2003

Turbulence, Magnetics, and Closure Equations

Jane Pratt
Harvey Mudd College

Magnetohydrodynamics With the Eddy Damped Quasi-Normal Markovian Closure

by

Jane Pratt
Lisette de Pillis, Harvey Mudd College, Advisor

Advisor: ______________________________________

Second Reader: ______________________________________

(Leaf Turner, Los Alamos National Laboratory)

June 24, 2003

Department of Mathematics
HARVEY MUDD COLLEGE

Abstract

Magnetohydrodynamics With the Eddy Damped Quasi-Normal Markovian Closure

by Jane Pratt

June 24, 2003

When a ferromagnet is heated, it loses its magnetism. Stars and planets have magnetic fields, as does the Earth. But it is known that the center of the Earth is very hot. Therefore, to sustain the large magnetic field of a planet, we cannot look to simple ferromagnetism like that of a bar magnet, but we have to look at the movement of electric charges within the Earth's molten core to generate magnetic field. This magnetic field sustainment against ohmic dissipation by turbulent flow is referred to as the turbulent dynamo effect. Theoretical research into the mechanisms that create the dynamo has been actively pursued for several decades, culminating recently in massive computer simulations of the Earth's core. Most of these studies have employed the equations of magnetohydrodynamics (MHD), a nonlinear theory of electrically conducting fluids.

The EDQNM (Eddy-Damped Quasi-Normal Markovian) closure is a statistical model designed so that the turbulence equations derived from Navier-Stokes dynamics can be closed and satisfy the realizability condition of positivity of the kinetic energy spectrum. In case of MHD turbulence, realizability requires more work. We have proved in an earlier work that equations analogous to those expected of the EDQNM closure for MHD without mean fields satisfy the appropriate realizability conditions (Turner and Pratt 1999).

In this work, we discuss requirements needed to make the MHD equations

realizable with mean fields, extending those of neutral fluid turbulence by Turner [1]. Finally, we discuss direct numerical simulations and the correspondence of the statistical theories with simulation results.

Table of Contents

List of Figures

Acknowledgements

I would like to thank Dr. Leaf Turner of Los Alamos National Laboratory and my thesis advisor Dr. Lisette de Pillis of Harvey Mudd College for their supervision of this work.

Part I

Introduction

Although the field of magnetohydrodynamics has been a source of interest for centuries, only recently has decisive progress been made toward understanding the mechanisms that create the magnetic fields of planets. We open this manuscript with a brief account of some significant events in the history of magnetohydrodynamics. The question of the origin of magnetic fields first arose in Western history when Gilbert, physician to Queen Elizabeth I of England, introduced the concept of the magnetic field of Earth over four hundred years ago. He showed that this magnetic field, as defined by the pointing of a freely suspended magnetic needle, was associated with the planet in the same way that the field of a lodestone is associated with the stone. Gilbert reduced the two phenomena to one, with the declaration that Earth is a huge lodestone. For the succeeding three centuries, this explanation was sufficient to explain all the known facts about terrestrial magnetism; then the Curie temperature was discovered, and it was found that ferromagnetic substances lose their magnetic properties at relatively modest temperatures of several hundred degrees centigrade. Since the interior of Earth was known to be extremely hot, the earth could not operate as a lodestone. Given the large, fluctuating magnetic fields of planets and stars, a primordial field like that of a lodestone could not be the only source of magnetism.

Not until the early part of the twentieth century were ideas found to solve this problem. In 1908, Hale discovered that sunspots form where the field exits or enters the Sun; they occur in these bipolar magnetic regions ([2] p537).

