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Finite Invariance of Cayley Calibration Form

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Abstract

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In the further development of the string theory, one needs to understand 3 or 4-dimensional volume minimizing subvarieties in 7 or 8-dimensional manifolds. As one example, one would like to understand 4-dimensional volume-minimizing cycles in a torus T^8. The Cayley calibration form can be used to find all volume-minimizing cycles in each homology class of T^8. In order to apply the Cayley form to 8-dimensional tori, we need to understand the finite symmetry of the Cayley form, which has a continuous symmetry group Spin(7). We have found one finite symmetry group of order eight generated by three elements. We have also studied the symmetry groups of tori based on the results of H.S.M. Coxeter, and have had a simple description of the four crystallographic groups in $O(8)$. They can be used to classify all finite symmetry groups of the Cayley form.

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Chapter 1

INTRODUCTION

1.1 Motivations and summary of results

Calibrated geometry is a branch of differential geometry that uses differential forms with certain properties to identify volume-minimizing cycles in its homology class of a Riemannian manifold. Cayley calibration form was first introduced by R. Harvey and H. B. Lawson in their 1982 paper "Calibrated Geometries" [6]. The Cayley form is a differential 4-form on an 8-dimensional vector space, which can be identified with the set of Cayley numbers, or the octonions, the only 8-dimensional alternative real normed division algebra. Since the octonion algebra is not associative, we can define associator on the octonion algebra, similar to the commutator on non-commutative algebra. The Cayley form is introduced by studying the non-associativity of the octonions. In their paper, Harvey and Lawson have found that the Cayley calibration form is invariant under the group Spin(7). They used the Cayley calibration form to study volume-minimizing 4-cycles in their homology classes in $\mathbf{R}^8$. Now, Dr. Weiqing Gu would like to use the Cayley calibration form to study volume-minimizing 4-cycles in 8-dimensional tori, since these 4-cycles will play important roles in development of the duality theory in the string theory. Because tori can only have finite symmetry groups, we would like to study the finite symmetry groups of the Cayley calibration form first. In our study, we have already found one finite symmetry group of the Cayley calibration form.

Theorem 1 *The Cayley form is invariant under the group generated by the following*

three elements, written in their matrix representations with the standard basis of $\mathbf{R}^8$*:*

$$
\begin{aligned}
\sigma_1 &= \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0 \end{pmatrix}\\
\sigma_2 &= \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{pmatrix}\\
\sigma_3 &= \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}.
\end{aligned}
$$

It is very hard to find finite symmetry groups of the Cayley calibration form, Φ directly. We have to change our strategy in order to classify all finite symmetry groups

of Φ. Since we are going to use the Cayley calibration form Φ on 8-dimensional tori, we may get some useful information about the finite symmetry groups of Φ from the study of the finite symmetry groups of tori. Since n-dimensional tori are quotient spaces of $\mathbf{R}^n$ with n-dimensional lattices, which is the set of all integer linear combinations of n linearly independent vectors, the symmetry groups of tori are the symmetry groups of lattices. Therefore, we want to know the finite symmetry groups of the lattices. Since the isometry groups of $\mathbf{R}^n$ are the orthogonal groups $O(n)$, we have the following definition:

Definition 1 (Crystallographic Group) *A crystallographic group is a subgroup of $O(n)$ which maps some n-dimensional lattices onto itself.*

It has been known that there are only four crystallographic groups in $O(8)$. Three of them preserve the lattice spanned by the standard basis of $\mathbf{R}^8$, and the other preserve the lattice spanned by the following basis: $\{r_1 = \frac{1}{2}(\sum_1^3 e_i - \sum_4^8 e_i),\ r_i = e_i - e_{i-1}, 2 \leq i \leq 8\}$. We will provide a simplified proof of this fact based on the proof given in Grove and Benson [5].

1.2 Calibrated geometry

In calibrated geometry, one uses differential forms on a differential manifold to find optimal (i.e. volume-minimizing) subvarieties. The basic idea of calibrated geometry is an elegant application of the Stoke's Theorem. Suppose that ϕ is a differential n-form on an m-dimensional manifold, M, with $n < m$ and that ϕ achieves a finite maximum on all n-dimensional subspaces of the tangent spaces at all points of M. If N is an n-cycle in M and the tangent spaces at all points of N maximize ϕ, then N minimizes the volume in its homology class. The proof is straight forward. For simplicity, let's assume the finite maximum is 1, and let N' be homologous to N. The

Stoke's Theorem implies that $\int_N \phi = \int_{N'} \phi$. We have

$$\begin{aligned} \text{volume}(N) &= \int_N 1 \\ &= \int_N \phi \\ &= \int_{N'} \phi \\ &\leq \int_{N'} 1 \\ &= \text{volume}(N'). \end{aligned}$$

Chapter 2

CAYLEY FORM

2.1 Octonions

Since the Cayley calibration form is defined using the octonions, we are going to review the definition of the octonions and some of its properties that will be useful in later sections. The octonion numbers is an 8-dimensional normed division algebra over $\mathbf{R}$. We can define the octonions from the quaternions by the Cayley-Dickson process. First, given a normed division algebra $\mathbf{B}$ over $\mathbf{R}$, let 1 be the identity element. We can define Re$\mathbf{B}$ to be the span of 1 and Im$\mathbf{B}$ to be the orthogonal complement of Re$\mathbf{B}$. Each element in $\mathbf{B}$ can be decomposed into its real and imaginary parts, i.e.

