiv \sum_{v \in F, v \neq v_j} f_i(v_j, v)[F] \equiv f_i(E_{j-1})[F] + f_i(E_j)[F] \mod 2,$$

since (v_j, v) is an edge only if it is E_{j-1} or E_j. Thus, $f_i(E_{j-1})[F] \equiv f_i(E_j)[F] \mod 2$, proving our second claim. $\square$

Consider the (12345) facet in the figure below. Let $n(j_1 j_2 j_3)$ denote the number of full facets with the labels j_1, j_2, and j_3 used by some nerve component. The above lemma tells us that

$$n(132) + n(134) + n(135) \equiv 0 \mod 2,$$

since the left side of the equation is just the number of full facets containing the labels 1 and 3. Moreover, we have

$$n(123) + n(124) + n(125) \equiv n(234) + n(235) + n(231) \mod 2.$$
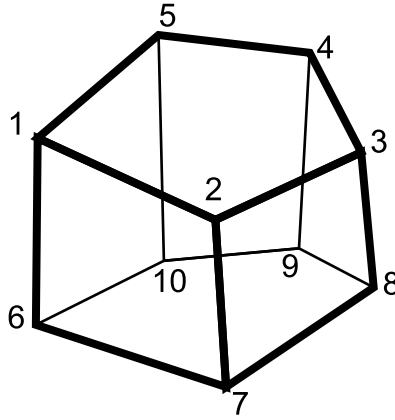
Since this result holds for any adjacent edges E_{j-1} and E_j, we have:

Corollary. *Given a facet $F = (v_0, v_1, \ldots, v_m)$ and a nerve component N_i of a 3-polytope, $f_i(E_j) \equiv f_i(E_l) \mod 2$ for all j, l.*

Thus, the parity of a facet is determined not by the number of full facets occurring there, but by the number of full facets containing the labels of an edge of that facet. This gives us a notion of parity for the whole polytope:

Lemma 5.5. *Let N_i be a nerve component of a 3-polytope, and let F_1 and F_2 be two facets of that polytope sharing a common edge E. Then, $f_i(E)[F_1] \equiv f_i(E)[F_2]$ mod 2.*

Proof. Let $I' = \ell(E)$ be the label set of the edge. Note that E borders only these two facets of the polytope. Consider the subnerve (N', ℓ) with $N' = N_i(I')$. Because the polytope has simplicial facets, $|I'| = k - 1$, and so by Observation 4.2, $|B_i'| \equiv f_i(E)[F_1] + f_i(E)[F_2] \equiv 0 \mod 2$. Thus, $f_i(E)[F_1] \equiv f_i(E)[F_2] \mod 2$. □

This means that, in the above figure,

$$n(271) + n(276) \equiv n(273) + n(278) \mod 2,$$

because the two facets (1267) and (2378) share the common edge (27). We can extend this equivalence throughout the polytope:

Corollary. *Let N_i be a nerve component of a 3-polytope P. Given any two edges E_1 and E_2 of P, if E_1 occurs on a facet F_1 of the polytope, and E_2 occurs on a facet F_2 of the polytope, then $f_i(E_1)[F_1] \equiv f_i(E_2)[F_2] \mod 2$.*

Thus, given a nerve component N_i, if we define the parity by $\rho(N_i) = f_i(E)[F]$ mod 2 for any edge E of any facet F, then $\rho(N_i) \equiv f_i(E_j)[F_l] \mod 2$ for any edge E_j of any facet F_l. This allows us to prove the full nerve component theorem for arbitrary 3-polytopes.