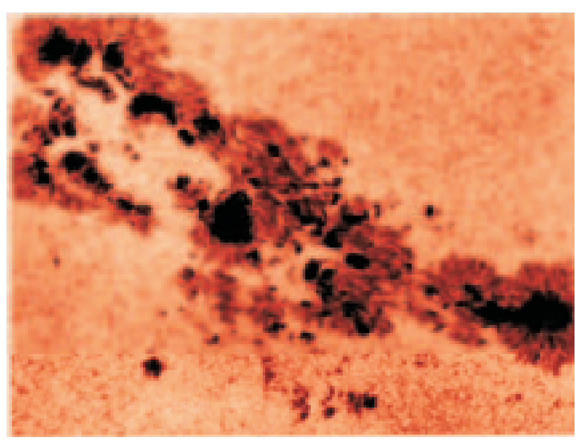

Figure 1: Sunspots

Soon after, Larmor noticed vortical designs in the filaments around the sunspots; he concluded that the sunspot (Fig. 1) was the upper end of a tornado and suggested that the swirling gases carried electric currents, generating magnetic field ([2] p537). This was not a bad guess, because although solar gases are not intrinsically magnetic, they are fully ionized except for a thin layer at the visible surface, so that the gases are excellent conductors of electricity. Larmor's conjecture sparked considerable theoretical progress. In 1934 Cowling proved his famous anti-dynamo theorem – that sustained dynamo action is impossible in an axisymmetric system or in a two dimensional system ([3] p221). This tells us that any axisymmetric mode of a field is bound to decay (see Fig. 2). The anti-dynamo theorem can be generalized to assert that fluid motions cannot maintain any magnetic field in which there is anywhere a closed line of force around which the neighboring lines of force circle ([2] p538-541). Finally, in 1945, Elsasser proved that no other mechanism was possible for induction of magnetic fields except the motions of the liquid metal core ([2] Ch 18). However, dynamo theory remained a little explored field. As late as 1964, Feynman wrote:

> Certainly in some places around the sun and stars there are effects of electromagnetic induction. Perhaps also (though it's not certain) the magnetic field of the earth is maintained by an analog of an electric generator that operates on circulating currents in the interior of the earth ([4] 16-9).

There are two steps to this magnetic field generation. First weak polar (poloidal) fields are sheared by the non-uniform rotation of a planet or star, drawing out the field lines in the azimuthal direction. Our Sun spins around many times in a complicated motion that is periodic with approximately an 11 year cycle; the result is that the azimuthal magnetic field component builds up to an intensity far in excess of the weak poloidal field. Second, magnetic fields are generated by turbulent convection ([5] 28). When this process was discovered, the field of magnetohydrodynamics (MHD) was born. Even now, Parker tells us that we have yet to understand some important physics of the Sun and stars ([5]).

Once Elsasser had proved that current bearing flows were the key to understanding planetary magnetics, new work in magnetohydrodynamics closely followed parallel work in hydrodynamics. Solar convection, pictured in Fig. 3, is

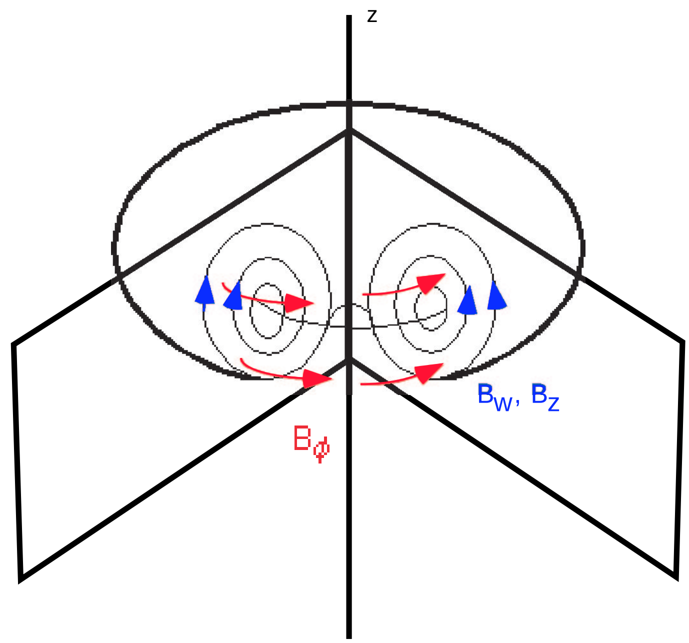

Figure 2: An Axisymmetric Field such as in Cowling's Theorem.

surely turbulent; the magnetic Reynolds number in our Sun is on the order of 10^{12} according to Parker ([5] 28), and so well within the turbulent regime.