$$x = x_r + x_i$$

for all $x \in \mathbf{B}$ and where $x_r \in \mathrm{Re}\mathbf{B}$ and $x_i \in \mathrm{Im}\mathbf{B}$. The conjugate of x is defined to be

$$\overline{x} = x_r - x_i.$$

Given the quaternions, $\mathbf{H}$, we can define the octonions as the direct sum of the quaternions, i.e. $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}$. The addition on $\mathbf{O}$ is defined naturally from the direct sum and the product on $\mathbf{H} \oplus \mathbf{H}$ is defined by

$$(a, b)(c, d) = (ac - \overline{d}b, da + b\overline{c}).$$

Let $1 = (1, 0) \in \mathbf{H} \oplus \mathbf{H}$ and $e = (0, 1) \in \mathbf{H} \oplus \mathbf{H}$. The standard basis of $\mathbf{O}$ becomes $\{1, i, j, k, e, ie, je, ke\}$, which is written as $\{e_i\}_{i=1}^{8}$ when the vector space

properties of **O** are emphasized. They multiply according to the following table: (The ith entry in the first column times the jth entry in the first row gives the ith row and jth column entry.)

	1	i	j	k	e	ie	je	ke
1	1	i	j	k	e	ie	je	ke
i	i	-1	k	-j	ie	-e	-ke	je
j	j	-k	-1	i	je	ke	-e	-ie
k	k	j	-i	-1	ke	-je	ie	-e
e	e	-ie	-je	-ke	-1	i	j	k
ie	ie	e	-ke	je	-i	-1	-k	j
je	je	ke	e	-ie	-j	k	-1	-i
ke	ke	-je	ie	e	-k	-j	i	-1

From this table, it is clear that the quaternions are not commutative and the octonions are not associative. For example, $ij = k \neq -k = ji$ and $i(ieje) = j \neq -j = (iie)je$. Since the octonions are not associative, we can define a trilinear form called the associator, similar to the commutator in the non-commutative algebra.

Definition 2 *Suppose $x, y, z \in \mathbf{O}$, then the associator of x, y, z is defined by*

$$[x, y, z] = (xy)z - x(yz).$$

Similar to the result in the non-commutative algebra that the commutator varnishes if the two arguments are equal, it is a matter of computation to show that in **O** the associator also varnishes if any two of its two arguments are equal. In this case, the algebra is alternative and the form is alternating on the algebra. Alternativity is a weak form of associativity.

An element in **H** has the general form $a + bi + cj + dk$ which can be written as $a + bi + (c + di)j$. This implies that **H** can be viewed as a 2-dimensional complex

algebra. A deeper and more generalized result is the Artin's Theorem, which we will not prove it here.

Theorem 2 (Artin) *The subalgebra A with unit generated by any two elements of* $\mathbf{O}$ *is associative.*

Octonion multiplication satisfies three important identities related to the associativity of multiplication, which can be proved by computation. They are called the Moufang identities:

$$\begin{aligned} (xyx)z &= x(y(xz)), \\ z(xyx) &= ((zx)y)x, \\ (xy)(zx) &= x(yz)x. \end{aligned}$$

As we have discussed before, the Cayley calibration form arises from the study of the non-associativity of multiplication on the octonions. We will introduce several forms that are alternating on the octonions.

Definition 3 (Cross Product) *Define* $x \times y = -\frac{1}{2}(\overline{x}y - \overline{y}x)$.

Proposition 1 $x \times y$ *is alternating.*

Proof:

$x \times x = -\frac{1}{2}(\overline{x}x - \overline{x}x) = 0$. Therefore the cross product is alternating.

Similarly, we can define the triple cross product and the fourfold cross product to be

$$x \times y \times z = \frac{1}{2}(x(\overline{y}z) - z(\overline{y}x)).$$

and

$$x \times y \times z \times w = \frac{1}{4}(\overline{x}(y \times z \times w) + \overline{y}(z \times x \times w) + \overline{z}(x \times y \times w) + \overline{w}(y \times x \times z)).$$

Proposition 2 *The triple cross product and the fourfold cross product are alternating.*

Proof:

For the triple cross product, we have

$$x \times y \times x = \frac{1}{2}(x(\overline{y}x) - x(\overline{y}x)) = 0,$$

$$\begin{aligned} x \times x \times z &= \tfrac{1}{2}(x(\overline{x})z - z(\overline{x}x)) \\ &= \tfrac{1}{2}(|x|^2 z - z|x|^2) \\ &= 0, \\ x \times y \times y &= \tfrac{1}{2}(x(\overline{y}y) - y(\overline{y}x)) \\ &= \tfrac{1}{2}(x|y|^2 - |y|^2 x) \\ &= 0. \end{aligned}$$

In the last two equations, we use the Artin's Theorem that a subalgebra of **O** generated by two elements is associative. Similarly, we can verify that the fourfold cross product is alternating.

The triple cross and the fourfold cross product have the following properties which can be proved by computation:

$$\mathrm{Re}(x \times y \times z \times w) = < x, y \times z \times w > .$$

Next, a set of equivariance properties can be proved using the Moufang identity.

Lemma 1 (Equivariance of Cross Products) *Suppose* $u \in Im\mathbf{O}$ *and* $|u| = 1$ *(so* $\overline{u} = -u$*). Then for all* $x, y, z, w \in \mathbf{O}$*:*

$$\begin{aligned} (xu) \times (yu) &= u(x \times y)\overline{u}, \\ (xu) \times (yu) \times (zu) &= (x \times y \times z)u, \\ (xu) \times (yu) \times (zu) \times (wu) &= u(x \times y \times z \times w)\overline{u}. \end{aligned}$$

2.2 *Cayley Calibration Form*

Definition 4 (Cayley Calibration Form) *The four form Φ defined by $\Phi(x,\ y,\ z,\ w)$ = $< x,\ y\ \times\ z \times\ w >$ is called the Cayley calibration form on* $\mathbf{O}$.

