Journal of Humanistic Mathematics

Volume 3 | Issue 2 July 2013

# DNR-Based Curricula: The Case of Complex Numbers

Guershon Harel
*University of California - San Diego*

# DNR-Based Curricula: The Case of Complex Numbers

**Cover Page Footnote**

The development of this paper was in part supported with funds from the National Science Foundation under Grants No. NSF 0934695 and NSF 1035503 (Harel PI). Any opinions, findings, and conclusions expressed are those of the authors and do not necessarily reflect the views of the National Science Foundation. I would like to thank Martin Simon, Jeffery Rabin, Sara-Jane Harel, and Barbara Edwards for their comments on an earlier version of this paper.

# DNR-Based Curricula: The Case of Complex Numbers

Guershon Harel

*Department of Mathematics, University of California, San Diego*
harel@math.ucsd.edu

**Abstract**

DNR is a research-based framework which seeks to understand fundamental problems of mathematics learning and teaching. This paper discusses DNR with a particular reference to a curricular unit on complex numbers. Originally designed for college-bound high school students, the unit is structured to progress along a path that roughly parallels the development of complex numbers in the history of mathematics. We have tested the unit in three teaching experiments with in-service and prospective secondary mathematics teachers. The results from these experiments demonstrate the ways of thinking afforded and targeted by the unit. The correspondence between these ways of thinking and the Standards Practices outlined by the Common Core State Standards are also discussed.

This paper discusses a curricular unit on complex numbers[1] and its theoretical foundation. Originally designed for college-bound high school students,[2] the unit was implemented in three teaching experiments with in-service and prospective secondary mathematics teachers. Selected learning outcomes from these experiments are also discussed in the paper, primarily for two purposes: (a) to demonstrate the emergence of ways of thinking afforded by the unit and (b) to illustrate the pedagogical approach to its implementation.

[1] The unit is available at http://www.math.ucsd.edu/~harel/projects/dnr.html.

[2] This is the last in a 10-unit high school algebra curriculum, called *High School Algebra Essentials*. I embarked on the development of this curriculum after the publication of two independent reviews of four current high school mathematics curricula [25]. The reviews reveal grave compromises of the mathematical integrity of the curricular content of these programs.

The theoretical underpinning of the unit is *DNR-based instruction in mathematics*[3] (DNR, for short). DNR is a research-based framework which seeks to understand fundamental problems of mathematics learning and teaching (e.g., [14, 15, 16]). In earlier work we have used this understanding to investigate existing products and develop new ones that would advance the quality of mathematics education (see for instance [17, 20, 22, 23, 24, 25]). In this respect, DNR is consistent with Stokes' model [40] for thinking about scientific and technological research which blends two motives: the pursuit for fundamental understanding and considerations of use.

The paper is organized around six sections:

- Section 1 outlines features of DNR-based curricula, focusing mainly on those that are most relevant to the curricular unit under discussion. This may be viewed as a brief introduction to DNR as well; for the reader willing to look elsewhere for more, we provide references.
- Section 2 presents a brief account of the historical development of complex numbers and relates it to specific curricular considerations.
- Section 3 describes the subjects and methodology of the three teaching experiments where we tested the unit.
- Section 4 outlines the content and structure of the unit, and describes students' mathematical behaviors during its implementation.
- Section 5 describes salient ways of thinking targeted and afforded by the unit, as well as their correspondence to the mathematical practices advocated by the Common Core State Standards [28]. In particular, we offer a definition of *structural reasoning*, and classify its manifestations in our teaching experiments into two styles of reasoning.
- Section 6 concludes with a summary and questions for further research.

[3] Here and in the rest of the paper, several words and phrases appear in italics. Often I provide working definitions and brief descriptions for these that should make the paper coherent and self-contained. At other times, the common meanings of the constituent words will suffice to follow the arguments of the paper. However for the reader interested or well-versed in mathematics education research, I should emphasize that these are technical terms that have precise meaning in their context and I use them as such.

## 1. DNR and DNR-based Curricula

DNR has been developed in a long series of teaching experiments, spread over almost three decades, in elementary, secondary, and undergraduate mathematics courses, as well as teaching experiments in professional development courses for teachers in each of these levels [18]. The study reported here is a recent project in this series. The development of DNR-based curricular material is inextricably linked to the development of DNR itself. Essentially, this is a cyclical approach which advocates development and simultaneous implementation, with resultant feedback forming the impetus for further consideration of the DNR assumptions and claims and for further development and refinement of the curricular material.

Briefly, DNR can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the DNR concepts and claims; *concepts*—constructs defined and oriented within these premises; and *claims*—assertions formulated in terms of the DNR concepts, entailed from the DNR premises, and supported by empirical studies. The system states three foundational principles: the *duality principle* (§§1.1), the *necessity principle* (§§1.2), and the *repeated reasoning principle* (§§1.3); hence, the acronym DNR. The other principles in the system are largely derivable from and organized around these three principles.

Elsewhere we have articulated eight premises for DNR, but we will explicitly refer to only four in this paper. One of these four, the *subjectivity premise*, asserts, after Piaget:

> **subjectivity:** Any observation humans claim to have made is due to what their mental structure attributes to their environment.

This assertion plays an axiomatic role in stating definitions and supporting claims in DNR. We also need it to set the tone for how the narrative of this report is to be interpreted. As Steffe and Thompson [39] argue, this subjective perspective is essential in teaching, in general, and in teaching experiment methodology, in particular:

> In our teaching experiments, we have found it necessary to attribute mathematical realities to students that are independent of our own mathematical realities. By "independent of our own"

> we mean that we attribute mathematical concepts and operations to students that they have constructed as a result of their interactions in their physical and sociocultural milieu [39, page 268].

In other words, when we describe observations of students' experiences we merely offer a model recounting our conception of what we have observed. Similarly, when we talk about concepts and skills, we mean an individual's (or community's) conceptualization of concepts and skills.

———o———0———o———

Having offered a lens through which to read this paper, I will now return to the three DNR foundational principles.

*1.1. Ways of Understanding and Ways of Thinking: The Duality Principle*

The notions of *way of understanding* and *way of thinking* have technical meanings in DNR (see [14]). For the purposes of this paper, it is sufficient to think of them as two different categories of knowledge. Ways of understanding refer to products, such as *definitions*, *conjectures*, *theorems*, *proofs*, *problems*, and *solutions*, whereas ways of thinking refer to the mathematical practices used to create such products. Examples of ways of thinking include *empirical reasoning*, *deductive reasoning*, *structural reasoning*, *heuristics*, and *beliefs about the nature of mathematical knowledge and the process of its acquisition.*

The content of mathematics is composed of both of these categories of knowledge. Mathematicians practice mathematics by applying a range of ways of thinking to reinterpret existing ways of understanding and create new ones. We state this position as the first of the eight premises of DNR [14], and call it the *knowledge of mathematics premise*:

> **knowledge of mathematics:** Knowledge of mathematics consists of all the ways of understanding and ways of thinking that have been institutionalized throughout history.

Implied in this premise is that instructional objectives should be formulated in terms of both ways of understanding and ways of thinking, not only in terms of the former, as is typically the case in traditional curricula.[4] In my experience, historical analyses and interactions with research mathematicians can be of great value in discerning and understanding the development of desirable ways of thinking, particularly, those that brand certain mathematical areas.

The first feature of DNR-based curricula is that they are designed on the basis of conceptual analyses that look for connections between ways of understanding (concepts and skills) and ways of thinking (practices, dispositions, and beliefs). The *duality principle*—the first of the three principles—expresses this feature in two dual statements:

**DUALITY I:** Students at any grade level come with a set of ways of thinking, some desirable and some undesirable, that inevitably affect the ways of understanding we intend to teach them.

**DUALITY II:** Students develop desirable ways of thinking only through proper ways of understanding.

The first part of the duality principle has several implications. First, it is of critical importance to take into account students' current ways of thinking in designing a curriculum, because these determine the content which students can and cannot learn and the quality of what they will learn. Second, long term planning for targeted ways of thinking is essential. The absence of such planning can have undesirable consequences, because, as is implied from the subjectivity premise, the ways of thinking students acquire now will affect the quality of the concepts and skills they will learn later. Third, the formation of ways of thinking is extremely difficult and those that have been established are hard to alter, (see for instance [7]). Hence, the development of desirable ways of thinking should not wait until students take advanced mathematics courses; rather, students must begin constructing them in elementary and secondary mathematics.

Implied in the second part of the duality principle is that verbally describing ways of thinking to students before they have developed them through

[4]Appreciatively, this implication has been adopted in the Common Core State Standards: "Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices [i.e., ways of thinking] to mathematical content [i.e., ways of understanding] in mathematics instruction" [28].

the acquisition of ways of understanding would likely have no effect. In particular, proceduralizing ways of thinking for students—as it is often done with Pólya's famous four steps (Read the problem, Write a plan for its solution, Execute the plan, and Test the solution, as in [31])–could just as likely have a negative effect on student performance as a positive one.

### *1.2. Learning, Intellectual Need, and the Necessity Principle*

Perhaps the most salient aspect of DNR is its description of learning:

> Learning occurs in a continuum of disequilibrium-equilibrium phases, and its content consists of (a) intellectual and affective needs that instigate or result from these phases and (b) the ways of understanding and ways of thinking that are utilized or newly constructed during these phases.[5]

This definition rests on several DNR premises (see [15]), but the part concerning disequilibrium-equilibrium is an outcome of the *knowing premise*, which follows from the Piagetian theory of equilibration [44].

> **knowing:** Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium.

With the above we intend to capture three central elements of learning: (a) that learning occurs through perturbations, both intellectual and emotional, (b) that these perturbations are an integral part of learning, and (c) that the knowledge utilized, not only that which is newly constructed, is a component of learning. The inclusion of the latter is justified by the fact that through the process of learning, old knowledge is re-learned, for example, by getting reorganized, further encapsulated, better internalized, etc.

One of the most critical pedagogical implications of this description of learning is the *necessity principle*—the second foundational principle of DNR:

> **NECESSITY:** For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to intellectual need.

[5]This is slightly modified from what appeared in [16] and [21]. See an interesting debate of this definition in [34].

Perhaps we should clarify here what we mean by intellectual need. We use the term *intellectual need* as a category distinct from *affective need*. *Affective need* is a broad class of emotions, global as well as local. Global need includes one's desires, volitions, interests, self-determination, and the like [8], while local need includes sensations of, for example, frustration, perseverance, and gratification one goes through during the learning process. *Intellectual need*, on the other hand, has to do with the disciplinary knowledge born out of one's current knowledge through engagement in problematic situations conceived as such by the person.[6] The notion of intellectual need is inextricably linked to another notion: *epistemological justification*. The latter refers to the learner's discernment of how and why a particular piece of knowledge came to be. It involves the learner's perceived cause for the birth of knowledge.

While both categories of needs are equally important in our theoretical framework, as is evident from the description of learning provided above, we have focused mainly on intellectual need in our work on DNR. We have emphasized only indirectly local aspects of affective need; global aspects of affective need have not been addressed at all. This is because we believe that the latter are largely factors of social values, norms, and priorities, as well as economic considerations, such as market supply and demand, none of which is under the direct influence and jurisdiction of teachers or curriculum developers. On the other hand, intellectual need—being the sole mechanism for knowledge construction—falls under the responsibility of those to whom the society mandates the devolution of knowledge to its youth. The local aspects of affective needs are unavoidable as products of learning, and therefore are addressed in curriculum design and development, by, for example, considerations of problem difficulty, length of curricular segments, style and level of mathematical writing, etc.

For curriculum design, the necessity principle requires that new ways of understanding and new ways of thinking should emerge from mathematical problems understood and appreciated as such by the students. And through the solution of such problems, students should realize the intellectual benefit of the targeted knowledge.

[6]For a technical definition of *intellectual need*, see [16]. For more discussion on intellectual need and epistemological justification, see [19].

DNR-based curricula, thus, do not appeal to gimmicks, entertainment, or contingencies of reward and punishment, but focus primarily on the learner's intellectual need by fully utilizing humans' remarkable capacity to be puzzled. Nor do DNR-based curricula compromise the mathematical integrity of their contents. In DNR, a curriculum is mathematical only if it adheres to and maintains the essential nature of the mathematics discipline. A "geometry curriculum," for example, is not geometry if deductive reasoning is not among its eventual objectives. DNR, however, recognizes that teaching correct mathematics is not necessarily correct teaching. A teacher may maintain the mathematical integrity of the content he or she is presenting but neglect the intellectual need of the students or be mistaken as to what constitutes such a need for them. In DNR-based instruction, the integrity of the content taught and the intellectual need of the student are equally central.

The necessity principle manifests itself regularly in the mathematician's research practices, though often it does not show up consistently in her teaching practices. The following quote from Poincaré hints at the presence of considerations of intellectual need and epistemological justification in the thought process of a great mathematician with deep pedagogical sensitivity.

> What is a good definition? For the philosopher or the scientist, it is a definition which applies to all objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils. [...]
>
> [But] what is understanding? Has the word the same meaning for everybody? Does understanding the demonstration of a theorem consist in examining each syllogism of which it is composed in succession and being convinced that it is correct and conforms to the rules of the game? In the same way, does understanding a definition consist simply in recognizing that the meaning of all the terms employed is already known, and being convinced that it involves no contradiction? ... Not for the majority [of people]. Almost all are more exacting; they want to know not only whether all the syllogisms of a demonstration are correct, but why they are linked together in one order rather than in another. As long as they appear to them engendered by caprice, and not by an intelligence constantly conscious of the end to be attained, they do not think they have understood. [30, pages 117–118]

The structural alignment of the unit we describe here with the historical development of its subject (see the preamble to §4) may give the impression that the only kind of necessity that could or should be presented to students is the actual history of the mathematical development of the concept. This is certainly not the case. In [19], I identify five categories of intellectual need: *need for certainty*, *need for causality*, *need for computation*, *need for communication*, and *need for structure*. Briefly, the *need for certainty* is the need to prove, to remove doubts. One's certainty is achieved when one determines, by whatever means he or she deems appropriate, that an assertion is true. Truth alone, however, may not be the only need of an individual; she may also strive to explain why the assertion is true. The *need for causality* is the need to explain—to determine a cause of a phenomenon, to understand what makes a phenomenon the way it is. This need does not refer to physical causality in some real-world situation being mathematically modeled, but to logical explanation within the mathematics itself. The *need for computation* includes the need to quantify and to calculate values of quantities and relations among them by means of symbolic algebra. The *need for communication* consists of two reflexive needs: the need for formulation—the need to transform strings of spoken language into algebraic expressions—and the need for formalization—the need to externalize the exact meaning of ideas and concepts and the logical justification for arguments. The *need for structure* includes the need to reorganize knowledge learned into a logical structure.