MHD benefited from the development of turbulence theory in the mid- twentieth century. The ground breaking work in turbulence happened about the same time as Elsasser's proof; in 1941 A. N. Kolmogorov published two papers on turbulence which are still of fundamental importance to turbulence theory.

0.1 Kolmogorov

Although Kolmogorov's work is not directly connected to the Navier-Stokes equations of fluid flow, which will be discussed in this thesis, it provides a background for most of the research that goes on in turbulence today. Kolmogorov's work can also be used as a check for more complicated models for the energy, because part of his theory models energy dissipation . The purpose of his theory was to formulate hypotheses compatible with the Navier-Stokes equation from which further predictions could be made ([6]).

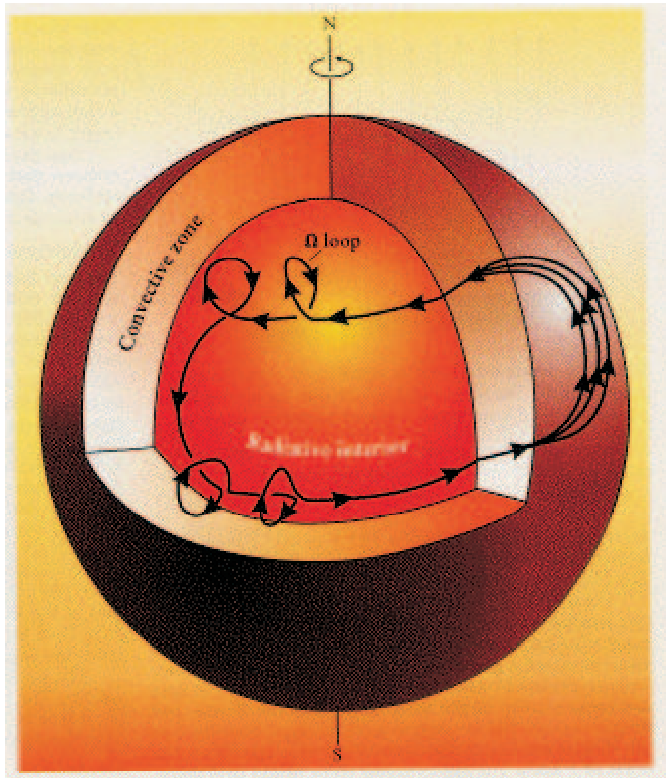

Figure 3: Turbulent convection in the Sun. ([5])

One result in particular of Kolmogorov's 1941 theory is of interest to us – the law regarding the energy spectrum in the inertial range where there are small eddies and large k (wavenumber) — the $k^{-5/3}$ law ([6] 264). The inertial (nonlinear) term in the Navier Stokes equations cannot create or destroy energy, but transfers energy from one wavenumber mode to another. If this inertial term were not in the Navier-Stokes equation, then each mode would decay separately and the spectrum would be fully determined by the initial conditions (i.e. we would get exponential decay). However, observation shows this is not the case. Kolmogorov postulated that inertial transfer of energy was a local phenomenon, so that the probability of a transfer over a large range in wavenumber space is small. He then divided the wavenumber spectrum into three ranges: the production range, the inertial range, and the dissipation range (pictured below). These regions correspond to low wavenumbers, intermediate wavenumbers, and high wavenumbers, respectively. In the inertial range, energy is shifted upwards toward higher wavenumbers, without any production of energy or dissipation taking place. In the dissipation range, viscous dissipation destroys the energy produced in the production range. Kolmogorov's postulate essentially says that these three ranges are independent. In particular, the way in which energy is put into the production range does not affect the spectra in the inertial and dissipation ranges. It is this postulate which justifies making calculations on homogeneous isotropic turbulence, a common starting point for most turbulence theories, even though nothing need be said in such a model about how the turbulent energy entered the system ([7] 25). It is worth noting that Kolmogorov's inertial law has been confirmed by experiment [1].