It is easy to check that $\Phi(1, i, j, k) = 1$ which is the maximum of Φ on all four unit vectors. $1, i, j, k$ generate the quaternions which is the associative part of the octonions. Harvey and Lawson have shown that Φ achieves its maximum on four vectors, if and only if those four vectors generate a quaternion-like structure. This is why we say that Cayley form arises from the study of non-associativity in the octonions.

Cayley form Φ can be written in the standard dual basis of $\mathbf{R}^8$. That is

$$\begin{aligned}\Phi &= \omega_{1234} - \omega_{1278} - \omega_{1638} - \omega_{1674} - \omega_{1265} - \omega_{1375} - \omega_{1485} \\ &+ \omega_{5678} - \omega_{5634} - \omega_{5274} - \omega_{5238} + \omega_{3478} + \omega_{2468} + \omega_{2367}\end{aligned}$$

where $\omega_{ijkl} = \omega_i \wedge \omega_j \wedge \omega_k \wedge \omega_l$ and $\{\omega_i\}_{i=1}^{8}$ are the dual basis of $\mathbf{R}^8$. An easy but tedious way to see the representation is to note that ω_{ijkl} form a basis for the vector space of all 4-forms on $\mathbf{R}^8$ and the projection of Φ onto each basis ω_{ijkl} is $\Phi(e_i, e_j, e_k, e_l)$.

2.3 *Spin(7)*

For each $u \in \mathbf{O}$, $R_u : \mathbf{O} \to \mathbf{O}$ is defined by $R_u(x) = xu$. It is clear that if $u \neq 0$, $R_u \in SO(8)$.

Definition 5 (Spin(7)) *Spin(7) is the subgroup of SO(8) generated by* $S_6 = \{R_u : u \in Im\mathbf{O}$ *and* $|u| = 1\}$.

Theorem 3 *The form Φ is fixed by the subgroup Spin(7).*

Proof:

It is enough to show that Φ is fixed by the generators of Spin(7). Suppose $u \in S_6$. Then use the equivariance relations:

$$\begin{aligned}\Phi(xu, yu, zu, wu) &= < xu, yu \times zu \times wu > \\ &= \mathrm{Re} xu \times yu \times zu \times wu \\ &= -\mathrm{Re} u(x \times y \times z \times w)u \\ &= -u^2 \mathrm{Re} x \times y \times z \times w \\ &= \Phi(x, y, z, w).\end{aligned}$$

2.4 An example of finite invariance of the Cayley form

We will prove Theorem 1 in this section. We can write those elements in the group as permutations of the basis vectors. We have

$$\begin{aligned}\sigma_1 &= (1,\ 2)(3,\ 4)(5,\ 6)(7,\ 8), \\ \sigma_2 &= (1,\ 3)(2,\ 4)(5,\ 7)(6,\ 8), \\ \sigma_3 &= (1,\ 8)(2,\ 7)(3,\ 6)(4,\ 5), \\ \sigma_4 &= (1,\ 4)(2,\ 3)(5,\ 8)(6,\ 7), \\ \sigma_5 &= (1,\ 7)(2,\ 8)(3,\ 5)(4,\ 6), \\ \sigma_6 &= (1,\ 6)(2,\ 5)(3,\ 8)(4,\ 7), \\ \sigma_7 &= (1,\ 5)(2,\ 6)(3,\ 7)(4,\ 8).\end{aligned}$$

Hence we know the actions of elements of this group on each basis vector. Since the Cayley form is linear, we can check the invariance on every permutation of four basis vectors of $\mathbf{R}^8$. For example, under σ_1, $\{e_1, e_2, e_3, e_4\}$ are sent to $\{e_2, e_1, e_4, e_3\}$. Then

$$\Phi(e_1, e_2, e_3, e_4) =< e_1, e_2 \times e_3 \times e_4 >=< e_1, e_1 >= 1$$

and

$$\Phi(e_2, e_1, e_3, e_4) =< e_2, e_1 \times e_3 \times e_4 >=< e_2, e_2 >= 1.$$

An example of a different type is that $\{e_5, e_2, e_7, e_8\}$ are mapped to $\{e_6, e_1, e_8, e_7\}$. Evaluation of Φ on those sets gives:

$$\Phi(e_5, e_2, e_7, e_8) =< e_5, e_2 \times e_7 \times e_8 >=< e_5, \frac{1}{2}(e_1 - e_1) >= 0,$$

$$\Phi(e_6, e_1, e_8, e_7) =< e_6, e_1 \times e_8 \times e_7 >=< e_6, \frac{1}{2}(e_2 - e_2) >= 0.$$

An easy way to prove this theorem, however, is to note that each generating matrix is self-adjoint. Therefore, instead of permuting the basis vectors of $\mathbf{R}^8$, we can permute the dual basis of $\mathbf{R}^8$ in the same way. If the Cayley form remain invariant under this permutation of dual basis of $\mathbf{R}^8$, then it is fixed by the corresponding linear transformation.

It is not hard to check that the Cayley form Φ remain the same if we permute the indices in ω_{ijkl} according to σ_1. Note that since wedge product is antisymmetric, the sign changes whenever two adjacent indices switch places. For example, ω_{1234} will remain the same, whereas ω_{1638} will be mapped to $\omega_{2547} = -\omega_{5247} = \omega_{5274}$ which is a term in Φ. Similarly, the Cayley form can be shown to be invariant under the other two generating elements.