In modern mathematical practice, these five needs are inextricably linked and often occur concurrently. The need for computation, in particular, is strongly connected to other needs. For example, the need to compute the roots of the cubic equations led to advances in exponential notation, which, in turn, helped abolish the psychological barrier of dealing with the third degree "by placing all the powers of the unknown on an equal footing" [43, page 38]. Collectively, these five needs are ingrained in all aspects of mathematical practice—in forming hypotheses, proving and explaining proofs, establishing common interpretations, definitions, notations, and conventions, describing mathematical ideas unambiguously, etc. They have driven the historical development of mathematics and characterize the organization and practice of the subject today.

DNR-based instruction is structured in such a way that these five needs drive student learning of specific topics. Personally experiencing the different needs that drive mathematical practice, students can construct a global understanding of the epistemology of mathematics as a discipline.

*1.3. Repeated Reasoning Principle*

Even if concepts and skills are intellectually necessitated, there is still the task of ensuring that students (a) internalize, (b) organize, and (c) retain this knowledge. This concern is addressed by the third foundational principle of DNR, called the *repeated reasoning principle*:

> **REPEATED REASONING:** Students must practice reasoning in order to internalize, organize, and retain ways of understanding and ways of thinking.

Research has shown that repeated deliberate practice is a critical factor in these cognitive processes [6, 9]. Repeated reasoning, not mere drill and practice of routine problems, is essential to the process of internalization—a conceptual state where one is able to apply knowledge autonomously and spontaneously—and reorganization of knowledge. The sequence of problems must continually call for reasoning through the situations and solutions, and they must respond to the students' changing intellectual needs.

———o———0———o———

To recap, instructional objectives of DNR-based curricula are formulated in terms of both ways of understanding and ways of thinking, not only in terms of the former as is traditionally the case, taking into account (a) the developmental interdependency between these two categories of knowledge (the *duality principle*), (b) students' intellectual needs (the *necessity principle*) and developmentally appropriate epistemological justifications corresponding to these needs, and (c) factors that facilitate internalization, organization, and retention of knowledge (the *repeated reasoning principle*). These elements serve as the foundations for the design and implementation of curricula, as we will see in the case of the unit on complex numbers.

## 2. A Brief Historical Account

The history of the development of complex numbers can be divided into three main stages: (1) the solution of the cubic equation, (2) the struggle to make sense of this solution, (3) the emergence of complex numbers out of this struggle (see, for example, [43]). In this section, I present a synopsis of this account, focusing solely on those aspects that were didactically transposed [2, 5] into the curricular content of the unit under discussion.

### *2.1. Solution to the Cubic Equation*

The history of the development of complex numbers begins with the 16th century mathematicians' discovery of a solution formula to cubic equations. Already in 1515 the Italian mathematician del Ferro obtained an algebraic solution to $x^3 + mx = n$. The solution was rediscovered by Tartaglia (1500–1557). Cardano (1501–1576), an Italian mathematician and scientist, received the solution from Tartaglia without any justification. In modern terms, Tartaglia's solution can be described as follows:

To solve the cubic equation $x^3 + mx = n$, first obtain $t$ and $u$ such that

$$\begin{cases} t - u = n; \\ tu = (m/3)^3. \end{cases}$$

Now solve this system to obtain:

$$t = \sqrt{(n/2)^2 + (m/3)^3} + (n/2), \qquad u = \sqrt{(n/2)^2 + (m/3)^3} - (n/2).$$

The solution to the cubic equation is then:

$$x = \sqrt[3]{t} - \sqrt[3]{u} = \sqrt[3]{\sqrt{(n/2)^2 + (m/3)^3} + (n/2)} - \sqrt[3]{\sqrt{(n/2)^2 + (m/3)^3} - (n/2)}.$$

Justifying this solution by substituting $x$ in the equation was far from trivial in the 16th century mathematics, for it requires the use of the identity

$$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3.$$

While this identity is easy to prove by means of algebraic rules, the mathematics of the 16th century was not equipped with such algebraic conceptualizations. Therefore any proof of the identity would have to involve a geometric interpretation, which would likely involve the dissection of a cube in three-dimensional space [43]. Despite this, Cardano was able to justify Tartaglia's solution and further offer a solution to the general cubic equation. In modern notation, Cardano's solution can be summarized as follows:

To solve the general cubic equation $x^3 + ax^2 + bx + c = 0$, carry out the following steps:

A. With the change of variable $y = x + \frac{a}{3}$, reduce the given equation into one without the second term:

$$y^3 + py + q = 0, \text{ where } p = b - \frac{a^2}{3} \text{ and } q = c - \frac{a}{3}b + 2\left(\frac{a}{3}\right)^2$$

B. Set $y = \sqrt[3]{t} + \sqrt[3]{u}$. With this substitution, the above equation becomes

$$(t + u + q) + \left(\sqrt[3]{t} + \sqrt[3]{u}\right)\left(3\sqrt[3]{tu} + p\right) = 0.$$

C. This new equation holds if

$$(t + u + q) = 0 \quad \text{and} \quad tu + \left(\frac{p}{3}\right)^3 = 0.$$

D. Solve this system to get:

$$t = -\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}, \quad u = -\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}.$$

E. Conclude:

$$y = \sqrt[3]{t} + \sqrt[3]{u} = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}}.$$

F. Substitute $p$ and $q$ from Step A in the above to obtain a solution $x$ to the given equation.

Given the rudimentary notational system of the 16$^{\text{th}}$ century mathematics, the solution of the cubic equation was a remarkable achievement—in our eyes today, as well as in the eyes of the mathematics community then.

*2.2. Intellectual Perturbations Associated with the Cubic Formula*

Cardano's contemporaries looking into this new result encountered three baffling behaviors.

The first of these behaviors was that the cubic formula, unlike the quadratic formula (which was known at the time), did not yield all the roots. The mathematicians likely knew, by experience or intuition, that the number of solutions can be three, but not knowing anything about the $n^{th}$ roots of unity, they saw only one solution, and in some cases no solution, in the expression yielded by the formula. Furthermore, even when a root is expected, it is not always yielded by the formula. For example, $x = 2$ is a root to $x^3 + 16 = 12x$, but the root obtained by the cubic formula is $-4$. Hence, Cardano's solution led to an important question: *How many roots must a cubic equation have?*

I surmise that this behavior was more unsatisfactory than baffling, because it can be explained by the fact that Cardano's formula constitutes a sufficient, not necessary, condition for the cubic equation. This can be seen in Step C above. While the equation in Step B holds if $(t + u + q) = 0$ and $tu + \left(\frac{p}{3}\right)^3 = 0$, the converse is trivially not true—a fact which is unlikely to have gone unnoticed by the mathematicians of the time. We can easily fix this step by using the identity $(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$, making the cubic equation logically equivalent to Cardano's formula (see §§4.1). But as we indicated earlier, the proof of this identity may not have been accessible to the mathematicians of the time.

The second baffling behavior is that often the formula yields complicated expressions for simple roots. Consider, for example, the equation $x^3 + x = 2$. By substitution, we can see that $x = 1$ is a root to this equation. In fact, this is the only root, since $f(x) = x^3 + x$, being the sum of two monotonically increasing functions, is an increasing function. On the other hand, the cubic formula yields

$$x = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}}$$

as a root of the equation. Hence,

$$\sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}} = 1,$$

a rather pleasantly surprising result.

The third and most perplexing behavior of the cubic formula is that in certain cases the formula yields meaningless expressions when "real" roots are known. For example, for $x^3 = 15x+4$, the formula yields $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$, where we can easily see that $x = 4$ is a root. The puzzlement about this behavior does not end here, as we will now see.

### *2.3. The Emergence of Complex Numbers*

Among those who continued to investigate the cubic formula was Bombelli (c. 1526–1573). His approach was to apply simplification procedures to expressions involving $a + b\sqrt{-1}$, treating them as if they were meaningful expressions. In doing so, he showed, for example, that the two cubic-root addends in the expression $x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$ yielded by the cubic formula as a root of the equation $x^3 = 15x+4$ are, respectively, $2+\sqrt{-1}$ and $2-\sqrt{-1}$; therefore, $x = \left(2+\sqrt{-1}\right) + \left(2-\sqrt{-1}\right) = 4$, as expected.

While Bombelli's work provided some assurance about the validity of the cubic formula, it led to further puzzlement: How is it possible that meaningless expressions turn under legitimate manipulations into meaningful results?

This question led to further investigations into the meaning and role of the expressions $a + b\sqrt{-1}$, which—understandably—were dubbed "complex numbers." It was not until Gauss' (1777–1855) near-complete proof of the Fundamental Theorem of Algebra that complex numbers received a "legitimate" status in mathematics. The theorem implies a remarkable result: that the field of complex numbers, viewed as an extension of the field of real numbers, contains all the roots of any polynomial equation; hence, no additional extensions are needed to solve polynomial equations.

### *2.4. Historical Selectivity and DNR*

The history of complex numbers, from Cardano's solution of the cubic equation to Gauss' proof of the Fundamental Theorem of Algebra a century later, is by far richer and more intricate that the account provided here. This history is filled with ingenious ideas. They include (mentioning only those that appear in the unit under discussion): Cardano's proof of the cubic formula; the simplification of the expressions $\sqrt[3]{a+b\sqrt{-1}}$ into the more manageable expressions $c + d\sqrt{-1}$; the audacity of treating the latter expressions as encapsulated meaningful entities; the observation that a real number can be viewed as a complex number, and that, consequently, the field of complex numbers is an extension of the field of real numbers; the geometric representation of complex numbers and its consequences to the geometric meaning of the binary operations on them; De Moivre's formula; the roots of unity and their geometry; and, above all, the Fundamental Theorem of Algebra. Clearly, the content trajectory chosen for our unit does not capture the "true and complete" development of complex numbers. This trajectory merely goes through carefully selected "landmarks" in the intricate paths marking the historical development of complex numbers.

More generally, historical and philosophical developments are analyzed in DNR with the sole purpose of better understanding the genesis of concepts; we then use results of such analyses to further explore their potential relevance to individuals' cognitive processes and to curriculum development and instruction [14]. For example, as I have discussed in [12], the philosophical debate during the Renaissance as to whether mathematics conforms to the Aristotelian definition of science sheds light on certain difficulties able students have with a particular kind of proof; in turn, the results of this investigation were implemented in developing a new approach to the teaching of proof. Thus, when certain curricular conditions are satisfied, selected aspects of the historical developments may be didactically transposed. However in many cases—as in the case of the concept of logarithms, for example—the actual historical development of a concept is no longer relevant enough to be the best presentation for modern students.

### *2.5. Curricular Considerations*

As I worked on the development of the unit, I sought to address several critical questions:

1. Which aspects of this history should be adopted and translated into curricular material?
2. How should these ideas be represented and sequenced as to anchor them in students' current knowledge, intellectually necessitate them, and provide opportunities for repeatedly reasoning with them and about them?
3. What desirable ways of thinking are afforded by this history?
4. Which of these ways of thinking can be made accessible to students for whom the unit is intended?

The choices made in answering these questions involved various considerations; they include, but are not limited to, the following:

1. The typical background knowledge and cognitive ability of college-bound high school students and beginning undergraduate students.
2. The time that can reasonably be allocated to the unit in the existing high school or lower-level college programs.[7]
3. Compatibility between the content of the unit and the content of current programs; in particular, relationship between ways of thinking afforded by the unit and the mathematical practices highlighted by the Common Core State Standards.

These questions and considerations served as my guide for the development and implementation of the unit, as will be discussed in the next three sections. The level of detail in what follows is necessary for two reasons: (a) to demonstrate an actual implementation of DNR in the classroom; and (b) to delineate the instructional objectives targeted and afforded by the unit.

## 3. Method of Implementation

In this section we describe the methodology we used in the three teaching experiments where we tested our DNR-based unit on complex numbers. The subjects are inservice secondary mathematics teachers and sophomore mathematics education majors; throughout the paper, they are referred to collectively and interchangeably as "students" or "participants."

The first teaching experiment (hereafter, Experiment 1) was with thirty-two inservice secondary mathematics teachers from a large Southwestern metropolitan region. Experiment 1 lasted twelve consecutive days, six contact hours per day, from 9:00 AM to 3:00 PM, with one hour lunch break. The main focus of the experiment was *teacher's knowledge base*, which in DNR, after Shulman [37, 38], is defined in terms of three components of knowledge: *knowledge of mathematics* (teachers' ways of understanding and ways of thinking), *knowledge of student learning* (teachers' understanding of fundamental principles of learning, such as how students learn and the impact of their previous and existing knowledge on the acquisition of new

[7]I recognize that this unit takes vastly more time than an existing high school program could spare, especially given that the solution of cubic equations is not part of the Common Core State Standards. However, the unit can be offered as an enrichment course to those who take Algebra II. The intersection between the content of this unit and Algebra II is substantial.

knowledge), and *knowledge of pedagogy* (teachers' understanding of how to teach in accordance with these principles). Classroom discussions of aspects pertaining to student learning and pedagogy always emerged out of the participants' reflections on personal and collective experiences with the unit content and problems in juxtaposition with their teaching experiences. Due to this broader focus, the time allocated to Experiment 1 was not sufficient to cover the unit in its entirety. In this paper, I limit the report of this experiment to the participants' mathematical behaviors, and only in relation to the material leading up to the cubic formula.[8]

The second and third teaching experiments (hereafter Experiment 2 and Experiment 3) were conducted at a major university, also located at a Southwestern metropolitan region. The participants of these two experiments were prospective secondary teachers in their sophomore year. Each experiment was carried out in a two-unit quarter course made up of ten 100-minute weekly lessons. Fifteen prospective teachers participated in Experiment 2, and twelve in Experiment 3. The focus in both experiments was restricted to mathematical aspects of the unit; student learning and pedagogy were not addressed. The entire unit was covered in each of these experiments.