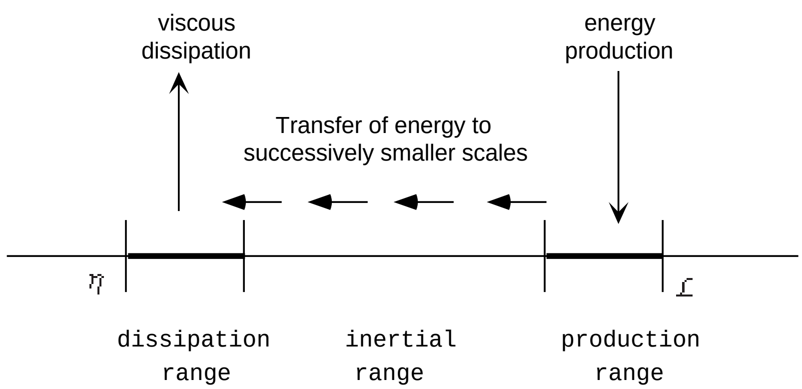

Figure 4: The Inertial Range. Here η is the dissipation length scale and $\mathcal{L}$ is the characteristic length scale of the flow ([9])

We discuss these historical results of Kolmogorov, because a comparison between Kolmogorov's results and those of statistical closures are frequently sought. Lesieur comments that eddy damped theories, which we will discuss at length

[1] This result is obtained from Kolmogorov's first universality assumption that at very high, but not infinite Reynolds numbers, all the small-scale statistical properties are uniquely and universally determined by a characteristic scale L (for instance $\frac{2\pi}{k}$), the mean energy dissipation rate ϵ and the dissipation length scale η, where $\eta = (\frac{\nu^3}{\epsilon})^{\frac{1}{2}}$ where ν is the kinematic viscosity ([8] Chapter 6). Through a simple dimensional argument, the first universality assumption implies the following universal form for the energy spectrum at large wavenumbers k is of the form $E(k) = \epsilon^{\frac{2}{3}} k^{-5/3} F(\eta k)$, where F is a function not determined by the law. This law for the energy spectrum, E, is an immediate consequence of Kolmogorov's famous two-thirds law (that the mean square velocity increment between two points is proportional to the distance between the points to the two-thirds power).

later in this paper, should lead to a Kolmogorov $k^{-5/3}$ inertial range spectrum for isotropic three-dimensional turbulence ([10] 219). Orszag also discusses the accuracy of Kolmogorov's Law in reference to statistical closures as a comparison ([6] 277, 295). Whether they eventually corroborate Kolmogorov's laws or not, researchers tend to compare their theories to those of Kolmogorov. Thus there are checks and comparisons to be drawn between experiment or simulation, Kolmogorov's famous work, and the statistical closures that we will be treating.

A second, more general product of Kolmogorov's theories is the energy cascade, a model for energy dissipation. This idea is one that Kolmogorov borrowed from Richardson (1922) and then refined and made more quantitative. According to Kolmogorov's 1941 theory, when energy is injected into a flow at rate ϵ, it 'cascades' down a hierarchy of eddies at the same rate until it is eventually removed by dissipation at the bottom ([8] 104). A cartoon of the energy cascade is shown below. [2]

0.2 Statistical Closures

All attempts at a statistical theory of turbulence are ultimately faced with the problem of closures. In general, the evolution equations obtained from the Navier-Stokes equation are infinite in number. To solve the set of equations some statistical quantity, for which no finite number of governing equations apply, must be truncated at some order; this is a closure.[3] The success of a closure depends on

---

[2]Richardson described the energy cascade in the verse:

> Big whorls have little whorls,
> Which feed on their velocity;
> And little whorls have lesser whorls,
> And so on to viscosity
> (in the molecular sense). ([9] 183)

[3]The Navier-Stokes equations are closed. Batchelor explains why the equations describing cumulants, which are derived directly from the Navier-Stokes equation, are unclosed. The Navier-Stokes equations are open in time, since any one realization must be integrated over the whole of future time before it can give exact values for the various cumulants. In contrast, the equations for the cumulants can be made independent of time. Also additional higher order cumulants give less and less additional information, so that only a finite number are necessary to understand the main properties of the flow [11].