Note that the group we found is highly non-trivial in Spin(7), though it looks simple. Right multiplication by i is an element in Spin(7) whose matrix representation

in the standard basis is

$$R_i = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The matrix σ_1 is the product of QR_i where

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Q is clearly in $SO(8)$. It is in Spin(7) as well, but it will be nice to know which R_u's generate Q.

Chapter 3

CRYSTALLOGRAPHIC GROUPS

As we mentioned in the introduction, we will discuss one approach based on the work of H.M.S. Coxeter [1] to find all the crystallographic groups. We first realized that each element in a crystallographic group must have a finite order. Lattices are discrete sets in $\mathbf{R}^n$. This means that there are only finite numbers of vectors in a lattice that have the same length since spheres are compact sets. Consider all the vectors in a lattice that have the same length with the basis vectors of the lattice. They form a finite set. The action of an element from the crystallographic group is to send one permutation of those vectors to another permutation. Since the number of permutations is finite, we will return to our original permutation. Therefore the order of the element is finite. This fact can help us to simplify the proof given by Grove and Benson [5] by assuming the crystallographic condition from start. First we will illustrate this approach with a simplified 2-dimensional case. Not only is this an easier case, we will also find that n-dimensional crystallographic groups can be reduced down to 2-dimensional ones.

3.1 2-dimensional lattices and their finite invariance

First, let's assume that the lattice is invariant under some subgroups of $SO(2)$ rather than $O(2)$. A 2-dimensional lattice is the set

$$L_2 = \{a_1 r_1 + a_2 r_2 \parallel a_1,\, a_2, \in \mathbf{Z},\ r_1,\, r_2 \in \mathbf{R}^2, r_1,\ r_2 \text{ are linearly independent.}\}.$$

Thus, r_1 and r_2 are the basis of the lattice L_2.

Now suppose that G is a subgroup of $SO(2)$ that preserves the lattice L_2. Let $T \in G$ where T can be written as $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. This matrix has a trace equal to $2\cos\theta$.

Since T preserves L_2, both $T(r_1)$ and $T(r_2)$ remains in L_2 and we have the following relations:

$$T(r_1) = a_{11}r_1 + a_{12}r_2$$
$$T(r_2) = a_{21}r_1 + a_{22}r_2$$

where $a_{ij} \in \mathbf{Z}$. This matrix has an integer trace. From linear algebra, we know that given two square matrices A and B, traceAB = traceBA. It is easy to see that trace$P^{-1}BP$ = trace$B$. This implies that the trace is a numerical invariant of a linear transformation. Therefore, $2\cos\theta$ is an integer. Thus, $\cos\theta$ is either $0, \pm\frac{1}{2}$ and ± 1 and θ can only be $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$. It is not hard to see that rotations by $0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$ form a group, and the lattice generated by the standard basis is invariant under this group. The other four rotations and 0 and π form a group as well with the lattice generated by the vectors $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.

From this example, we can make the following generalization about an n-dimensional lattice L_n. If $T \in O(n)$ preserves this lattice and if we use the basis of this lattice to write the matrix representation of T, then it is clear that the matrix has only integer entries. Thus, its trace is an integer.

3.2 Reflections and $O(n)$

Since all crystallographic groups are subgroups of $O(n)$, we need an important fact that $\mathbf{O}(n)$ is generated by reflections. In other words, every linear transformation in $\mathbf{O}(n)$ can be written as a composition of finitely many reflections. (This proof is based on Curtis [2].)

We can see this clearly in three dimension. An element in $O(3)$ can be written as a

product of a reflection and a rotation. A rotation in a 3-dimensional vector space has a fixed axis. Therefore, a three dimensional rotation can be regarded as a rotation of a plane. A two dimensional rotation can be easily decomposed as a product of two reflections. Suppose we have a rotation of angle α. The two reflection axises can be chosen as the two line through the origin, making an angles of $\frac{1}{4}\alpha$ and $\frac{3}{4}\alpha$ with the x-axis respectively. The fact that we can always write two dimensional rotations as products of two reflections is crucial here. The generalization is that through conjugations, we can view rotations in any dimension as compositions of rotations of two dimensional subspaces. Thus we can write any rotations as compositions of finite reflections. The concept of maximal tori in $SO(n)$ makes this idea more precise.

First, $SO(2)$ is a set of matrices of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. It is isomorphic to the circle S^1. The direct product of two $SO(2)$ is a subgroup of $SO(4)$ which is isomorphic $S^1 \times S^1$, a torus. Elements in this torus are of the form:

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix}.$$

In this form, it is clear that this rotation in $\mathbf{R}^4$ is a composition of two 2-dimensional rotations. Tori in $SO(n)$ are subgroups that are isomorphic to m-tori. A torus is maximal if it is not contained in any other tori in $SO(n)$. The maximal tori in $SO(n)$ are m-tori, where m is the largest integer less than or equal to $\frac{n}{2}$. If an element can be conjugated to a torus element in $SO(n)$, then it can be viewed as a composition of two dimensional rotations.

We need the following lemma from linear algebra to decompose high dimensional rotations to 2-dimensional rotations.