The instructional methods used in the three experiments included small working groups, whole class discussions, and lectures by the instructor.[9] Typically, lessons began with participants working collaboratively in small groups on problems from the unit. Composition of the groups varied in each lesson, with no particular strategy. Participation in the small working group was entirely voluntary. To foster meaningful participation of all group members in the small groups, students were urged to first work individually on the problem assigned before the group discussion. After some time commensurate with the difficulty of the problem and engagement level of the participants, representatives of various groups presented their solutions and responded to questions and comments from their classmates or teacher.

[8]The data collected in this experiment was rather extensive. It includes, a 90-minute pretest, a 90-minute posttest, a daily 20-minute survey, a 90-minute extensive survey at the end of the experiment, and an extensive field notes recording each of the classroom discussions taken by another researcher, Dr. Osvaldo Soto. The analysis of this data is under way, and will be reported in a separate publication.

[9]We do not see lecturing as a method that contradicts our problem-oriented learning approach. See the discussion in §§4.5.

At the start of the unit and during various times thereafter, the following general information was conveyed to the participants:

> "The unit consists of three parts, which mark three evolving stages in the development of its content. Your work in this unit revolves around four kinds of activities:
>
> 1. In-Class Problems (ICP)
> 2. Questions (Q) and Answers (A)
> 3. Probes (P)
> 4. Homework Problems (HP)
>
> **ICP:** The ICPs serve two purposes. Some ICPs serve as a starting point for a new idea introduced in a lesson; their goal is to bridge new knowledge to be learned with the knowledge you have already learned. Other ICPs aim at enhancing your ability to read and understand mathematical text. In these problems, you are asked to *study* a solution to a problem or a proof of an assertion.
>
> - *To study a mathematical text means to understand the underlying ideas in the text and the mathematical reason for each claim made in the text.*
> - *To test your understanding, reproduce from memory the complete solution or the proof, and, in doing so, use your own words and choose your own symbols (different from those used in the text).*
>
> **P:** Often, as you read a mathematical text, you will be asked to respond to Probes (marked by P). They appear in the form of queries, such as "why," "how," and "explain." It is necessary that you respond to these queries before continuing reading. This will help you better understand the text and, for a long run, improve your ability to read mathematical texts.
>
> **Q&A:** The aim of the Questions is to motivate new concepts and ideas, by helping you see that new knowledge always comes about out of a need to solve a problem, resolve a puzzle, explain a phenomenon, etc. Each Question is then followed by an Answer, which often contain Probes.

**HP:** Last, but not least, are the Homework Problems (HP) on each of the lessons. To master a mathematical subject, one must practice the reasoning particular to that subject. The ultimate goal of these problems is to make you a better mathematical reasoner, by gradually improving your ability to prove assertions, solve problems, think in general terms, and be fluent in computing with understanding. Some of the HPs aim at extending your knowledge beyond what is presented in a particular lesson, to prepare you for subsequent lessons."

Printouts of segments of each lesson were distributed to the students. However, we did this strategically, to avoid introduction of material prematurely before students had fully realized the need for an idea, for example. To illustrate, Lesson 1 consists of four ICPs. The first three are listed below, and the fourth is a solution to ICP 3b.

**ICP 1:** The sum of the volumes of two cubes is 16, and the product of the side of one cube by the side of the other cube is 4. What are their dimensions?

**ICP 2:** Repeat Problem 1, where the sum of the volumes is 27/4 and the product of the sides is 9/4.

**ICP 3:** We can create as many problems like Problems 1 and 2 as we want by varying the volumes of the cubes and the product of their sides.

a. Create two such new problems and solve them.

b. Is it possible that among such problems there are ones with no solutions?

The three problems were not distributed at once. Students first worked on ICPs 1 and 2. Following the discussion of their solutions to these problems, they were handed ICP 3, which aims to necessitate the abstraction of ICPs 1 and 2. Following the discussion of ICP3, students were handed ICP 4, which is a solution to ICP 3. Consistent with this strategy, the unit lessons are not titled, so as to not reveal their instructional goals in advance; such revelation might restrict student reasoning and actions.

Due to the time intensity of Experiment 1 and the family obligations of its participants (inservice secondary teachers), no homework was assigned outside the classroom; instead, participants worked collaboratively in small groups during class on the homework problems for each of the lessons covered. In Experiment 2 and Experiment 3, on the other hand, homework was assigned in each lesson, collected, graded, and returned. Occasionally, upon a participant's request, some homework problems were revisited in class.

The observations of the participants' mathematical behaviors I report in this paper are based on various sources: retrospective notes taken immediately after each class, synopses of interactions within working groups, student questions and responses during class discussions, works presented to the entire class by the group representatives; homework assignments, and, in the case of Experiment 2 and Experiment 3, a final examination.

Not surprisingly, no significant differences between Experiment 2 and Experiment 3 were observed; one can expect this, given the similarities in population of the two experiments, class size, and length and intensity of intervention. There were major differences between Experiment 1 and these experiments. Nonetheless, there were shared behaviors among the three experiments in relation to the lessons covered. The discussion in the rest of the paper focuses mainly on these behaviors.

### Interlude: What do we mean by curriculum?

While it should be clear from the discussion thus far what the term "curriculum" means in DNR, it is worth articulating it explicitly. For this we refer to Thompson: "[Mathematics curriculum] is a selected sequence of activities, situations, contexts, and so on, from which students will, it is hoped, construct ... particular [mathematical] way[s] of [understanding and ways of] thinking" [44, page 191].[10] To this, I add—perhaps the obvious—the classroom discourses over the curriculum material. The crucial implication of this definition, as Thompson put it, is that (a) one "must make explicit the nature of the knowledge that [one] hope[s] is constructed and (b) make a case that the chosen activities will promote its construction" [44, page 192].

[10] As I mentioned in §1, and discussed in length in [14, 15, 16], the term "way of thinking" has a particular technical meaning in DNR. For the sake of accuracy, I should point out that Thompson's meaning of this term is different from, yet complementary to, the DNR meaning. This, however, has no effect on what is being said here.

In other words, instructional objectives are not inherent to curricular content and its presentation; rather, both the curriculum developer and the teacher must be cognizant of and explicit about these objectives and the teaching actions needed to achieve them.

The following two sections, §4 and §5, demonstrate this characterization of "curriculum." Specifically, in §4 I outline the learning activities of the unit explicitly. Then in §5 I lay out the salient ways of thinking targeted by these activities, and further explain why the chosen activities are likely to promote their construction.

## 4. Structure and Development

The unit studied here consists of twelve lessons[11] organized around three stages corresponding to the historical development of its subject (cf. §2):

**Stage 1 (Lessons 1–5)** delineates the ideas underlying the development leading up to the cubic formula (cf. §§2.1).

**Stage 2 (Lessons 6–8)** deals with the puzzling behaviors of the cubic formula (cf. §§2.2).

**Stage 3 (Lessons 9–12)** deals with resolutions of these puzzles by constructing new numbers (the complex numbers), investigating their algebraic and geometric meanings, and articulating their remarkable role in understanding polynomial equations (i.e., the Fundamental Theorem of Algebra) and in solving various mathematical problems (cf. §§2.3).

Although we will organize the discussion of the unit around these stages, they are not labeled as such in the unit. The unit is composed of an uninterrupted sequence of intellectual perturbations followed by their mathematical resolutions, what we call *perturbation-resolution pairs*. The twelve lessons merely correspond to twelve breaking points, determined primarily by considerations of lesson flow and duration. Since it is neither necessary, nor

[11] The division of the unit into lessons does not dictate a particular pace; it merely marks the sequence of evolving segments that build up the entire unit. The pace of a lesson is to be determined by the teacher, who knows best her or his students' current knowledge and abilities.

possible due to space limitation, to discuss the complete sequence in detail, we condense it into nine perturbation-resolution pairs, which we use here as the basis of our discussions of the unit and its implementation.

Readers interested in the specifics of the unit will benefit from having access to it as a complementary reference for the rest of this section. Readers who are not as interested in the particular details of the complex numbers unit may skip to §§4.4 where we provide a bird's eye view of the unit, and continue from there with our discussion of DNR as it applies to the unit.

We have one final comment before proceeding to discuss the three stages. The narrative reporting on student experiences in the rest of this paper is about students' past behaviors as they engaged in activities (still) present in the unit. To capture both past and present, I chose to use a past tense in connection to student behaviors and a present tense in connection to the unit text.

### *4.1. Stage 1: Delineating the Ideas Underlying the Development Leading Up to the Cubic Formula*

As we noted in §§2.1, the solution formula in Cardano's proof is only a sufficient condition for the cubic equation. The proof developed in the unit is a slight modification of Cardano's proof, in that it uses the identity $(u+v)^3 = u^3+3u^2v+3uv^2+v^3$, thereby making Cardano's solution formula equivalent to the cubic equation, as shown below:

To solve the equation

$$x^3 + px + q = 0,$$

where $p, q \neq 0$, let $x = u + v$. By cubing both sides of this equation, expanding, and factoring $uv$, we get:

$$x^3 - 3uvx - (u^3 + v^3) = 0.$$

These two equations are equivalent if and only if:

$$\begin{cases} uv = -\frac{p}{3}; \\ u^3 + v^3 = -q. \end{cases}$$

And this system is equivalent to the system,

$$\begin{cases} v = -\frac{p}{3u}; \\ 3^3(u^3)^2 + 3^3qu^3 - p^3 = 0. \end{cases}$$

The quadratic equation in this latter system is equivalent to

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}} \quad \text{or} \quad u = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}}.$$

Using the symmetry between the variables $u$ and $v$ in the former system, we get that the latter system is equivalent to:

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}} \quad \text{and} \quad v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}}.$$

Hence $x$ is a solution to our cubic equation if and only if:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}}.$$

This short proof is more intricate that it might seem. It rests on the ingenious idea of conceiving of a solution $x$ as a sum of two numbers $u + v$ for the purpose of reducing the cubic equation into a system of equations, which, in turn, is reducible to a quadratic equation. The critical questions from a DNR perspective are: How do we intellectually necessitate the various elements of this proof to the students for whom the unit is intended? What ways of thinking are afforded by the proof? How do we advance these ways of thinking among students? We begin to answer these questions here, in the rest of this subsection, as we further discuss Stage 1 (Lessons 1-5). Some of our answers will be further delineated and will evolve as the paper unfolds.

———o———0———o———

The essence of Lessons 1-5 can be captured in the following sequence of four perturbation-resolution pairs:

**Perturbation 1: How do we solve these new systems of equations?** The unit is launched in Lesson 1 with four ICPs (see §3 for the specific statements of these problems). The first two are word problems involving products and sums of cubes of unknowns. The third problem is formulated so as to compel students to abstract the first two problems into a family of problems represented by a system of equations of the form (1) below, and investigate the conditions under which a solution exists.

$$\begin{cases} uv = P; \\ u^3 + v^3 = Q \end{cases} \tag{1}$$

The fourth problem provides a solution to this system, and asks the students to study the solution and compare it to their own.

This content was chosen because it anchors the unit in students' prior knowledge and ushers them into the proof of the cubic formula. Students come to the unit knowing how to translate word problems into systems of equations and solve the latter by elimination or by substitution of variable. Typically they are familiar with simple systems, such as $2\times 2$ and $3\times 3$ linear systems. Thus the main difference here is the form of the system. As can easily be seen, system (1) is of the same form as that of the first system in the proof of the cubic formula presented above.

**Resolution: Reduce the system down to a quadratic equation (RQE).** The students were first handed ICP 1 and ICP 2. As they worked individually on these problems, we observed that most of them chose a trial-and-error approach. There were a few students, however, who began by translating ICP 1 into a system of equations of the form (1). As soon as the class transitioned to small working groups, these students learned about the trial-and-error approach and consequently abandoned the algebraic approach. Following a brief classroom discussion of the solution to ICP 1 and 2, the teacher handed ICP 3 to the students.

To solve ICP 3, one is compelled to generalize ICP 1 and ICP 2 into a family of problems, and to reason about the latter in terms of parameters. It took some negotiation for the students to understand this problem, especially its second part, due—as some of them indicated—to its unusual formulation. By and large, most students constructed a system of equations of the form (1) and pursued to solve it by substitution of variable (e.g., $v = \frac{P}{u}$). However, many students (around 40% in Experiment 1 and 30% in Experiment 2 and Experiment 3) were unable to independently observe that the polynomial equation resulting from the substitution is quadratic in $u^3$ (i.e., $(u^3)^2 - Qu^3 + P^3 = 0$). Surprising as it may be, this is an important observation, because it indicates that this activity can potentially promote the way of thinking of encapsulating an expression ($u^3$ in our case) into a single entity. As will be discussed in §5, this way of thinking is one of the various aspects of *structural reasoning*.

Once the students completed successfully their solution to ICP 3, they were handed ICP 4. The latter presents a complete solution to ICP 3, with one element not present in the students' solutions. The new element is the use of symmetry between the variables of system (1), in order to save computation. Observing and utilizing symmetry is another aspect of *structural reasoning*, to be discussed in §5.