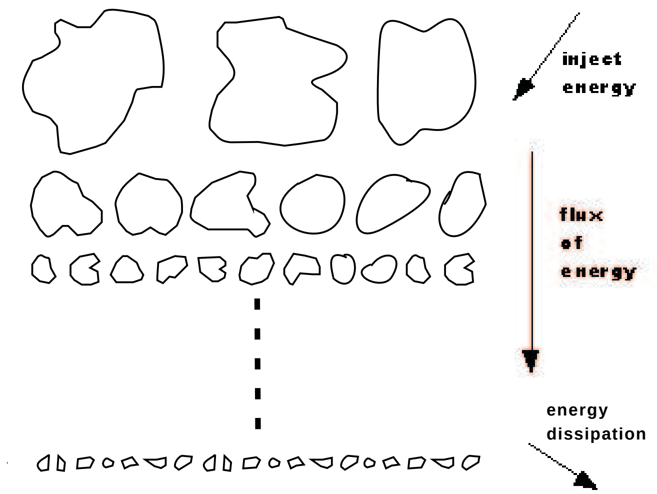

Figure 5: The Energy Cascade. ([9] 183)

both the flow configuration and the statistical order at which results are desired. Most closures can be classified as either one-point or two-point closures. One-point closures start with the Reynolds equations (see Appendix B), and calculate mean characteristics like velocity profiles, and spreading rates; they typically utilize the Prandtl mixing length theory and are efficient for engineering interests. They are called one-point closures because they are used to calculate correlations using information only at a single point in space. The most common one-point closure is the Reynolds Averaged Navier Stokes (RANS). Two-point closures deal with correlations at two different points of the space. The common two-point closures are Kraichnan's Direct Interaction Approximation (DIA), the Test Field Model (TFM), and Orszag's Eddy Damped Quasi-Normal Markovian (EDQNM) closure developed in 1970 ([6]).

Millionshtchikov (1941) and Chou (1940) independently proposed the quasi-normal approximation [10]. The idea of this approximation is simply to assume that the fourth order cumulants of the velocities are zero, without any assumptions on the third order cumulants[4]. Orszag proposed that the fourth order cumulants be approximated in terms of second order cumulants multiplied by a damping term,

[4]see Appendix for further discussion of cumulants

the eddy damping function. For turbulence, EDQN (Eddy Damped Quasi Normal) closures are physically accurate, but do not guarantee realizability (positivity of the energy spectrum). In order to assure realizability, the equations must also be Markovian ([6]). The Markovian aspect of the closure is the property that the rate of change of the energy spectrum depends only on the current values of the energy spectrum.

0.3 This Work

This thesis will apply the statistical closures of turbulence to magnetohydrodynamics. In this work we extend ideas that I worked on under the supervision of Leaf Turner during the summers of 1999 and 2000 at Los Alamos National Laboratory; this work is in preparation to be published. However, this earlier work, detailed in section IV of this manuscript, stopped short of treating the mean field equations of magnetohydrodynamics. Mean fields contribute in important ways to magnetic field generation. The motivation for the work in this thesis is to adapt the EDQNM closure so that it provides a reasonable model for magnetohydrodynamics when mean fields are present. First and most important, the modified closure must still be realizable for Navier Stokes turbulence, when the magnetic field is zero. For neutral fluids, the way that mean fields work with the EDQNM closure has been worked out by Turner [1]. To be a useful closure, it must also be accurate when checked against a direct numerical simulation. After these crucial steps are confirmed we address the modified closure works with full-blown magnetohydrodynamics.

This paper will be organized as follows. We will begin by introducing the equations of magnetohydrodynamics and turbulence. We then discuss the Eddy Damped Quasi-Normal Markovian statistical closure. At this point we present a proof developed by Pratt and Leaf Turner ([12]), showing that when no mean fields are present, EDQNM gives a realizable set of equations for the evolution of the spectral energy of turbulent magnetohydrodynamics. We will then proceed to investigate matters of mean fields, and numerical calculations. It is our hope that this thesis will also make the ideas of MHD and turbulence accessible to students with a basic background in mathematics.