Lemma 2 *Let $A \in \mathbf{O}(n)$. There exists an A-invariant subspace W of $\mathbf{R}^n$ with dim W equal to 1 or 2.*

Proof:

Let $S = A + A^t$. S is clearly symmetric. Let w be an eigenvector for S. Consider w and wA. If w and wA are linearly dependent, i.e. $wA = \lambda w$, then $W = \{rw | r \in \mathbf{R}\}$ is A-invariant and dim $W = 1$. If w and wA are linearly independent, then let $W = \text{span}(w, wA)$. Since $wS = \lambda w$, i.e. $wA + wA^t$ and $A^t = A^{-1}$ for $A \in \mathbf{O}(n)$, we have

$$wA^2 + w = \lambda wA.$$

Let $x \in W$, so $x = \alpha w + \beta wA$. To show W is A-invariant, we need to show $xA \in W$. Now,

$$\begin{aligned}(\alpha w + \beta wA)A \qquad &= \alpha wA + \beta wA^2 \\ &= \alpha wA + \beta(\lambda wA - w) \\ &= (-\beta)w + (\alpha + \beta\lambda)wA \\ &\in W.\end{aligned}$$

Theorem 4 *The conjugates of our standard maximal torus in* $\mathbf{O}(n)$ *cover* $\mathbf{SO}(n)$.

Proof:

If A has an invariant 1-dimensional subspace, we can choose v_1 to be a unit vector in it. We can look at its orthogonal complement and choose another invariant 1-dimensional subspace and so on. Thus we will generate an orthononormal set $\{v_1, v_2, \dots, v_k\}$ of eigenvectors for A. Let $W = \text{span}(v_1, \dots, v_k)^\perp$, the orthogonal complement of $\text{span}(v_1, \dots, v_k)$. A is still orthogonal on W, but A has no more 1-dimensional invariant subspaces. Choose a stable 2-space W_1 and let v_{k+1}, v_{k+2} be an orthonormal basis for W_1. After finding all the stable 2-dimensional subspaces of W. We will get an orthonormal basis $\{v_1, \dots, v_k, v_{k+1}, v_{k+2}, \dots, v_{n-1}, v_n\}$. Let P map e_i to v_i . Then $P \in \mathbf{O}(n)$ and PAP^{-1} is the matrix representation of A in the basis, $\{v_i\}$. In this case, it is clear that PAP^{-1} will have 1 on its first k diagonal

entries, and on the rest of the diagonal entries, we will have matrices of the form, $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Therefore, A is in a conjugate of a maximal torus.

This theorem shows that it is enough to prove that each element in the maximal torus can be generated by reflections. First, we need a more precise definition of reflections. Let u be a unit vector in $\mathbf{R}^n$ and let

$$u^{\perp} = \{x \in \mathbf{R}^n \mid < x, u >= 0\}$$

be its orthogonal complement. The projection of a vector v onto u is $< v, u > u$ and the projection of v into $u^{\perp}$ is $v- < v, u > u$. We define the reflection of v in $u^{\perp}$ to be

$$S_u = v - 2 < v, u > u.$$

We call u the root of the reflection S_u. Let's choose an orthonormal basis, $u_1, \dots, u_n$ with $u_1 = u$. Since

$$S_u(u_1) = u_1 - 2 < u_1, u_1 > u_1 = u_1 - 2u_1 = -u_1$$

and

$$S_u(u_i) = u_i - 2 < u_i, u_1 > u_1 = u_i - 0 = u_i$$

for $i \neq 1$, the matrix representation of S_u using this basis is a matrix whose off diagonal elements are zeros and whose first diagonal element is -1 and rest of the diagonal elements are 1. Thus, S_u is in the orthogonal group, but not in the special orthogonal group.

In addition, it is clear that any vector in $u^{\perp}$ is mapped to itself by S_u. In other words, the S_u-invariant subspace has dimension $n-1$. Then any two reflections S_u and S_v share an S_uS_v-invariant subspace of dimension $n-2$. Therefore, S_uS_v is really an element in $O(2)$. On the other hand, they both have determinants equal to -1, so S_uS_v has a determinant equal to 1. This implies that S_uS_v can be treated as a 2-dimensional rotation, i.e an element of $SO(2)$.

Since the groups $\mathbf{SO}(n)$ can be covered by maximal tori, we only need to show that each element in the torus can be written as a finite product of reflections.

Theorem 5 $\mathbf{O}(n)$ *is generated by reflections.*

Proof:

First we prove this for elements of a maximal torus T. Let m be the largest integer $\leq \frac{n}{2}$. An element in T has $B_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{pmatrix}, \ldots, B_m = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix}$ on the diagonal. Let $S_i \in T$ be the identity matrix except in block B_i. Any element of T is the finite product of elements $S_1, \ldots, S_m$. Each S_i is a 2-dimensional rotation, so they can be written as a product of two reflections. Hence, any element in T can be written as a product of finitely many reflections.

Then given $A \in \mathbf{SO}(n)$, choose B such that

$$BAB^{-1} \in T.$$

Since BAB^{-1} equals to a finite product of reflections, $R_1 \ldots R_m$, we can express A as $(B^{-1}R_1B)(B^{-1}R_2B)\ldots(B^{-1}R_mB)$. Therefore, it is a finite product of reflections with those reflections written in a different basis. Finally, given $C \in \mathbf{O}(n)$, $C = AD$ where $A \in \mathbf{SO}(n)$ and D is a reflection. Therefore, every elements in $\mathbf{O}(n)$ can be written as a finite product of reflections.