Full Nerve Component Theorem (3 dimensions). *Given the reduced simplicial graph N of a 3-polytope P with nerve components $N_1, \ldots, N_m$, there is some full nerve component N_i.*

Proof. Let $f(E)[F] = \sum_{i=1}^{m} f_i(E)[F]$ for some E on a facet F of the polytope. Then,

$$\sum_{i=1}^{m} \rho(N_i) \equiv \sum_{i=1}^{m} f_i(E)[F] \equiv f(E)[F] \mod 2.$$

Now, consider the simplicial graph of the facet F, call it N_F. Then, in the 2-subnerve (N_F', ℓ) with $N_F' = N_F(I')$ induced by the label set $I' = \ell(E)$, we must have $|B_F'| \equiv |C_F'| \mod 2$, by the corollary to Proposition 3.14. Note that $|C_F'| = f(E)[F]$, because both represent the number of full facets on the F facet containing the label set of E. Since $|B_F'|$, the number of full edges on the edge E, is odd by the 1-dimensional Sperner's Lemma, we see that $f(E)[F] \equiv 1 \mod 2$. Hence, $\rho(N_i) \equiv 1 \mod 2$ for some i, and this gives the full nerve component. $\square$

This, of course, proves the generalization of Sperner's Lemma in 3 dimensions.

Generalized Sperner's Lemma (3-polytopes). *In an arbitrary 3-polytope P having n vertices, subdivided with a Sperner Labeling, there are at least $n-k$ fully-labeled elementary simplices.*

In the next chapter, we will take on the case of a polytope with non-simplicial facets. Additional technicalities arise in this case, making the proof more difficult and requiring some new ideas.

Chapter 6

METHODS FOR FURTHER GENERALIZATION

In this chapter, we suggest two purely combinatorial methods which can be used to prove the full generalization of Sperner's Lemma:

Generalized Sperner's Lemma (polytopal conjecture). *In an arbitrary k-polytope P having n vertices, subdivided with a Sperner Labeling, there are at least $n - k$ full cells.*

The first method, which uses induction on the number of vertices, is short and non-constructive. The second method uses the machinery of nerve graphs. The difficulty in the case of arbitrary polytopes arises from the non-simplicial facets, which make the use of induction difficult.

6.1   A Non-Constructive Method

The first method we give relies on induction on the number of vertices n. The base case is covered by Sperner's Lemma. For the general case, we create a new polytope by identifying two main vertices, use induction, and then find an additional full cell in the original polytope.

In an arbitrary polytope, we have not found a way to easily determine when a vertex v_1 can be relabeled by the label of some other vertex v_2, such that the result is equivalent to a polytope with 1 less vertex. By "equivalent," we mean that the polytope and its subdivision can be deformed into another polytope, with one less vertex, such that the new labeling scheme describes a Sperner Labeling on the new polytope.

For example, taking the case of a square pyramid below, we can change all the 2 labels in the triangulation to 1's. Although we still have a square pyramid, we can just push in the vertex to obtain a polytope (here, a tetrahedron). We can then apply induction.

polytope
P
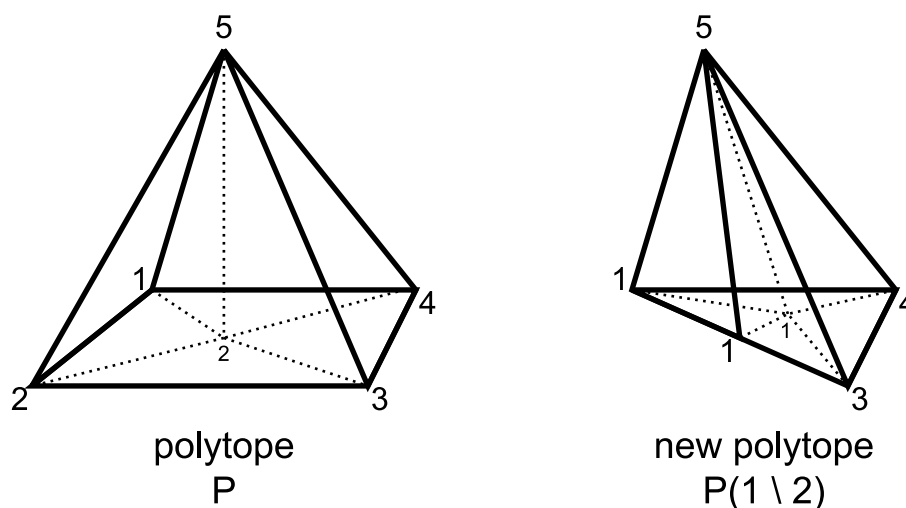
new polytope
P(1 \ 2)