The solution to ICP 3 was followed by a discussion comparing the trial-and-error approach and the algebraic approach. The conclusion of this discussion was that while the trial-and-error approach is advantageous when the relations among the problem quantities are simple, as in ICP 1 and ICP 2, it is ineffective when these relations are complex. And, in addition, the trial-and-error approach, in and of itself, does not resolve the question concerning uniqueness of a solution, whereas the algebraic approach does.

**Perturbation 2: What can we do if the system is not reducible to a quadratic equation?** As students practiced the RQE (reduction to quadratic equation) technique on a family of systems involving products and cubes of unknowns, they encountered one system for which the technique leads to an irreducible 6-degree polynomial equation. This is system 1(f) in the Homework Problems on Lesson 1, which is of the form:

$$\begin{cases} \frac{uv}{u+v} = P; \\ u^3 + v^3 = Q \end{cases} \tag{2}$$

with $P = -2$ and $Q = 32$.

Students searched for ways to reduce the resulting 6-degree polynomial equation into a quadratic equation. There were also a few of the students (one in Experiment 1, none in Experiment 2, and 1 in Experiment 3) who tried to reduce the equation into a cubic equation.

**Resolution: Reduce the system down to a cubic equation (RCE).** After these failed attempts, students were asked (in Lesson 2) whether the three expressions, $uv$, $u + v$, and $u^3 + v^3$, in system (2) reminded them of a known identity. In each of the teaching experiments, there were only a few students who made the connection between these expressions and the identity $(u+v)^3 = u^3+3u^2v+3uv^2+v^3$, but none of them was able to utilize the identity to solve the system.[12]

[12] As a group, students indicated that while they didn't remember the identity—and

The class was then shown how an appropriate use of this identity reduces system (2) into a cubic equation (see Lesson 2). Following this, students were asked (in the Homework Problems on Lesson 2) to solve a series of systems of equations aimed at helping them practice the RCE (reduction to cubic equation) technique. The systems are designed to be reducible to "easily solvable" cubic equations, i.e., equations that can be solved by finding one root of the equation through trial and error or by the Rational Root Theorem, and then finding the rest of the roots by the Division Theorem.[13]

**Perturbation 3: What can we do if RCE (reduction to cubic equation) leads to a cubic equation that is not "easily solvable?"** Among the systems in the Homework Problems on Lesson 2, there is one which was designed to be reducible to a cubic equation that is not "easily solvable." This is the system (3) which was included in the subsequent discussion in the opening of Lesson 3:

$$\begin{cases} uv(u+v) = 8; \\ u^3 + v^3 - 2u - 2v = -31. \end{cases} \tag{3}$$

Students successfully used the RCE (reduction to cubic equation) technique to reduce this system into the cubic equation $x^3 - 2x + 7 = 0$, but since the equation has no rational roots, they were not able to solve it by trial-and-error or the Rational Root Theorem. This difficulty in turn led to the question "How do we solve cubic equations in general?" That is, "Is there a solution formula for cubic equations, as in the case of quadratic equations?"

It should be highlighted that the goal of the lessons up to this point goes beyond this question. Through these lessons, students learn the RQE and RCE techniques, a combination of which forms the underlying idea of the proof of the cubic formula, as we will now see.

consequently did not see its connection to the expressions involved in the system—they recognized it once it was shown to them. The inability to retrieve an object but merely recognize it speaks against those who seek to deemphasize computational fluency in mathematics curricula—a point I will return to in §5.

[13]Most of the students were familiar with the Rational Root Theorem, though without a proof. In each of the teaching experiments, the theorem was proved, though it does not appear in the unit. The Division Theorem was familiar to all students, mainly as a technique for division of polynomials. A generic proof (a proof in the context of a particular example) was presented in the first experiment, and a general proof was given in the other two experiments (see Homework Problems on Lesson 8).

**Resolution: Develop a formula for cubic equations; first focus on those without the second term.** Since the problem at hand is solving the equation $x^3-2x+7=0$, Lesson 3 focuses on equations of the same form; namely: $x^3+Ax+B=0$. In the first segment of the lesson, the teacher began by reminding students of how the completing-the-square method transforms the quadratic equation $x^2+Ax+B=0$ to an equation of the form $(x+T)^2+L=0$, which in turn leads to the quadratic formula. The question is then raised as to whether an analogous approach can be used to transform the general cubic equation $x^3+Ax+B=0$ into an equation of the form $(x+T)^3+L=0$—an easily solvable equation. Students pursued this approach in their small working groups. Some of them concluded on their own, others after reading the text of Lesson 3, that such a transformation is not possible for all cases. However, a sizable number of students (eight in Experiment 1 and four in each of Experiment 2 and Experiment 3) needed additional help to understand this conclusion.

The goal of this segment was for students to realize that the "reduction to a simpler problem" strategy—another manifestation of *structural reasoning*, to be discussed in §5—is merely a heuristic, a rule of thumb, not a law. In this respect, the failure to analogize the development of the cubic formula to that of the quadratic formula achieved its intended goal.

In the second segment of Lesson 3, the teacher followed this failed attempt by returning to the equation $x^3-2x+7=0$, the initial focus of the lesson. He began by asking the students to revisit the solutions to systems (1) and (2) and try to use the RQE and RCE techniques to devise a way to solve the equation. After some time, when none of the students came up with the solution, he suggested that they use the substitution $x=u+v$. Even after this hint, only one student (in Experiment 1) was able to use the teacher's suggestions to successfully develop a solution to the equation, and her solution required reorganization and bridging of several gaps. In all cases, therefore, the teacher ended up presenting the complete solution to the equation. The solution amounts to a generic proof of Cardano's formula along the ideas of the proof presented in the opening of this subsection.

In the Homework Problems on Lesson 3, students are asked to practice this solution process on a relatively large number of cubic equations of the same form. Through the repeated reasoning involved in solving these problems, students (with almost no exception) successfully arrived at the solution formula for cubic equations of the form $x^3 + Ax + B = 0$. Once they internalized the process leading up to this formula, they were given another set of problems (Homework Problems on Lesson 4) to apply the formula directly.

**Perturbation 4: What can we do with cubic equations with a second term?** As students proceeded with the Homework Problems on Lesson 4, they encountered an obstacle in Problem 1(j). Neither the cubic formula known this far, nor the re-application of the technique applied to develop it (i.e., the combination of RQE and RCE techniques) are successful for solving equations involving a second term. By now, students had gotten accustomed to the way of thinking of reducing one unfamiliar form into a familiar one. In each experiment, students independently asked "how can we transform equations of the form $x^3 + Bx^2 + Cx + D = 0$ into equations of the form $x^3 + Ax + B = 0$?" This is the topic of Lesson 5.

**Resolution: Reduce equations with a second term into ones without a second term.** Lesson 5 begins by revisiting the quadratic equation $x^2 + Ax + B = 0$ and showing how the change of variable $x = y + \frac{-A}{2}$ reduces the equation to one without the first term. Then students can see how, similarly, the change of variable $x = y + \frac{-B}{3}$ in the cubic equation $x^3 + Bx^2 + Cx + D = 0$ leads to a cubic equation without the second term, resulting in a cubic equation of a desired form: $x^3 + Ax + B = 0$. (Later, in Homework Problem 6 for Lesson 5, students are asked to generalize this observation and prove the result by using the Binomial Theorem.[14]) With this knowledge in hand, students derived the cubic formula for the most general cubic equation in the Homework Problems on Lesson 5.

### *4.2. Stage 2: Realizing the Puzzling Behaviors of the Cubic Formula*

The second stage in the development of complex number is realized in three lessons (Lessons 6-8) which mainly lay the groundwork for two specific perturbations that will finally be resolved in Stage 3. Lesson 6 discusses three puzzling behaviors (labeled in the unit as *Surprises*) of the cubic formula:

1. Unlike the quadratic formula, the cubic formula does not (seem to) give all the solutions to the equation,

[14]The Binomial Theorem was also shown in class, but without proof, since most of the students did not know mathematical induction and only a few were exposed to combinatorics.

2. The cubic formula often produces complicated expressions for simple roots, and
3. The cubic formula often produces meaningless expressions as roots when it is known that (real) roots exist; and yet, when such expressions are manipulated using "legitimate" algebraic rules, they often turn into (meaningful) numbers.

Clearly, of these three surprises, the second can hardly be called a perturbation, since students have experienced cases where complicated expressions are manipulated into simple ones. Accordingly, we list only the first and third surprises as perturbations. The second "surprise," however, has an important role, as we will see shortly.

**Perturbation 5: Why does the cubic formula fail to yield all the roots?** From the vantage point of our students at this stage, the cubic formula always yields at most one root—the same vantage point of the mathematicians of the $16^{\text{th}}$ century. For our students, however, this behavior should have been even more puzzling since the proof for the cubic formula they learned establishes logical equivalency between the formula and the equation. This is not what happened. For the sake of continuity, we postpone discussion of this observation to §§5.2 where it belongs.

A full resolution of this perturbation appears in §§4.3, following the Fundamental Theorem of Algebra and the geometric representation of complex numbers.

**Perturbation 6: Why does the cubic formula often yield meaningless expressions as roots even when it is known that (real) roots exist, and how is it possible that when such expressions are manipulated using algebraic rules they often turn into meaningful numbers?** The cubic formula often produces meaningless roots involving expressions of the form $a + b\sqrt{-1}$ (where $a$ and $b$ are real numbers), even in cases where it is known in advance that all the roots are meaningful (i.e., real numbers). Here we take advantage of the second behavior of the cubic formula; namely, that often under certain algebraic manipulations, complicated expressions yielded by the formula are simplified into a single number. For example, the root

$$\sqrt[3]{-5 + \frac{26}{3\sqrt{3}}} - \sqrt[3]{5 + \frac{26}{3\sqrt{3}}}$$

turns out to be 2 when simplified (Lesson 6). This suggests the following thought experiment: Apply similar manipulations to such meaningless roots by treating, at least temporarily, the expressions $a + b\sqrt{-1}$ as if they were meaningful algebraic expressions. The result turns out to be pleasantly surprising. The meaningless roots often turn into meaningful ones. But, reflecting on this result, a new perturbation arises: **How is it possible that logical algebraic operations turn meaningless roots into (meaningful) numbers? Can it be that the expressions are meaningful after all? If so, what are their meanings?**

The resolution to this perturbation is the main focus of Stage 3 (§§4.3). It is appropriate, however, to describe here a conversation that occurred in both Experiment 2 and Experiment 3 (where Stages 2 and 3 were fully covered) which demonstrates that the way complex numbers are traditionally introduced in elementary algebra is abrupt and rather contrived.

Several students in these experiments were exposed to complex numbers in their classes in high school, and so when the question about the meaning of the expressions $a+b\sqrt{-1}$ first emerged, a conversation along the following lines ensued:

> **Students:** $\sqrt{-1}$ is the complex number $i$.
>
> **Teacher:** What does that mean?
>
> **Students:** It means that $i^2 = -1$.
>
> **Teacher:** So we define $i$ to be a number such that $i^2 = -1$. This is fine, but for what purpose?
>
> **Students:** To solve the equation $x^2 + 1 = 0$.
>
> **Teacher:** That is true. But consider this: We create a new number $i$ to turn an equation with no solution into one with a solution. Why then don't we do the same for other equations, such as $x+1 = x+2$ or its equivalents? Why don't we create numbers for such equations to turn them into equations with a solution?
>
> **Students:** That was what we were told in school. Really, why don't we?—Why do we treat $x^2+1 = 0$ differently from all other equations that don't have solutions?

Thus, students' prior knowledge about complex numbers was not dismissed, but was confronted in a manner that created a need with the students to better understand how complex numbers came about and the role they serve in mathematics. Understandably, these students did not have the necessary mathematical knowledge to counter this argument. Nor could they see that the two types of equations are fundamentally different. While a proposed solution (in any nontrivial algebraic structure such as a field) to $x + 1 = x + 2$ would lead to logical inconsistencies, the proposed solution $i$ to $x^2 + 1 = 0$ does not.

*4.3. Stage 3: An Investigation into the Meanings of Complex Numbers*

In the previous stage, the class as a whole came to see the need to launch an investigation into the possible meanings of the expressions $a+b\sqrt{-1}$. The following investigations are prefaced by a statement along the following lines:

> It took a long time in the history of mathematics to construct meanings for these expressions and fully appreciate their contribution to mathematics and science. Perhaps this is the reason they were dubbed complex numbers.
>
> The main questions comprising the forthcoming investigations are: Are complex numbers "genuine?" That is to say, do they fulfill our expectations about numbers? The numbers we know represent quantities, such as length, width, area, weight, temperature, speed, work, etc., and they can be located on the number line. These numbers can be added, subtracted, multiplied, and divided, and these operations obey certain rules, and they too have quantitative and geometric meaning. And, most important, with the use of real numbers we can solve problems of all kinds. Do the complex numbers have these qualities?

This launches two investigations into these questions; the first is algebraic (Lessons 8-10) and the second geometric (Lessons 11 and 12).

The goal of the algebraic investigation is two-fold. It begins (in Lesson 8) with the idea that the field of complex numbers is an extension of the field of real numbers. Without mentioning the term "field," the notion of extension is conveyed (through problems) as a relation where

(a) each real number is a complex number and

(b) each of the operations defined on complex numbers agrees with its counterpart operation on the real numbers.

The focus of Lesson 8 is to establish these properties, familiarize students with complex numbers, and advance their computational fluency by solving various types of problems (Homework Problems on Lesson 8). Of particular importance are problems asking to show that a certain object is a complex number (e.g., a solution to a given equation). To maintain the flow of our presentation, we defer the discussion of other subtle issues involved to §§5.2 and §§5.3 where it belongs.