Part II

The Magnetohydrodynamic Equations

We begin with a development of the coupled symmetric magnetohydrodynamic equations. To obtain a magnetic induction equation in the form we want, we begin with Faraday's law, the solenoidal condition on B, Ampere's law, and Ohm's law:

$$\frac{\partial \mathbf{B}}{\partial t} = -\bigtriangledown \times \mathbf{E}, \quad \bigtriangledown \cdot \mathbf{B} = 0 \,, \tag{1}$$

$$\mathbf{j} = \bigtriangledown \times \frac{\mathbf{B}}{\mu_0} \,, \tag{2}$$

$$\mathbf{j} = \sigma \mathbf{E} + \sigma \mathbf{u} \times \mathbf{B} \,, \tag{3}$$

where **E** and **B** represent electric and magnetic fields, **j** is the current density, **u** is velocity field, and σ is the electrical conductivity of the fluid ([13] 16). Elimination of **j** and E from these three equations gives one of the MHD equations:

$$\frac{\partial \mathbf{B}}{\partial t} = \bigtriangledown \times (\mathbf{u} \times \mathbf{B}) + \eta \bigtriangledown^2 \mathbf{B} \,. \tag{4}$$

where η is the magnetic diffusivity (or "resistivity") of the fluid $\eta = (\mu_0 \sigma)^{-1}$ ([14] 7). It is estimated that for the sun, η is on the order of $10^4 cm^2/s$ ([5] p29). Notice that $q\mathbf{u} \times \mathbf{B}$ is the magnetic Lorentz force on a charge q. If one wanted to consider the case of compressible flow, one can use a differential equation for the mass conservation of the fluid, that is the equation for the density in the footnote. The mass conservation equation leads to a compressible induction equation also given in the footnote.[5]

The momentum equation for the dynamics of our magnetic fluid, the Navier-Stokes equation, is [6]

---

[5]Using the conservation of mass equation:

$$\frac{\partial}{\partial t}\rho + \bigtriangledown \cdot (\rho u) = 0 \,,$$

we find the induction equation for compressible fluids (Childress 7):

$$\frac{D}{Dt}(\frac{B}{\rho}) - \frac{\eta}{\rho} \bigtriangledown^2 \mathbf{B} = (\frac{\mathrm{B}}{\rho}) \cdot \bigtriangledown \mathrm{u} \,.$$

[6]Landau and Lifshitz give the compressible Navier-Stokes equation in the slightly modified form ([15] 214):

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times (\bigtriangledown \times \mathbf{u}) + \nu \bigtriangledown^2 \mathbf{u} - \bigtriangledown(\mathrm{p} + \frac{1}{2}\mathrm{u}^2) - \mathbf{B} \times \bigtriangledown \times \mathbf{B} \,. \tag{5}$$

where p is pressure, and as usual bold indicates a vector quantity. Here ν represents kinematic viscosity, and $(\bigtriangledown \times \mathbf{B}) \times \mathbf{B}$ is the Lorentz force density ([16] 244). Notice that this is valid only for incompressible fluids; to treat compressible fluids, one should use the equation in the footnote. For ease of notation we will label $(p + \frac{1}{2}u^2)$ as $p^\star$.[7]

In order to obtain a symmetric form we use the two dimensionless variables $\mathbf{W}^{(\pm)} = \mathbf{u} \pm \mathbf{B}$, which were first introduced by Elsasser ([16] 46). In terms of these Elsasser variables the dynamical equations (4) and (5) take the more symmetrical form

$$\begin{aligned} \frac{\partial \mathbf{W}^{(+)}}{\partial t} &= \frac{1}{2}[\mathbf{W}^{(+)} \times (\bigtriangledown \times \mathbf{W}^{(-)}) + \mathbf{W}^{(-)} \times (\bigtriangledown \times \mathbf{W}^{(+)}) \\ &- \bigtriangledown \times (\mathbf{W}^{(+)} \times \mathbf{W}^{(-)})] \\ &- \bigtriangledown p^\star + (\frac{\nu+\eta}{2}) \bigtriangledown^2 \mathbf{W}^{(+)} + (\frac{\nu-\eta}{2}) \bigtriangledown^2 \mathbf{W}^{(-)} \,, \end{aligned} \tag{6}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (u \cdot \bigtriangledown)u = -\frac{1}{\rho} \bigtriangledown p - \frac{1}{4\pi\rho} B \times (\bigtriangledown \times B) + \frac{\nu}{\rho} \bigtriangledown^2 u + \frac{1}{\rho}(\zeta + \frac{1}{3}\nu) \bigtriangledown (\bigtriangledown \cdot u) \,.$$

Here ζ and ν are the two coefficients of viscosity of the fluid. $\frac{\nu}{\rho}$ is the kinematic viscosity, denoted simply by ν in the main text. Notice that this equation does include all of the terms necessary to consider compressibility, and the last term in this equation disappears in the limit of incompressible flow.