3.3 Coxeter Groups

A finite subgroup $\mathbf{O}(n)$ generated by reflections is a Coxeter group. Crystallographic groups are finite. Since Coxeter groups are finite, the sets of generators are finite as well. The finiteness of the Coxeter groups imposes extra conditions on the sets of generators, that the "angle" between the roots of two reflections S_i and S_j has to be $\pi - \pi/p$ where p is the order of S_iS_j. This is clearly true in two dimension. The composition of two reflections is a rotation with an angle twice the angle between

the two mirror lines of the two reflections. Since we have a finite group, the order of the rotation is finite as well. Hence, we must have the angle between the roots equal to π/p for some natural number p. If we impose the extra condition that the group has to be crystallographic as well, then those natural numbers can only be $1, 2, 3, 4, 6$. Note that from now on we will assume that all reflections are in some Coxeter crystallographic groups.

Theorem 6 *If S_{r_i} and S_{r_j} are generating reflections of a Coxeter crystallographic group, then $< r_i, r_j > /\|r_i\|\|r_j\| = -\cos(\pi/p_{ij})$ with p_{ij} being the order of S_iS_j as a group element and p_{ij} can only be* $1, 2, 3, 4, 6$.

Proof:

We have already shown that $S_{r_i}S_{r_j}$ is a 2-dimensional rotation. Moreover, being in crystallographic groups means that it must have a finite order as well, so let p_{ij} be its order. Then the angle of rotation is $\frac{2\pi}{p_{ij}}$. In two dimension, the angle of the rotation, which is a composition of two reflections, is twice the angle between the two mirror lines of the reflections, which is π minus the angle between the two root vectors. Therefore, $< r_i, r_j > /\|r_i\|\|r_j\| = -\cos(\pi/p_{ij})$.

Next, we will need the key insight from Coxeter to associate a Coxeter group with a graph, hence called a Coxeter graph, and a square matrix, which can also be identified with a positive definite quadratic form.

Suppose G is a Coxeter group with $\{r_1, \dots, r_n\}$ as generating reflections. We define its Coxeter graph as follow. Each node in the graph represents a root. If $< r_i, r_j > \neq 0$, then we will connect the ith node with the jth node. Since $< r_i, r_j > /\|r_i\|\|r_j\| = -\cos(\pi/p_{ij})$, we label this branch with p_{ij}. We will omit the label if $p_{ij} = 3$. We can associate the Coxeter group with a symmetric matrix, whose diagonal elements are all 1's and whose off-diagonal ij elements equal to $-\cos(\pi/p_{ij})$. It is clear that this matrix can represent a quadratic form. Since there is one to one correspondence between the Coxeter groups, their Coxeter graphs, their matrices and

their quadratic forms, we will sometimes use the Coxeter graphs to represent the other three terms. Our goal is to find all the graphs that can represent some Coxeter groups. Note that if a Coxeter group can not be decomposed into a direct product of two Coxeter groups of lower dimensions, then it is called irreducible. It is a fact that if a Coxeter group is irreducible, then its generating roots cannot be separated into two mutually orthogonal sets, i.e. its Coxeter graph is always connected. And we can always find n linearly independent roots whose reflections generate this Coxeter group in $\mathbf{O}(n)$.

The reason that we want to associate a Coxeter group with a quadratic form is that the quadratic form has to be positive definite and hence the associated matrix must have a positive determinant.

Lemma 3 *The Coxeter graph of a Coxeter group is positive definite.*

Proof:

If roots are taken to be unit vectors, then the matrix A defining the associated quadratic form has ijth entry $< r_i, r_j >$. If $0 \neq x = (\lambda_1, \dots, \lambda_n)$ in $\mathbf{R}^n$, then $\sum_i \lambda_i r_i \neq 0$ in V since the set of roots is linearly independent. Thus,

$$\begin{aligned} Q(x) &= \textstyle\sum_{ij} < r_i, r_j > \lambda_i \lambda_j \\ &= < \textstyle\sum_i \lambda_i r_i, \sum_j \lambda_j r_j > \\ &= \| \textstyle\sum_i \lambda_i r_i \|^2 > 0, \end{aligned}$$

so Q is positive definite.

Thus, in order to be a Coxeter graph, a graph must have a positive determinant. It is crucial to calculate the determinant of a graph easily. The next lemma provides a recursive relation to simplify the calculation.

Lemma 4 *Suppose G is a labeled graph having a node a_1 that is adjacent to only one other node a_2. Denote the subgraph $G - \{a_1\}$ by G_1 and the subgraph $G - \{a_1, a_2\}$ by*

G_2, and write p for the label p_{12}. Then

$$\det G = \det G_1 - (\cos^2 \pi/p) \det G_2.$$

Proof:

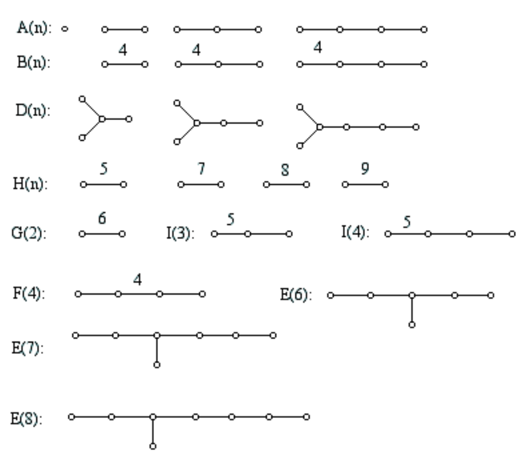
The matrix G has the form

$$\begin{pmatrix} 1 & -\cos \pi/p & 0 \\ -\cos \pi/p & 1 & * \\ 0 & * & G_2 \end{pmatrix}$$

with $G_1 = \begin{pmatrix} 1 & * \\ * & G_2 \end{pmatrix}$. If G' denote the matrix obtained from G by deleting the first row and the second column then clearly $\det G' = (-\cos \pi/p) \det G_2$. If we expand $detG$ along the first row, we find

$$\begin{aligned} \det G \quad &= \det G_1 + (\cos \pi/p) \det G' \\ &= \det G_1 - (\cos^2 \pi/p) \det G_2. \end{aligned}$$