Lesson 8 is followed by an effort to bring students to appreciate (without a proof) the essential claim of the Fundamental Theorem of Algebra; namely, that no further extension of the complex number field is needed in order to solve any polynomial equation. Two lessons (9 and 10) are devoted to this effort, which is encapsulated into the following perturbation:

**Perturbation 7: Is the extension of $\mathbb{R}$ into $\mathbb{C}$ enough?** The lead to this perturbation is formulated in a (seemingly) simple question: It is easy to show that the roots of any quadratic equation with real number coefficients are complex numbers, but what about a quadratic equation with complex number coefficients? Must its roots also be complex numbers? We develop the affirmative conclusion to this question by first showing that the roots of the equation $z^2 - i = 0$ are complex, and then using this result to answer the question concerning the general case $Az^2+Bz+C = 0$ where $A$, $B$, and $C$ are complex numbers. The Homework Problems on Lesson 9 consist mostly of quadratic equations with complex coefficients, but they also contain simple cubic equations. The goal of these problems is to bring students closer to understanding and appreciating the value of the Fundamental Theorem of Algebra.

In the opening of Lesson 10, the teacher summarized the results obtained in Lesson 9, and then asked the class, "What question would naturally follow from these results?" In both Experiment 1 and Experiment 2, some students promptly responded with "Are the roots of all cubic equations complex numbers?" And when the teacher pushed further, other students extended the question to polynomial equations of higher degrees, and eventually to all polynomial equations. The teacher then recapitulated this discussion as follows:

> We want to know whether the roots of any cubic equation are members of $\mathbb{C}$. If the answer is negative, then our extension of $\mathbb{R}$ into $\mathbb{C}$ is incomplete, because for cubic equations whose roots are neither real nor complex, we would need to continue our

> investigation, and hope to be able to extend $\mathbb{C}$ further in order to accommodate the new roots. We can ask the same question about any polynomial equation: Are the roots of any polynomial equation members of $\mathbb{C}$? If not we will have to keep extending $\mathbb{C}$ with every discovery of a new type of root.

**Resolution: Remarkably, the answer to this question is yes and so no further extension of $\mathbb{C}$ is needed.** At this point, students were presented with the remarkable result that the answer to this question is affirmative; namely, there is no need to further extend $\mathbb{C}$, as is implied by the Fundamental Theorem of Algebra: *Any polynomial equation* $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$*, where* $c_0, c_1, \cdots, c_n$ *are complex numbers, has a solution and all its solutions are members of* $\mathbb{C}$. The following was also conveyed to the students:

> We (the class) proved this theorem only for the case of quadratic equations. As you have seen, the proof was not trivial, and nor were the proofs you provided for some simple particular cubic equations in the Homework Problems on Lesson 9. What if we had to approach the problem in a similar manner for a general polynomial equation of any degree? This is unlikely to succeed. Indeed attempts to do so even for special cases, such as the general fifth degree polynomial equation, failed. Not until the $19^{\text{th}}$ century did mathematicians provide a satisfactory proof. By now, many proofs of the Fundamental Theorem of Algebra have been found, but all use ideas outside the scope of high school mathematics.
>
> This theorem is remarkable because it frees us from the need to further extend $\mathbb{C}$ in order to solve polynomial equations of different degrees. Historically, it was this theorem that gave the complex numbers a credible status—they are the only roots that any polynomial equation has! We can therefore understand why mathematicians dubbed this theorem, The Fundamental Theorem of Algebra.

Lesson 10 concludes with a resolution to Perturbation 5 (why does the cubic formula fail to yield all the roots?). Using the Fundamental Theorem of Algebra, we prove that any polynomial in $\mathbb{C}$ of degree $n$ has at most $n$ distinct roots, and if we count all the roots of the polynomial, including those that appear more than once, then the number is exactly $n$.

But this naturally leads to the question:

**Perturbation 8: How do we find all the roots of a cubic equation?** A resolution to this perturbation is one of the results of the geometric investigation encapsulated into the following perturbation.

**Perturbation 9: Do complex numbers and the operations on them have a geometric meaning?** Lesson 11 begins by establishing that the geometry of the real numbers is not suitable for complex numbers. The latter can neither all be located on the number line, nor ordered, and so the question is: What is the geometry of complex numbers?

**Resolution:** Some students suggested that we assign to each complex number $a + bi$ the ordered pair $(a, b)$. Clearly some of the students knew this representation from prior experience; others just came up with it on their own. This idea is then followed by the question: Does this assignment allow us to identify unambiguously complex numbers with points in the coordinate plane, and points in the coordinate plane with complex numbers? The question is answered affirmatively by articulating the phrase “to identify unambiguously” to mean that the assignment

$$a + bi \longmapsto (a, b)$$

is a one-to-one and onto function, and then proving these claims.

This part of the lesson was rough for many of the students, as was evident by their difficulty to recapitulate the meaning of “one-to-one” and “onto,” and correctly explain why it is needed to show that the assignment $a+bi \longmapsto (a, b)$ possesses these properties. This, of course, is due to students’ weak understanding of the concept of function, as has been widely documented (see, for example, [1, 29]).

Once this geometric meaning of complex numbers has been established, the next important question is: What does this meaning entail for the geometric meaning of the four operations on complex numbers? The word “entail” is critical here, since we are now constrained by the correspondence $a + bi \longmapsto (a, b)$ and the definitions of these operations. Through a probe into this question, the polar representation of the complex numbers emerges naturally, and the following meanings for addition and multiplication are correspondingly established: For any complex numbers $z_1$ and $z_2$, (1) the points $z_1$, $z_2$, $z_1 + z_2$, and 0 form a parallelogram and (2) the radius vector of the product $w = z_1 z_2$ is the product of the radius vectors of $z_1$ and $z_2$, and the angle of this product is the sum of the angles of $z_1$ and $z_2$.

Lesson 12 concludes the unit with a discussion of the $n^{th}$ roots of unity and their geometry, thereby resolving the remaining perturbation regarding the number of roots the cubic formula yields (see Perturbation 5 of §§4.2).

### *4.4. A Bird's Eye View of the Unit*

Figure 1 below depicts the main junctions in the development that leads up to the cubic formula. As indicated, the unit starts with a family of systems of equations, focusing on two types: (1) and (2). An arrow from one cell to another cell, going through a third cell, indicates that "the content of the first cell is reducible to the content of the second cell by using the content of the third cell."

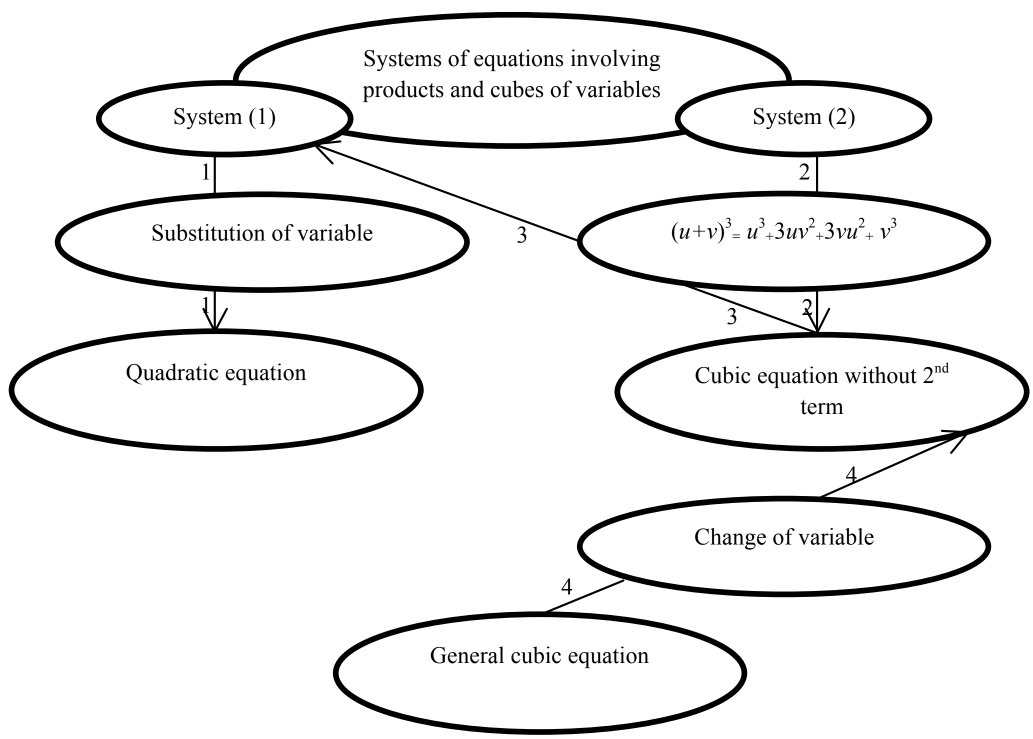


Figure 1: The development leading up to the cubic formula as described in Stage 1.

The labels (1)-(4) for the arrows correspond to the order of development:

(1) Systems (1) are easily solvable by RQE (reduction to quadratic equation) through Substitution of Variable, but

(2) Systems (2) are more intricate, in that they are solvable by RCE (reduction to cubic equation) through the identity $(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$.

(3) Cubic equations without the second term are reducible to system (1) through the same identity, which, by (1), is reducible to a quadratic equation. And finally,

(4) Any general cubic equation is reducible to a cubic equation without the second term through Change of Variable, which, by (3), is reducible to a cubic equation.

Figure 2 below provides a bird's eye view of the development of complex numbers as described in Stages 2 and 3. The cells and arrows are as described for Figure 1.

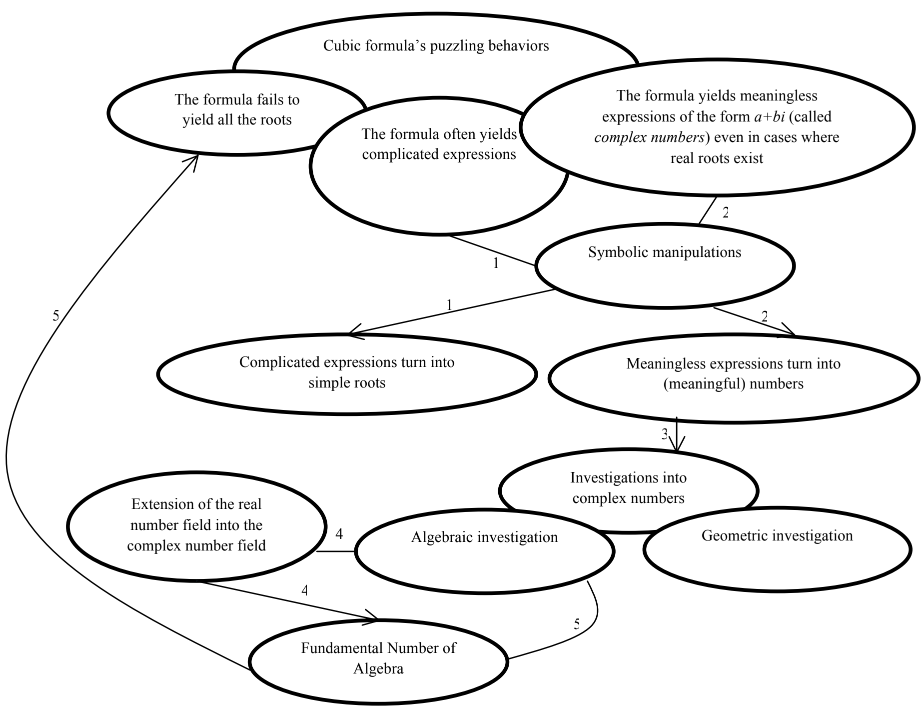


Figure 2: The development of complex numbers as described in Stages 2 and 3.

*4.5. To Tell or Not to Tell?*

Two remarks on DNR-based interventions are in order:

The first remark addresses the question of whether instructors should tell a mathematical idea directly or let students develop it on their own—a question widely and heatedly discussed in the mathematics education research community during the late 1980s and early 1990s.[15] According to DNR, the question to ask is not "to tell or not to tell?" but "when to tell and when not to tell?" For example, in the state of development discussed in Perturbation-Resolution 3 (§§4.1), students were judged to have had an intellectual need to solve the cubic equation under consideration, and had already learned the techniques needed to solve it, so they were deemed ready and capable to comprehend the solution presented to them by the teacher.

A more technical term used in the mathematics education literature to describe this state of readiness is *zone of proximal development* (ZPD) (cf. [45]). In DNR, the notion of ZPD is strongly connected with intellectual need: One is judged to be in the ZPD relative to a particular concept if one is judged (by her or his teacher) to have developed an intellectual need for that concept. The teacher's decision "to tell" and "what to tell" rests largely on this judgment.

This position is based in part on one of the DNR premises, called the *teaching premise* (see [16]):

> **teaching:** Learning mathematics is not spontaneous. There will always be a difference between what one can do under expert guidance or in collaboration with more capable peers and what he or she can do without guidance.

This premise stresses the indispensable role of the teacher in the classroom. Most emphatically, the teacher is the sage on the stage, not the guide on the side. The teaching premise is particularly needed in a framework oriented within a constructivist perspective, like DNR, because one might minimize the role of expert guidance in learning by (incorrectly) inferring from such a perspective that individuals are responsible for their own learning or that learning can proceed naturally and without much intervention.

[15] Editor's note: This conversation is still ongoing among mathematics instructors. For a different perspective, see Charles Coppin's essay in this issue of the *Journal*.

The teaching premise rejects this claim, and, in line with Vygotsky [45], insists that expert guidance in acquiring scientific knowledge—mathematics, in our case—is indispensable to facilitate learning.

This raises a question about the position of DNR about lecturing. To put the question in more realistic terms, are there situations where a DNR teacher would, for example, present a theorem and its proof without attending first to its epistemological justification? The DNR answer to this question is in the affirmative. If the teacher judges her students to be accustomed to probing into the epistemological justification of mathematical content presented to them, then lecturing is not only allowable but desirable. Indeed lecturing compels such students to seek epistemological justification for the mathematics they read or hear. The habit of seeking epistemological justification is a way of thinking which develops over time as students gradually and repeatedly experience the intellectual need for the mathematical content they are taught. I speculate that for most students this way of thinking reaches a satisfactory level of maturity only in graduate-level courses. For this reason, I believe lecturing must not be eliminated from the mathematics classroom. But it should certainly be introduced gradually, as students come to experience the various forms of intellectual needs discussed in §§1.2.