Given a triangulated (n, k)-polytope P with a Sperner Labeling, we let $P(a_1 \setminus a_2)$ be the object obtained by changing all a_2 labels in the triangulation to a_1. The following conjecture would allow us to prove the full generalization.

Conjecture. *Given a triangulated (n, k)-polytope P with a Sperner Labeling, there exist main vertices a_1, a_2, b_1, and b_2 of the polytope such that $a_2 \notin \{a_1, b_1, b_2\}$ and $b_2 \notin \{a_1, a_2, b_1\}$, and both $P(a_1 \setminus a_2)$ and $P(b_1 \setminus b_2)$ are equivalent to $(n-1, k)$-polytopes with Sperner labelings.*

In the square pyramid above, we could take $a_1 = 1, a_2 = 2, b_1 = 3, b_2 = 4$, for example. Now, assuming this conjecture is true, we can prove the full generalization. We begin with a lemma which, for any edge of the polytope, gives us a full cell using both the labels of that edge.

Lemma 6.1. *If (a_1a_2) is an edge of an (n,k)-polytope, then there is at least one full cell in the polytope labeled with both a_1 and a_2.*

Proof. We prove this by induction on n. The base case, where $n = k + 1$, holds by Sperner's Lemma.

Now, select b_1 and b_2 according to the above conjecture, so that $P(b_1 \setminus b_2)$ is equivalent to an $(n - 1, k)$-polytope. Then, by induction on n, there exists a full cell with both labels a_1 and a_2, since, by the above conjecture, we did not remove either of these labels to obtain $P(b_1 \setminus b_2)$. Of course, this full cell must also be full in the original polytope. $\square$

Based on the previous conjecture, we can now give the short, non-constructive proof of Sperner's Lemma.

Generalized Sperner's Lemma (polytopal conjecture). *In an arbitrary k-polytope P having n vertices, subdivided with a Sperner Labeling, there are at least $n - k$ full cells.*

Proof. This proof proceeds by induction on n, the number of vertices of the polytope. The base case, where $n = k+1$, is covered by Sperner's Lemma itself, which promises at least 1 full cell.

Otherwise, let $P(a_1 \setminus a_2)$ be the object given by our conjecture above. By induction, since $P(a_1 \setminus a_2)$ is equivalent to a triangulated polytope with a Sperner Labeling, there are at least $n - k - 1$ full cells in this object. These must, of course, correspond to full cells in the original polytope.

Moreover, there must exist some full cell using both a_1 and a_2, since they are connected main vertices, and this cannot be a full cell in $P(a_1 \setminus a_2)$. This full cell, together with the $n - k - 1$ cells found above, gives a total of $n - k$ full cells in the original polytope. $\square$

6.2 A Constructive Method

We now suggest a method for a constructive proof of the full generalization which uses nerve graphs. The key to proving the main theorems in the past few chapters has been the existence of a full nerve component, and we propose the following:

Conjecture. *Let $N_1, N_2, \ldots, N_m$ be the nerve components of a triangulated polytope P with a Sperner Labeling. Then, there exists a full nerve component N_i.*

Thus, our proposed constructive proof would be very similar to that used in Sections 4.5 and 5.3. In Section 4.5, where we proved the generalization for simplicial facets, we were able to define the parity of a nerve component using a single facet. In Section 5.3, where we proved the generalization for dimension 3, we were able to define the parity of a nerve component using a facet and an edge. With an appropriate definition of the parity of a nerve component for an arbitrary polytope, we should be able to prove the full generalization.