Figure 3.1: Graphs that can represent Coxeter groups



This lemma will allow us to compute the determinant of Coxeter graphs easily. We can either use induction or reduce a graph to simpler graphs. (see Figure 3.1 for all the labeled graphs that can represent a Coxeter group.) $A(n)$ will be our basic graph, so we will compute the determinant of $A(n)$ first. Since $\det A(1) = 1$ and $\det A(2) = 3/4$, we can guess that $\det A(n) = \frac{n+1}{2^n}$. We can use induction and the recursive formula to prove it. By the lemma 4, we have

$$\begin{aligned} \det A(n) &= \det A(n-1) - \tfrac{1}{4} \det A(n-2) \\ &= \tfrac{n}{2^{n-1}} - \tfrac{1}{4}\tfrac{n-1}{2^{n-2}} \\ &= \tfrac{n+1}{2^n}. \end{aligned}$$

Therefore, $A(n)$ are all positive definite and they can represent Coxeter groups. Now, we can also show a set of graphs that can represent Coxeter groups.

$$\begin{aligned}
\det B(n) &= \det A(n-1) - (\tfrac{\sqrt{2}}{2})^2 \det A(n-2) \\
&= \tfrac{n}{2^{n-1}} - \tfrac{n-1}{2^{n-1}} \\
&= \tfrac{1}{2^{n-1}}, \\
\det D(n) &= \det A(n-1) - \tfrac{1}{4} \det A(n-3) \\
&= \tfrac{n}{2^{n-1}} - \tfrac{n-2}{2^{n-1}} \\
&= \tfrac{1}{2^{n-2}}, \\
\det I(3) &= \det A(2) - \alpha^2 \det A(1) \\
&= \tfrac{3}{4} - \alpha^2 \\
&= \tfrac{3-\sqrt{4}}{8}, \\
\det I(4) &= \tfrac{1}{2} - \tfrac{3\alpha^2}{4} \\
&= \tfrac{7-3\sqrt{5}}{32}, \\
\det F(4) &= \det B(3) - \tfrac{1}{4} \det A(2) \\
&= \tfrac{1}{16}, \\
\text{n} = 6,\ 7,\ 8 & \\
\det E(n) &= \det D(n-1) - \tfrac{1}{4} \det A(n-2) \\
&= \tfrac{1}{2^{n-3}} - \tfrac{n-1}{2^{n}} \\
&= \tfrac{9-n}{2^{n}}.
\end{aligned}$$

In fact, those graphs are the only graphs that can represent some Coxeter groups. If more nodes and branches are added to those graphs, the results cannot represent a Coxeter group anymore. We need another key lemma to prove this fact.

Lemma 5 *A (nonempty) subgraph H of a positive definite marked graph G is also positive definite.*

Proof:

We can order the nodes $a_1,\ a_2, \ldots,\ a_m$ of G in such a way that $a_1, \ldots,\ a_k$ are the nodes of H. If $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ are the matrices of G and H respectively, then $\alpha_{ij} \leq \beta_{ij}$ for all i and j between i and k, since H is a subgraph of G. Let Q_G and Q_H denote the corresponding quadratic forms. If Q_H is not positive definite, choose $x \neq 0$ in $\mathbf{R}^k$ for which $Q_H(x) \leq 0$. If $x = (\lambda_1, \ldots, \lambda_k)$, set $y = (|\lambda_1|, \ldots, |\lambda_k|, 0, \ldots, 0) \in \mathbf{R}^m$. Then $y \neq 0$, and

$$\begin{aligned} 0 \geq Q_H(x) &= \textstyle\sum_{i,j} \beta_{ij}\lambda_i\lambda_j \\ &\geq \textstyle\sum_{i,j} \beta_{ij}|\lambda_i||\lambda_j| \\ &\geq \textstyle\sum_{i,j} \alpha_{ij}|\lambda_i||\lambda_j| = Q_G(y) > 0. \end{aligned}$$

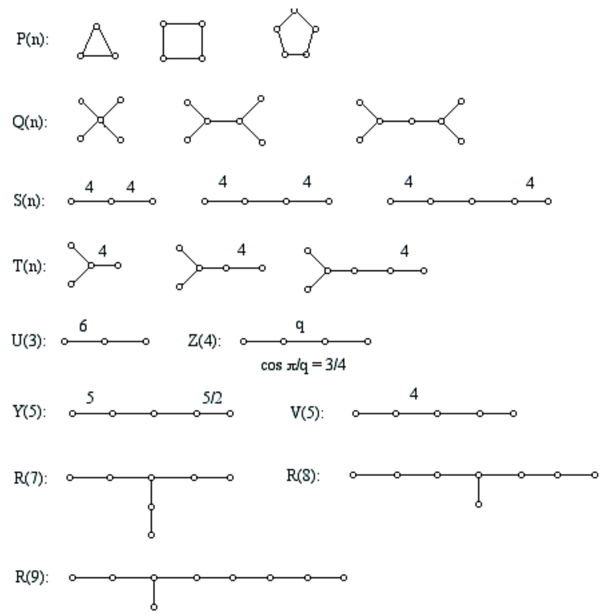
We reach a contradiction. Therefore H is positive definite as well.