Our second remark concerns the question: When is it suitable to point out a particular way of thinking to the students? Consonant with the duality principle, as was discussed in §§1.1, throughout the teaching experiments a way of thinking was identified for the students only after the teacher had witnessed signs of its presence in student actions. For example, at the end of each lesson in Stage 1, the teacher summarized the students' effort by giving it the general characterization of "attempting to reduce a difficult problem into a familiar one." The teacher's goal in such summary statements was to gradually institutionalize the way of thinking targeted by a specific lesson as a desirable mathematical practice.

## 5. Ways of Thinking Afforded by the Unit and Targeted by the Teaching Experiments

In this section I lay out the salient ways of thinking targeted by the learning activities described in §§4.1-4.3, and explain why the chosen activities are likely to promote their construction. We focus on three ways of thinking, *structural reasoning* (§§5.1), *deductive reasoning* (§§5.2), and *reflective reasoning* (§§5.3).

*5.1. Structural Reasoning*

To approach *structural reasoning*, we must first understand the term *structure*. Among the different meanings the American Heritage dictionary gives to this term, the following seems to be the most relevant to mathematics: "Structure [is] something made up of a number of parts that are held or put together in a particular way." The phrase "held or put together" must not be restricted to special configuration; rather, it is to be thought of more broadly, as the relation(s) one conceives among different parts or objects. In this respect, the following examples convey the notion of "structure" as defined here.

- An algebraic expression is of a particular structure when it is viewed as a string of symbols put together in a particular way to convey a particular meaning.
- One might observe that the words and phrases comprising various word problems are put together in a "similar way," conveying to the person that the problems are of a common textual structure even though their story lines are entirely different (what is known in the literature as "problem isomorph" [10, 26, 32]).
- The algebraic representations (e.g., systems of equations) of a collection of problems are viewed as having comparable expressions, thereby being of the same structure.
- When two objects within a particular family (e.g., integers, real numbers, functions, etc.) are put together in a particular way (e.g., related to each other by the standard multiplication operation or the standard division operation) they form a structure.
- One might observe that relations among objects within various families (e.g., the family of numbers, versus the family of functions, versus the family of matrices, etc.) have common properties, thereby forming a *general*, or *representational structure*.

Although the term "structural reasoning" is suggestive, it is not easy to define due to its numerous manifestations in mathematical practice. The notion of "structure" articulated here is needed for the following definition, which I am offering as starting point for further discussion, refinement, and perhaps revision:

> "Structural reasoning" is a combined ability to (a) observe structures, (b) act upon structures purposefully, and (c) reason in terms of general structures, not only in terms of their instances.

The third among these three abilities marks a highly advanced stage in the development of structural reasoning, and, as will be discussed in the next subsection, it is not a target for our curricular unit. On the other hand, the first two abilities are afforded and targeted by the unit. We do this through activities aiming at six instances of structural reasoning:

1. Generalizing problems,
2. Reducing an unfamiliar structure into a familiar one,
3. Encapsulating expressions into single entities,
4. Recognizing symmetry,
5. Recognizing structures by carrying out operations in thought, and
6. Reasoning in terms of abstract mathematical structures.

We now zero in on these individually.

*5.1.1. Generalizing problems.*

Elsewhere [13], I analyze the act of generalizing in terms of two of its characteristics: *result pattern generalization* and *process pattern generalization*. Observing that 2 is an upper bound for the sequence

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \cdots$$

because the value checks for the first several terms is an instance of *result pattern generalization*. Concluding this fact by attending to the underlying structure of the sequence, thereby observing the invariant relationship between two neighboring terms of the sequence, is *process pattern generalization*. Thus, process pattern generalization is a way of thinking in which one attends to regularity in the process, though of course it might be initiated by regularity in the result. On the other hand, result pattern generalization is a way of thinking in which one attends solely to regularity in the result—obtained by substitution of numbers, for instance. In [13], I relate process pattern generalization to the development of the principle of mathematical induction. In what follows, I will discuss process pattern generalization in relation to problem generalization.

Consider ICPs 1, 2 and 3 (stated explicitly in §3). A typical formulation of ICP 3 in traditional textbooks would be:

**ICP 3-Traditional:** The sum of the volumes of two cubes is $Q$ and the product of the side of one cube by the side of the other cube is $P$. Find all the values of $P$ and $Q$ for which such cubes exist.

Clearly, from the student's perspective this formulation is likely to be a mere replacement of numeric quantities by parameters, a conceptualization akin to result pattern generalization, and the generalization from ICPs 1 and 2 to ICP 3-Traditional is done by the problem poser, not the problem solver. In addition, the student is cued—essentially told—as to the direction to take: to find the values of the given parameters for which a solution exists. The formulation of ICP 3 in our unit, on the other hand, aims at promoting process pattern generalization by intellectually compelling the students, without dictating to them the direction to take, to generalize ICPs 1 and 2 into a family of problems and express the structure of this family by a system of equations with suitable parameters.

The difference between the traditional formulation of ICP 3 and that of our original version can be captured by the notion of *holistic problem*:

> A holistic problem is one where the solver must figure out from its statement all the elements needed for its solution.

The worst form of non-holistic problem is one in which the problem is broken down into small parts, each of which attends to one or two isolated one-step tasks. Current high school textbooks are populated mainly with this kind of problem (see [17]).

*5.1.2. Reducing an unfamiliar structure into a familiar one.*

The unit offers many opportunities for students to act purposefully on structures, by reducing one structure into another structure. For example, through the process that leads up to the development of the cubic formula, students learn to

a. Solve systems of the form (1) by reducing them into quadratic equations;

b. Solve systems of the form (2) by using a familiar structure (the identity $(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$) to reduce them into cubic equations of the form $x^3 + Ax + B = 0$, and

c. Solve equations of the latter form by reducing them into systems of the form (1).

d. Reduce cubic equations with a second term ($x^3 + Bx^2 + Cx + D = 0$) into cubic equations without the second term ($x^3 + Ax + B = 0$), for which they have developed a solution formula.

e. Reduce any $n$-degree polynomial with an $x^{n-1}$ term into an $n$-degree polynomial without an $x^{n-1}$ term (Homework Problem 6 on Lesson 5)

f. Learn that reduction of one structure to a familiar one is not always successful. Specifically, the attempt to reduce the cubic equation $x^3 + Bx^2 + Cx + D = 0$ into the familiar equation $(x - A)^3 + L = 0$ as is successfully done with quadratic equations (i.e., a direct analogue of the completing square method), fails.

It should be clear that here the terms "familiar" and "unfamiliar" refer to capacity for action rather than mere recognition. For example, one might recognize both equations $x^2 + Ax + B = 0$ and $x^3 + Ax + B = 0$, but know to act on (i.e., solve) only the first.

*5.1.3. Encapsulating expressions into single entities.*

The use of substitution to solve systems of type (1) leads students to recognize the equation $(u^3)^2 - Qu^3 + P^3 = 0$ as a quadratic equation in $u^3$. For this, students had to encapsulate $u^3$ into a single entity. Similarly, in reducing system (2) into a cubic equation (Lesson 2), students had to conceive of $u + v$ as a single entity $x$, the unknown of the equation.

Note that this way of thinking is, in fact, a particular case of *reducing an unfamiliar structure into a familiar one*. For example, the structure $u^3$ as $u \cdot u \cdot u$ is reduced to a single entity ($u^3$), acted upon purposefully with the operation $(\cdot)^2$.

*5.1.4. Recognizing symmetry.*

Attention to the symmetry between the unknowns $u$ and $v$ in system (1) leads to the conclusion that the solution set for $u^3$ and the solution set for

$v^3$ should be equal. Together with the constraint $u^3 + v^3 = Q$, this observation enables one to determine the values for $u$ and $v$ (Lesson 1, ICP 4). This approach is very different from, and more economical than, the back-substitution approach students are familiar with. Attention to symmetry, as in this case, is akin to structural reasoning, because it involves an observation of a structure (the symmetry) and a purposeful action—to utilize the symmetry to save computations.

In our experiments no student noticed the symmetry between $u$ and $v$. However, some students expressed pleasure upon learning the new approach.

*5.1.5. Recognizing structures by carrying out operations in thought.*

This refers to the ability to carry out in thought algebraic operations without actually performing them. A simple example of this way of thinking is in Lesson 5. Instead of dealing with the cubic equation $ax^3+bx^2+cx+d = 0$, students are told that one can save one parameter and deal instead with the equation $x^3 + Bx^2 + Cx + D = 0$. Students are then expected to carry out mentally the division by $a$ and relabel the resulting parameters to obtain the latter equation, without performing these actions. The unit is highly populated with instances of this way of thinking. Here is another example, from Lesson 1. Since $P$ and $Q$ are positive and satisfy the condition

$$\left(\frac{Q}{2}\right)^2 - P^3 \geq 0,$$

the structure of the expressions

$$u = \sqrt[3]{\frac{Q}{2} + \sqrt{\left(\frac{Q}{2}\right)^2 - P^3}} \quad \text{and} \quad v = \sqrt[3]{\frac{Q}{2} - \sqrt{\left(\frac{Q}{2}\right)^2 - P^3}}$$

dictates that they must be positive as well.

Carrying out operations in thought, without the need to actually perform them, is a characteristic of structural reasoning because, as these cases illustrate, to do so one must observe a general structure, and furthermore, act on it directly without attending to its referents. In the second case, for example, one must use the general fact that for any non-negative number of the form $c = a^2 - b$, $a$ must be greater than $\sqrt{c}$.

*5.1.6. Reasoning in Terms of Abstract Mathematical Structures*

The third ability in the definition of structural reasoning is *reasoning in terms of general structures, not only in terms of their instances*. One significant aspect of this way of thinking is the ability to reason in terms of abstract mathematical structures, such as "group," "ring," and "field." I believe that this way of thinking is beyond the natural scope of the unit under study here. Nonetheless, I briefly discuss it here, for two reasons. The first reason is that this way of thinking is needed for a complete picture of the various *styles of reasoning*, the topic of the next subsubsection. The second and more important reason is to offer the conjecture that the unit may serve as an entry point to this highly advanced way of thinking, as I will now explain.

What does it take to help students reason in terms of abstract mathematical structures? Two combined abilities seem to be cognitive prerequisites to this way of thinking: *reasoning in terms of conceptual entities* and *reasoning in terms of operations on conceptual entities*.

The work of Tall and Vinner implies that one's conceptualization of a general mathematical structure rests on the nature of the *concept image* [42] he or she has of it. A mathematically mature concept image would include, among other things, examples (and non-examples) of the general structure. Initially, a general structure is abstracted by a person from realities concrete to that person, whereby they become *instance structures* for her or him.[16] Once this abstraction has taken place, other families of objects may become instance structures of the general structure. Dubinsky's APOS theory (as in [3]) strongly supports the claim that a necessary condition for this to happen is that these objects are conceived by the person as conceptual entities (in the sense of [11])—just as an adult might conceive of whole numbers, for instance. If, for example, a teacher illustrates the meaning of a general theorem in linear algebra with its specific instance in the vector space of

[16]Two comments: 1. The term "reality" refers to the individual's mental constructs, which may or may not have any meaning in one's physical or social environment. 2. The use of the plural "realities" raises the question: Is it possible to abstract a general structure from a single reality? Dubinsky (personal communication, 1992) suggests that the answer to this question may be in the affirmative. He bases his conjecture on the history of the concept of *ring*, which, according to him, was developed from a single reality, that of the integers. If this is indeed so, it raises the question as to the intellectual need for reasoning in terms of a general structure with a single instance structure.

functions, and the students have not yet transitioned from thinking of a function as a process to thinking of it as a conceptual entity, it is unlikely that they will be able to gain insight into the teacher's illustration.

A fundamental conceptual difference between a general mathematical structure and its instance structure is that while the latter consists of objects and relations, the former consists of *representations* of objects and relations. One can represent objects and relations in numerous ways, but in this case the representations are of a unique kind: they are derivatives of a small number of representations called axioms, which capture the most essential properties of the operations on, or relations among, the objects of each instance structure. A critical consequence of this feature is that when working within a general structure one is compelled to reason in terms of its axioms. To appreciate the magnitude of the cognitive demand involved in reasoning in terms of axiom representations, rather than in terms of "actual" objects and relations, one only needs to compare one's experience with Euclid's *Elements* to that of Hilbert's *Foundations of Geometry*. The axioms in the *Elements* are merely a description of one (idealized) physical reality, whereas the axioms in the *Foundations of Geometry* are representations of relations in an endless number of realities [27].

The rationale for the above conjecture—that the unit serves as an entry point to reasoning in terms of abstract mathematical structure—is this. First, we made a special effort in the unit to help students conceive of complex numbers as *numbers*—as conceptual entities. Students in Experiment 2 and Experiment 3 witnessed firsthand how complex numbers are products of human construction, not ready-made expressions endowed by a divine being. The status of these objects as numbers grew gradually, from meaningless expressions to solutions to polynomial equations. In particular, they saw how complex numbers indeed behave like the numbers with which they are familiar. The complex numbers, they saw, can be added, subtracted, multiplied, and divided. Furthermore, these four operations obey rules and have geometric meanings, which enable one to derive important conclusions and solve mathematical problems. Second, through all this, students came to experience the emergence of the structure of complex numbers as an *extension* of the structure of the real numbers—the latter being an instance structure of the former—and, furthermore, realize the remarkable value of this extension, as is expressed in the Fundamental Theorem of Algebra.