Chapter 7

CONCLUSION

Although much was answered in this paper, a number of questions remain and there are several areas for possible future work relating to the generalizations we proved.

7.1 Remaining Questions

There are a number of outstanding questions which can still be answered regarding Sperner's Lemma, its generalizations, and nerve graphs.

For example, what can we say about the parity of the number of full cells? In Section 5.1 we proved that for the plane, where $k = 2$, the number of full cells has the same parity as $n - 2$. Can we say that in the general case, the number of full cells has the same parity as $n - k$? There are a number of cases for which this parity claim does not hold. For example, consider the figure:
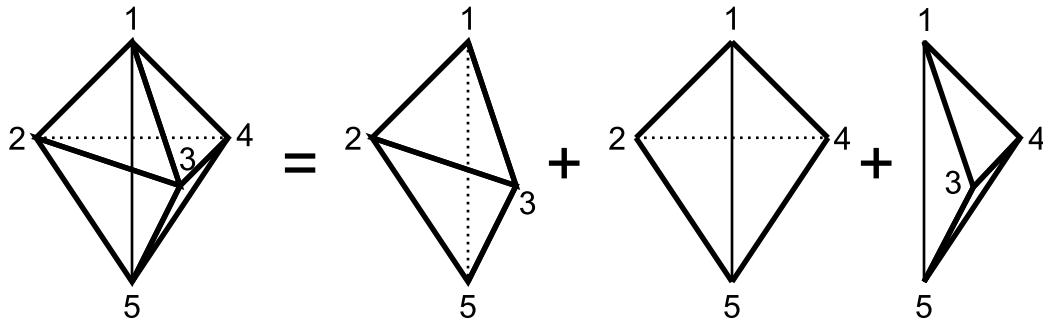
Here, there are three full cells, but $n - k = 5 - 3 = 2$. The reason for this oddity is that the figure is equivalent to three separate tetrahedrons glued together, each with

a separate triangulation and Sperner Labeling. Note that the vertices 1 and 5 are connected together by the triangulation, rather than on the exterior of the polytope. We propose that, in the absence of such "bridges" across the polytope, the number of full cells in the polytope has the same parity as $n - k$.

Also, what can we say about subdivisions other than triangulations? This was discussed in Section 5.2, where we conjectured that a planar polygon with n labels subdivided into q-gons will have at least $n - q + 1$ full cells. It would be interesting to study how this can be further extended to non-simplicial subdivisions of polytopes.

Finally, it also remains to complete the constructive and the non-constructive proofs of the full generalization which were presented in Chapter 6. We conjectured in that chapter that an arbitrary triangulated polytope with a Sperner labeling has a full nerve component, and this full nerve component would provide us with all $n - k$ full cells. Perhaps the topological argument proving the generalization for polytopes with simplicial facets (see [11]) could also be extended to prove the full generalization.

7.2   *Possible Applications*

There are a number of questions unrelated to the details of the proof which also arise, and could prove to be productive research topics. For example, there are several generalizations of Sperner's Lemma in literature, and variations on the lemma. It would be worthwhile to apply the theory of nerve graphs to other generalizations of Sperner's Lemma. Numerous other generalizations are given in [3], [8], and [18]. It would be interesting to see how many of these generalizations fall out of our polytopal generalization.

Also, there are several other combinatorial theorems, such as Tucker's Lemma, which are similar to Sperner's Lemma (see [13] and [17]). It is likely that nerve graphs, or a similar type of labeled graph, would be useful in proving such theorems and their generalizations.

It would also be nice to have an application for our generalization. There are many areas of mathematics which use Sperner's Lemma, such as game theory and fixed point theory (see [9] and [15]). For example, it would be very nice to find a fair division question that can be answered using this generalization, in the same way that Sperner's Lemma can be applied to these kinds of problems (see [15]). Also, since Sperner's Lemma is equivalent to the Brouwer Fixed Point Theorem, perhaps our generalization may prove a stronger fixed point result.

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