To use this lemma, we will show a set of small graphs that cannot represent any Coxeter groups. (See Figure 3.2 for those graphs.) The following calculations show that they all have determinants equal zero. Hence they cannot be associated with a positive definite form.

In the case of P_n, the sum of all the rows is the zero vector so the rows are dependent and $\det P_n = 0$. The fact that $\det Z_4 = 0$ and $\det Y_5 = 0$ have to be done by direct calculation. For all others, we apply previous recursive lemma on the determinants. We have

$$\begin{aligned} \det Q_n &= \det D_{n-1} - \tfrac{1}{4}\det D_{n-3} = 0, \\ \det S_n &= \det B_{n-1} - \tfrac{1}{2}\det B_{n-2} = 0, \\ \det T_n &= \det B_{n-1} - \tfrac{1}{4}\det B_{n-3} = 0, \\ \det U_3 &= \det A_2 - \tfrac{3}{4} det A_1 = 0, \\ \det V_5 &= \det B_4 - \tfrac{1}{4}\det A_3 = 0, \\ \det R_7 &= \det E_6 - \tfrac{1}{4}\det A_5 = 0, \\ \det R_8 &= \det E_7 - \tfrac{1}{4}\det D_6 = 0, \\ \det R_9 &= \det E_8 - \tfrac{1}{4}\det E_7 = 0. \end{aligned}$$

Figure 3.2: Graphs that can not represent Coxeter groups



Theorem 7 *If G is a connected positive definite Coxeter graph, then G is one of the graphs A_n, B_n, D_n, H_2^n, G_2, I_3, I_4, F_4, E_6, E_7, or E_8. Therefore, the only possible 8-dimensional Coxeter groups are A_8, B_8, D_8 and E_8.*

Proof:

First, G has no cycles as subgraphs since P_n is not positive definite. If H_2^n is a subgraph of G for any $n \geq 7$, then $G = H_2^n$, for otherwise U_3 would be a subgraph of G. Similarly, $G = G_2$ if G_2 is a subgraph of G. We may then assume, that any branch of G is marked 3, 4, or 5. Suppose that B_2 is a subgraph of G (it cannot occur more than once, otherwise some S_n would be a subgraph). Then G cannot have a branch point, for otherwise some T_n would be a subgraph. If H_2^5 is also a subgraph, then we may have $G = H_2^5$, $G = I_3$, or $G = I_4$. Otherwise G would have either Z_4 or Y_5 as a subgraph. If B_2 is a subgraph but H_2^5 is not, then G may be B_n, for some $n \geq 2$, or F_4. Otherwise V_5 would be subgraph of G.

Suppose that all branches of G are unlabeled. G can have at most one branch point, and only three branches can branch off from any node, for otherwise some Q_n would be a subgraph of G. If G has no branch point, then $G = A_n$ for some n. If G has a branch point, then either $G = D_n$ for some n, or $G = E_6$, E_7, or E_8, for otherwise R_7, R_8, or R_9 would be a subgraph.

Finally, since we only allow p_{ij} to be equal to $1, 2, 3, 4, 6$, we are left with only A_8, B_8, D_8 and E_8.

3.4 8-dimensional Crystallographic Groups

From previous discussion, we know that we can have at most four different crystallographic groups: A_8, B_8, D_8 and E_8. They are indeed crystallographic groups and Grove and Benson [5] provides their constructions. A_8 is generated by reflections with the following roots: $\{r_1 = e_1 - e_8, r_i = e_i - e_{i-1}, 2 \leq i \leq 8\}$. B_8 is generated by reflections with roots: $\{r_1 = e_1, r_i = e_i - e_{i-1}, 2 \leq i \leq 8\}$. D_8 is generated by reflections with the following roots: $\{r_1 = e_1 + e_2, r_i = e_i - e_{i-1}, 2 \leq i \leq 8\}$. E_8 is generated by reflections with roots: $\{r_1 = \frac{1}{2}(\sum_1^3 e_i - \sum_4^8 e_i), r_i = e_i - e_{i-1}, 2 \leq i \leq 8\}$. A_8, B_8 and D_8 all accept the standard lattice spanned by the standard basis of $\mathbf{R}^8$, i.e. $\{e_i\}_{i=1}^8$. The actions of A_8 are to permute all the basis vectors. For example, S_{r_1} will map e_1 to e_2 and e_2 to e_1 while keeping other basis vectors fixed. B_8 in fact includes $A(8)$ as a subset and it can also send one basis vector to its additive inverse one at a time. For example, S_{r_1} maps e_1 to $-e_1$ and keeps other basis vectors fixed. D_8 lies between A_8 and B_8. Not only can it exchange two basis vectors, it can also send each one to the other's additive inverse. For example, S_{r_1} sends e_1 to $-e_2$ and e_2 to $-e_1$. The root vectors of E_8 also generate the invariant lattice of E_8.

3.5 Future Work

In this research, we have found one finite symmetry group of the Cayley calibration form and classified 8-dimensional crystallographic groups and their associated lattices. Those 8-dimensional crystallographic groups can be used to classify finite symmetry groups of the Cayley calibration form. In the future work, on one hand we are going to study the behaviors of the Cayley form under the linear transformations corresponding to A_8, B_8, D_8 and E_8. We can simplify the procedures by utilizing the linearity of the problem and generating reflections of A_8, B_8, D_8 and E_8. On the other hand, we are going to find an isomorphic image of the finite group we found inside some of those four crystallographic groups and understand how the image lies in these groups. Since it is easier to do computation in those four groups than in Spin(7), a study of the subgroups in these four groups may lead to new finite symmetry groups of the Cayley calibration form.

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