### *5.1.7. Styles of Reasoning*

The six aspects of structural reasoning discussed above in §§5.1.1-5.1.6 can be classified into two not-mutually-exclusive categories (see Figure 3).

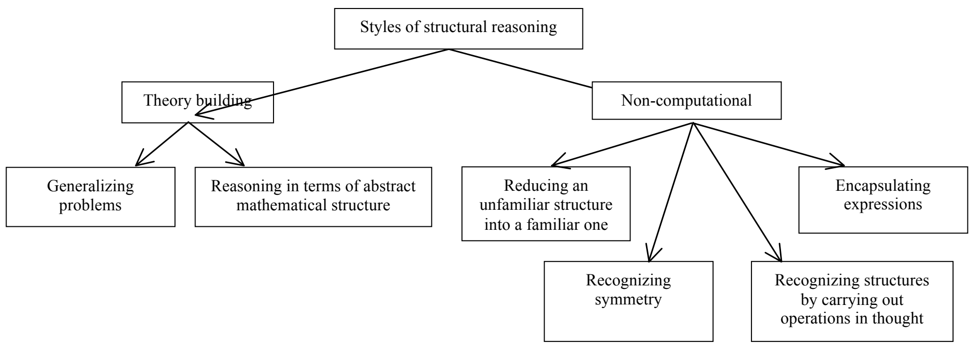


Figure 3: The classification of structural reasoning.

The two ways of thinking, *generalizing problems* (§§5.1.1) and *reasoning in terms of abstract mathematical structures* (§§5.1.6), form one category, one we will call *theory building style of reasoning.*[17] Such reasoning is typical of the work of a theoretician. This style of reasoning is characteristically outward; it aims at generalizing and capturing common features among seemingly different phenomena.

The other four ways of thinking, *reducing a new structure into a familiar one* (§§5.1.2), *encapsulating expressions into a single entity* (§§5.1.3), *recognizing symmetry* (§§5.1.4), and *recognizing structures by carrying out operations in thought* (§§5.1.5), form the second category. We will label this category the *non-computational style of reasoning*. Such reasoning is typical of mathematicians who seek to solve problems by fewer calculations and more conceptualization. Their attention, in contrast to that of the theoreticians, is inward; they aim to dig inside the structure of particular objects. The non-computational style of reasoning is a propensity to minimize computations, not eliminate them.

[17]The notion of "theory building" discussed here was inspired by a conversation I had with Hyman Bass in 2012 in Be'er Sheva, Israel.

It is the repeated reasoning through computation that is likely to advance these styles of reasoning, particularly, the non-computational style. The repeated demand for meaningful computation is throughout the unit, and we have observed a change in students' *computational fluency*: the dual abilities of (a) decontextualizing—abstracting a given situation and representing it symbolically and manipulating the representing symbols as if they have a life of their own, without necessarily attending to their referents and (b) contextualizing—pausing as needed during the manipulation process in order to probe into the referential meanings for the symbols involved in the manipulation. Furthermore, the symbol manipulations involved in the solutions promote the understanding that symbols are not manipulated haphazardly but with purpose, to achieve a particular familiar form, thereby enhancing structural reasoning, particularly through the habit of attempting to reduce new structures into familiar ones.

### *5.2. Deductive Reasoning*

Deductive reasoning is the most prevalent way of thinking throughout the unit. With the exception of the Fundamental Theorem of Algebra, whose proof is outside the background and expected mathematical ability of the intended consumers of the unit, all assertions made are proved. The persistent emphasis on proof throughout was a source of both intellectual challenge and emotional satisfaction for the students. The kinds of and reasons for challenges students typically have with proofs are many (see, for example, [41]), and I will not recount them here. I will, however, discuss two of the challenges our students encountered during the teaching experiments: *definitional reasoning*, and *reasoning in terms of quantifiers, connectives, and conditional statements*.

**Definitional reasoning.** *Definitional reasoning* is the way of thinking by which one characterizes objects and proves assertions in terms of mathematical definitions. In Van Hieles' 1980 model of geometric reasoning, only at the highest stage can secondary school students reason in terms of definitions (see [4]). College students too experience difficulty reasoning in terms of definitions. For example, asked to define "invertible matrix," many linear algebra students stated a series of equivalent properties (e.g., "a square matrix with a non-zero determinant," "a square matrix with full rank," etc.) rather than a definition. The fact that they provided more than one such property is an indication they are not definitional reasoners [12].

Lesson 8 demonstrated once again how difficult it is for students to reason in terms of definitions. The lesson begins by reiterating the definition of complex numbers as the set $\mathbb{C}$ of all the expressions of the form $a+bi$, where $a$ and $b$ are real numbers; clearly this includes $i$, which is defined as the expression $0 + 1i$. Viewing $a + bi$ merely as an algebraic expression, the definitions of the four arithmetic operations on complex numbers seemed unproblematic to the students. However, collectively the class had difficulty understanding and appreciating leading questions concerning these operations. One of these questions concerns the distinction between what is defined and what is logically derived. Specifically, the unit raises the following question:

> Each of the four new operations is defined only on a pair of complex numbers, but we want to be able to add, multiply, subtract and divide between real numbers and complex numbers. Can we view real numbers as complex numbers?

This question is answered affirmatively, in that any real number can be viewed as a complex number by *agreeing* that the expression $u+0i$ is $u$; this is what allows us then to view $\mathbb{R}$ (the set of real numbers) as a subset of $\mathbb{C}$. We had hoped that this would give rise to the distinction between *agreeing* (i.e., defining) and *deriving logically.* The instinctive reaction of most students on the other hand was to *derive* the equalities $0 + 1i = i$ and $u + 0i = u$, since, according to them, $1i = i$ and $0i = 0$. Only a few students in each of the teaching experiments seemed to have understood and appreciated the nuances of this claim.

Other questions concern the properties of the operations defined:

> With the view of the real numbers as complex numbers, do the four operations defined on complex numbers agree with the real-number operations?
>
> And do the operations on complex numbers satisfy all the rules of the operations on the real numbers?

Student responses seem to indicate that these questions were unproblematic to them. One of the students put it as follows: "I don't see how it can be otherwise since we defined the operations on complex numbers in the correct way."

Overall, the pedagogical moral from this experience is that Lesson 8 is the most subtle since it deals with questions requiring definitional reasoning.

The unit, however, provides an opportunity to tackle these questions within a concrete context. It is up to the teacher how to use Lesson 8—to delve into these subtle questions or merely use the lesson to familiarize the students with complex numbers and advance their computational fluency.

**Reasoning in terms of quantifiers, connectives, and conditional statements.** As we observed in Perturbation 5 (§§4.2), from the viewpoint of the students, the cubic formula always yields only one root even in cases when three (real) roots are known. In discussing this observation, the teacher saw an opportunity to bring up the notion of logical equivalence and the difference between a *necessary condition* and a *sufficient condition*. Accordingly, he suggested to the class to review carefully the proof of the cubic formula. Perhaps, he pronounced to the students, some of the steps in the process leading up to the cubic formula are not reversible (i.e., do not constitute equivalent statements), and so, if it turns out that the formula constitutes only a sufficient condition to the equation, then the fact that not all the roots are yielded by the formula should not be a surprise. In doing so, the teacher surmised that this review, if successful, can sever the perturbation, because the proof presented does establish an equivalence between the equation and the formula (see the discussion of the proof in §§4.1).

This teacher's initiative revealed that students were having difficulty with logical equivalence. To begin with, most students had difficulty understanding the given task. Even after some discussion of the task, there were some who viewed the verification process as superfluous. Examples to the contrary provided by the teacher (as in the cases of squaring both sides of an equation) were largely unhelpful to the students, for they deemed them to be a digression from the question at hand. Nevertheless, the questions as to why the cubic formula does not provide three roots, even in cases when all the roots are known, remained of interest to the class as a whole.

Students also had difficulties with quantifiers and logical connectives. Despite the fact that the proof to the cubic formula was first presented generically, through the solution to the equation $x^3 - 2x + 7 = 0$ (see Perturbation-Resolution 3 in §§4.1), students initially had difficulty fully understanding the long chain of inferences involved. In essence, the solution consists of five steps. (As can be seen in the unit ICP 5, the steps are not articulated in set-theoretical notation, and their narrative is not as terse as it appears here.)

1. For any $u$ and $v$, the sum $x = u + v$ is a solution to the equation
$$x^3 - 3uvx - (u^3 + v^3) = 0.$$
2. Therefore, for any $u$ and $v$ satisfying the conditions, $uv = \frac{2}{3}$ and $u^3 + v^3 = -7$, the above cubic equation is equivalent to $x^3 - 2x + 7 = 0$.
3. Implied from steps 1 and 2 is that the solution to the equation $x^3-2x+7=0$ is the intersection of two sets: $\{u+v|x^3 - 3uvx - (u^3+v^3) = 0\}$ and $\{u+v|uv = \frac{2}{3}$ and $u^3+v^3 = -7\}$.
4. Since for $x = u+v$, $x^3-3uvx-(u^3+v^3) = 0$ is an identity, the second set in step 3 is a subset of the first.
5. This implies that the solution to $x^3 - 2x + 7 = 0$ is the second set.

In all, the students seemed to have difficulties following a protracted chain of inferences. Their questions and responses reflected a limited understanding of quantifiers and logical connectives. In addition, they had difficulty with the meaning and conditions for two polynomials to be equal. However, this activity, in particular, and the many similar activities throughout the unit, in general, provided invaluable opportunities to advance students' understanding of the elementary concepts of logic.

### *5.3. Reflective Reasoning*

Reflecting on the meaning and consequences of a particular result is undoubtedly a routine practice by mathematicians, and it is among the ways of thinking targeted by the unit. The following are examples of this effort:

1. In Lesson 2, the failure to solve system (2) is handled by revisiting the structure of the system with the hope of uncovering cues;
2. In Lesson 3, the failure to solve the equation $x^3 - 2x + 7 = 0$ by known means invokes the question of whether a solution formula for cubic equations exists; also in the same lesson, when the attempt to develop a cubic formula by analogizing the development of the quadratic formula fails, students are encouraged to reflect on the RQE and RCE techniques, which helps them to see their utility in developing a formula for the cubic equation.
3. In Lesson 8, students are asked to reflect on the process they apply to divide one polynomial by another, as they read the proof of the Division Theorem.

These are examples of *retrospective reflection*. But the unit also includes *forward reflection*, or expectations of future actions. For example:

4. In Lesson 8 students are encouraged to articulate what it takes for the expressions $a + bi$ to be conceived of as numbers before the formal introduction of complex numbers. This requires them to ponder the facts that these expressions and the operations on them must represent something meaningful to us and they are useful in solving mathematical or physical problems.

### *5.4. Relation to the Common Core State Standards (CCSS)*

> [To] promote a better understanding of the Practice Standards [one must give] them *mathematical substance* rather than adding to the *verbal* descriptions of what mathematics is about. Seeing mathematics in action is a far better way of coming to grips with these Standards but, unfortunately, in an era of Textbook School Mathematics one does not get to see mathematics in action too often. [46, page 2]

This statement is consistent with one of the implications of the duality principle we discussed in §§1.1. Describing ways of thinking verbally to students before they have developed them through the acquisition of ways of understanding would likely have no or negative effect. On the other hand, the definition of *curriculum* discussed in the interlude before §4 implies that curriculum developers and teachers must be cognizant of and explicit about these objectives and the actions needed to achieve them. An important contribution of this paper is that it does exactly that.

The Common Core State Standards (CCSS) lists eight mathematical practices, called *Standards for Mathematical Practices*, but makes no connection between them and the mathematical content that is supposed to promote them. The term "standard for mathematical practice" corresponds roughly to the DNR term "way of thinking" we have been using here. Table 1 on the next page depicts some of these practices and their corresponding DNR ways of thinking, along with samples of the unit lessons that afford them.

Table 1: Correspondence between DNR Ways of Thinking and CCSS Standards for Mathematical Practices.

| **Ways of Thinking** | **Standards for Mathematical Practices** | **Lesson Samples** |
|---|---|---|
| **Structural Reasoning** | | |
| Generalizing problems | Abstract a given situation and represent it symbolically | Lessons 1, 5 |
| Reducing an unfamiliar structure into a familiar one | Try special cases and simpler forms of the original problem in order to gain insight into its solution | Lessons 5, 9, 10, 12 |
| | Look closely to discern a pattern or structure | Lessons 2, 11, 12 |
| | Consider analogous problems | Lessons 1, 3, 5 |
| Encapsulating an expression into a single entity | See complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. | Lessons 1, 2, 3, 4 |
| Recognizing symmetry | Look closely to discern a pattern or structure | Lessons 2, 11, 12 |
| Recognizing structures by carrying out operations in thought | | Lessons 1, 5 |
| Reasoning in terms of abstract mathematical structures | | Precursors in Lesson 8 |
| **Deductive Reasoning** | | |
| Definitional reasoning | Use clear definitions | Lessons 8, 11 |
| | Understand and use stated assumptions, definitions, and previously established results in constructing arguments. | Lesson 8 |
| Reasoning in terms of logical connectives, conditional statements, and logical equivalencies | Build a logical progression of statements | Lesson 8, 9, 10 |
| | Justify conclusions | Lessons 3, 10 |
| Protracted chain of inferences | | Lessons 3, 4 |
| **Reflective Reasoning** | | |
| Retrospective reflection | Notice if calculations are repeated, and look both for general methods and for shortcuts | Lessons 1, 5 |
| | Maintain oversight of the process, while attending to the details. | Lesson 2, 7 |
| | Monitor and evaluate one's progress and change course if necessary. | Lesson 2, 3 |
| Forward reflection | | Lesson 8 |

## 6. Closing Words on Designing DNR-Based Curricula

The goal of this article was three-fold: (1) to introduce the DNR framework as a theoretical foundation for curriculum development; (2) to give a specific instance of a DNR-based unit on complex numbers; and finally (3) to argue that the DNR framework provides a means for developing desirable ways of thinking that correspond to the Standards for Mathematical Practice outlined by the Common Core State Standards.

The design of the curricular unit on complex numbers was inspired by and roughly follows the development of its subject in the history of mathematics. Consonant with DNR, instructional objectives of the unit are formulated in terms of both ways of understanding and ways of thinking, not only in terms of the former as traditionally is the case. The design of the unit factors in three major considerations: (a) the developmental interdependency between ways of understanding and ways of thinking, as dictated by the duality principle, (b) the intellectual needs of the students and the epistemological justifications suitable to their background knowledge and current mathematical abilities, as implied by the necessity principle, and (c) ways to facilitate internalization, organization, and retention of knowledge, as it is called by the repeated reasoning principle.

In accordance with the DNR definition of learning and the instructional principle of intellectual need, i.e., the *necessity principle*, we designed the unit around alternating sequences of intellectual perturbations and their corresponding resolutions. The development leading up to the complex numbers and the investigation into their meaning provide students with repeated opportunities for applying familiar ways of understanding and ways of thinking and for acquiring new ones. Consistent with the *repeated reasoning principle*, the reoccurrence of these opportunities was by design—to help students organize, internalize, and retain the knowledge they learn. To this end, each lesson concludes with a set of practice-of-reasoning problems aimed at helping students internalize and organize the accumulated ways of understanding and ways of thinking they have learned in and up to that lesson. Some of these problems are rather demanding, as readers may witness for themselves. The *duality principle* too manifests itself throughout the unit. Students' prior ways of thinking are taken into account, and those that are targeted are developed through the solution of problems understood and appreciated as such by the students.

The unit is divided into three stages, corresponding to the historical development of complex numbers: (1) the solution of the cubic equation, (2) the struggle to make sense of this solution, and (3) the emergence of complex numbers out of this struggle, and the recognition of their utility and power in solving mathematical problems. Accordingly, the twelve lessons of the unit are organized around three stages (called Parts in the unit). Stage 1 is composed of Lessons 1–5; its aim is to delineate the ideas underlying the development of the cubic formula. Stage 2 is composed of Lessons 6–8; its aim is to draw attention to the puzzling behaviors of the cubic formula. Stage 3 is composed of Lessons 9–11; its aim is to resolve these puzzles by constructing a new set of numbers (the field of complex numbers), investigating their algebraic and geometric meanings, and articulating their remarkable value to understanding polynomial equations (i.e., the Fundamental Theorem of Algebra).

The questions I faced in the process of translating the history of development of complex numbers into a curriculum are generalizable and relevant to the development of any curriculum. Specifically, the questions are:

1. How should ideas underlying the historical development of a subject be represented and sequenced in a curricular unit so as to anchor them in students' current knowledge, intellectually necessitate them, and provide opportunities for reasoning about them and with them repeatedly?
2. What desirable ways of thinking are potentially afforded by this history?
3. What is the typical background knowledge and cognitive ability of the student populations for whom the unit is intended (from high school seniors to college freshmen and sophomores)?
4. How much time can reasonably be allocated to this unit in the existing high school or college programs?
5. How compatible is the content of the unit with the content of current programs and national reforms?

Ways of thinking afforded by the unit include: (1) *structural reasoning*, with its various instantiations, manifested in two styles of reasoning: *theory building* and *non-computational*; (2) *deductive reasoning*, focusing on *definitional reasoning* and *reasoning in terms of quantifiers, connectives, and conditional statements*; and (3) *reflective reasoning*, with its two instantiations, *retrospective reflection* and *forward reflection*.

I conclude with questions for further research:

1. What are the learning trajectories [33] of the ways of thinking we claim are advanced by the unit through the DNR-based interventions employed in the three teaching experiments described above? We agree with Martin Simon that understanding this process is crucial to curriculum development (see [35, 36]).
2. What is the extent and depth of students' acquisition of these ways of thinking and the various ways of understanding introduced in the unit? Except for the fact that the students were largely successful in solving all the problems in the unit, I have no evidence of their ability to transfer the knowledge they acquired through the unit to other mathematical settings.
3. What is the impact of the instructional interventions reported here on retention of these ways of understanding and ways of thinking?

An investigation into these questions may lead to a refinement or possibly a revision of the unit on complex numbers discussed in this paper.

**Acknowledgments:** I would like to thank Martin Simon, Jeffery Rabin, Sara-Jane Harel, and Barbara Edwards for their comments on an earlier version of this paper. The development of this paper was in part supported with funds from the National Science Foundation under Grants NSF#0934695 and NSF#1035503 (Harel PI). Any opinions, findings, and conclusions expressed are those of the author and do not necessarily reflect the views of the National Science Foundation.

## References

[1] Daniel Breidenbach, Ed Dubinsky, Julie Hawks and Devilyna Nichols, "Development of the process conception of function," *Educational Studies in Mathematics*, Volume **23** Number 3 (June 1992), pages 247–285.

[2] Guy Brousseau, *Theory of didactical situations in mathematics*. Kluwer Academic Publishers, Boston, MA, 1997.

[3] Anne Brown, David J. DeVries, Ed Dubinsky and Karen Thomas, "Learning binary operations, groups, and subgroups," *Journal of Mathematical Behavior*, Volume **16** Number 3 (1997), pages 187–239.

[4] William F. Burger and J. Michael Shaughnessy, "Characterizing the van Hiele Levels of Development in Geometry," *Journal for Research in Mathematics Education*, Volume **17** Number 1 (January 1986), pages 31–48.

[5] Yves Chevallard, *La Transposition Didactique*, La Pensée sauvage, Grenoble, 1985/1991 (expanded second edition).

[6] Robert G. Cooper, "The role of mathematical transformations and practice in mathematical development," in *Epistemological Foundations of Mathematical Experience*, Leslie P. Steffe, ed., Springer-Verlag, New York, 1991, pages 102–123.

[7] Robert B. Davis, *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*, Croom Helm / Ablex Publishing Corporation, Norwood NJ, 1984.

[8] Edward L. Deci and Richard M. Ryan, *Handbook of Self-Determination Research*, The University of Rochester Press, Rochester NY, 2004.

[9] K. Anders Ericsson, Ralf Th. Krampe, and Clemens Tesch-Romer, "The role of deliberate practice in the acquisition of expert performance," *Psychological Review*, Volume **100** Number 3 (1993), pages 363–406.

[10] Mary L. Gick and Keith J. Holyoak, "Schema induction and analogical transfer," *Cognitive Psychology*, Volume **15** Issue 1 (January 1983), pages 1–38.

[11] James G. Greeno, "Conceptual Entities," in *Mental Models*, Dedre Gentner and Albert L. Stevens, eds., Lawrence Erlbaum Associates, Hillsdale NJ, 1983, pages 227–252.

[12] Guershon Harel, "Students' understanding of proofs: a historical analysis and implications for the teaching of geometry and linear algebra," *Linear Algebra and Its Applications*, Volumes **302–303** (December 1999), pages 601–613.

[13] Guershon Harel, "The Development of Mathematical Induction as a Proof Scheme: A Model for DNR-Based Instruction," in *Learning and Teaching Number Theory*, Stephen R. Campbell and Rina Zaskis, eds.,

Journal of Mathematical Behavior Monographs, Ablex Publishing Corporation, Norwood NJ, 2001, pages 185–212.

[14] Guershon Harel, "What is Mathematics? A Pedagogical Answer to a Philosophical Question," in *Proof and Other Dilemmas: Mathematics and Philosophy (Current Issues in the Philosophy of Mathematics From the Perspective of Mathematicians)*, Bonnie Gold and Roger A. Simons, eds., Mathematical American Association, 2008, pages 265–290.

[15] Guershon Harel, "DNR perspective on mathematics curriculum and instruction: Focus on proving, Part I," *ZDM: The International Journal on Mathematics Education*, Volume **40** Issue 3 (2008), pages 487–500.

[16] Guershon Harel, "DNR perspective on mathematics curriculum and instruction, Part II," *ZDM: The International Journal on Mathematics Education*, Volume **40** Issue 5 (2008), pages 893–907.

[17] Guershon Harel, A review of four high school mathematics programs, University of California, San Diego, 2009, http://www.strategicteaching.com/review_of_4_hs_programs.pdf, accessed on July 30, 2013.

[18] Guershon Harel, "DNR-Based Instruction in Mathematics as a Conceptual Framework," in *Theories of Mathematics Education: Seeking New Frontiers*, Barath Sriraman and Lyn English, eds., Springer-Verlag, Berlin-Heidelberg, 2010, pages 343–367.

[19] Guershon Harel, "Intellectual need," In *Vital Direction for Mathematics Education Research*, Keith R. Leatham, ed., Springer Science+Business Media, New York, 2013, pages 119–151.

[20] Guershon Harel, "DNR-based curricula: the case of deductive geometry," in preparation.

[21] Guershon Harel and Boris Koichu, "An operational definition of learning," *Journal of Mathematical Behavior*, Volume **29** Issue 3 (September 2010), pages 115–124.

[22] Guershon Harel, Jeff Rabin, Laura J. Stevens, and Evan Fuller, A Module on Conic Sections for Pre-Service Mathematics Teachers, 2010.

[23] Guershon Harel, Jeff Rabin, Laura J. Stevens, and Evan Fuller, A Module on Rate of Change for Pre-Service Mathematics Teachers, 2010.

[24] Guershon Harel and Laura J. Stevens, Holistic Problems with Pedagogical Commentaries, 2011, http://www.math.ucsd.edu/~stevens/DNRproblems_final.pdf, accessed July 30, 2013.

[25] Guershon Harel and W. Stephen Wilson, "The State of High School Textbooks," *Notices of the American Mathematical Society*, Volume **58** Number 6 (June/July 2011), pages 823–826.

[26] J. R. Hayes and H. A. Simon, "Psychological differences among problem isomorphs," in *Cognitive Theory, Volume 2*, J. Castellan, Jr., D. B. Pisoni, and G. R. Potts, eds., Eribaum, Hillsdale NJ, 1977, pages 21–41.

[27] Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*. The MIT Press, Cambridge MA, 1981.

[28] National Governors Association Center for Best Practices, Council of Chief State School Officers, *Common Core State Standards (Mathematics)*, National Governors Association Center for Best Practices, Council of Chief State School Officers, Washington DC, 2010, available at http://www.corestandards.org/Math, accessed July 30, 2013.

[29] Michael Oehrtman, Marilyn Carlson and Patrick Thompson, "Foundational reasoning abilities that promote coherence in students' function understanding," in *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*, Marilyn Carlson and Chris Rasmussen, eds., Mathematical Association of America, Washington DC, 2002, pages 27–41.

[30] Henri Poincaré, *Science and Method*, Dover Publications, New York, 1952.

[31] György Pólya, *How to Solve It: A New Aspect of Mathematical Method*, Princeton University Press, Princeton NJ, 1945.

[32] Stephen K. Reed, "Constraints on the abstraction of solutions," *Journal of Educational Psychology*, Volume **81** Issue 4 (December 1989), pages 532–540.

[33] Martin A. Simon, "Reconstructing mathematics pedagogy from a constructivist perspective," *Journal for Research in Mathematics Education*, Volume **26** Number 2 (March 1995), pages 114–145

[34] Martin A. Simon, "Issues in Theorizing Mathematics Learning and Teaching: A Contrast between Learning through Activity and DNR Research Programs," *The Journal of Mathematical Behavior*, Volume **32** Issue 3 (September 2013), pages 281–294.

[35] Martin A. Simon and Ron Tzur, "Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory," *Mathematical Thinking and Learning*, Volume **6** Issue 2 (2004), pages 91–104.

[36] Martin A. Simon, Ron Tzur, Karen Heinz and Margaret Kinzel, "Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction," *Journal for Research in Mathematics Education*, Volume **35** Number 5 (November 2004), pages 305–329.

[37] Lee S. Shulman, "Those who understand: Knowledge growth in teaching," *Educational Researcher*, Volume **15** Number 2 (February 1986), pages 4–14.

[38] Lee S. Shulman, "Knowledge and teaching: Foundations of the new reform," *Harvard Educational Review*, Volume **57** Number 1 (February 1987), pages 1–22.

[39] Leslie P. Steffe and Patrick W. Thompson, "Teaching experiment methodology: Underlying principles and essential elements," in *Handbook of Research Design in Mathematics and Science Education*, Anthony E. Kelly and Richard A. Lesh, eds., Lawrence Erlbaum Associates, Hillsdale NJ, 2000, pages 267–307.

[40] Donald E. Stokes, *Pasteur's Quadrant: Basic Science and Technological Innovation*, Brookings Institution Press, Washington DC, 1997.

[41] Despina A. Stylianou, Maria L. Blanton, and Eric J. Knuth, eds., *Teaching and Learning Proof Across the Grades: A K-16 Perspective*, Studies in Mathematical Thinking and Learning, Routledge (co-published with NCTM), New York, 2010.

[42] David Tall and Shlomo Vinner, "Concept image and concept definition in mathematics with particular reference to limits and continuity," *Educational Studies in Mathematics*, Volume **12** Number 2 (May 1981), pages 151–169.

[43] Jean-Pierre Tignol, *Galois' Theory of Algebraic Equations*, Longman Scientific & Technical, Essex, England, 1980.

[44] Patrick W. Thompson, "Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula," in *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Edward A. Silver, ed., Lawrence Erlbaum Associates, Hillsdale NJ, 1985, pages 189–243.

[45] Lev S. Vygotsky, *Mind in Society: The Development of Higher Psychological Processes*, (Michael Cole, Vera John-Steiner, Sylvia Scribner, and Ellen Souberman, eds. and trans.), Harvard University Press, Cambridge MA, 1934/1978.

[46] Hung-Hsi Wu, Teaching geometry according to the common core standards, 2012, http://math.berkeley.edu/~wu/Progressions_Geometry.pdf, accessed July 30, 2013.