[7] We also note here that, for the incompressible case, the Navier-Stokes equation is also often tackled using the vorticity Ω to split Navier-Stokes into the equations below ([4] 40-5). These equations are only valid when ρ is spatially constant.

$$\begin{aligned} \frac{\partial}{\partial t}\Omega + \bigtriangledown \times (\Omega \times u) &= \frac{\nu}{\rho} \bigtriangledown^2 \Omega \,, \\ \Omega &= \bigtriangledown \times \mathbf{u} \,, \\ \bigtriangledown \cdot \mathbf{u} &= 0 \,. \end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathbf{W}^{(-)}}{\partial t} &= \frac{1}{2}[\mathbf{W}^{(+)} \times (\bigtriangledown \times \mathbf{W}^{(-)}) + \mathbf{W}^{(-)} \times (\bigtriangledown \times \mathbf{W}^{(+)}) \\
&- \bigtriangledown \times (\mathbf{W}^{(-)} \times \mathbf{W}^{(+)})] \\
&- \bigtriangledown p^{\star} + (\frac{\nu + \eta}{2}) \bigtriangledown^2 \mathbf{W}^{(-)} + (\frac{\nu - \eta}{2}) \bigtriangledown^2 \mathbf{W}^{(+)} .
\end{aligned} \tag{7}
$$

Notice that half of the sum of these Elsasser variables gives us back our original Navier-Stokes equation (5). Half of the difference gives us the induction equation (4). The various contributions to these equations are evident. The convective terms and Lorentz force are quadratic. The dissipative terms are linear.

Landau considers the interesting dependence of these equations on the fluid viscosity and the magnetic diffusivity. He proposes that if $\nu = \eta$ then a steady magnetic field can exist. If $\nu/\eta \geq 1$ then a magnetic field can spontaneously grow ([15] 235). This idea is helpful, because in the case where $\eta \geq \nu$, the coefficient of one of the linear terms would be negative, giving a "negative friction". This is an interesting case where energy is being added to one W term and taken away from another W, and their relationship is determined by the relative sizes of their viscous coefficients.

At this point we expand $\mathbf{W}^{(+)}$ and $\mathbf{W}^{(-)}$ into an orthonormal basis L, such that L is solenoidal, with spectral coefficients X and Y:

$$
\begin{aligned}
\mathbf{W}^{(+)}(\mathbf{r}, \mathrm{t}) &= \sum_i X_i(t)\mathbf{L}_i(\mathrm{r}) , \\
\mathbf{W}^{(-)}(\mathbf{r}, \mathrm{t}) &= \sum_i Y_i(t)\mathbf{L}_i(\mathrm{r}) .
\end{aligned} \tag{8}
$$

Basis functions used in practical calculations should conform to boundary conditions and geometries. For instance, if we were to use a plane-wave basis, for a hypothetical scalar field, we might write

$$
\begin{aligned}
W^{(+)}(\mathbf{r}, \mathrm{t}) &= \sum_{\mathbf{k}} X_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}} , \\
W^{(-)}(\mathbf{r}, \mathrm{t}) &= \sum_{\mathbf{k}} Y_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}} .
\end{aligned} \tag{9}
$$

where the X and Y terms would be labeled as $W^{(+)}(\mathbf{k}, \mathrm{t})$ and $W^{(-)}(\mathbf{k}, \mathrm{t})$. We will not require any specific basis, but will consider the resulting equations as differential equations. We will approximate them and consider what conclusions might be drawn regarding realizability. Clearly $\mathbf{W}^{(+)}$ and $\mathbf{W}^{(-)}$ are functionals:

$$\frac{d\mathbf{W}^{(+)}}{dt} = F(\mathbf{W}^{(+)}, \mathbf{W}^{(-)}) \, , \tag{10}$$

$$\frac{d\mathbf{W}^{(-)}}{dt} = F(\mathbf{W}^{(-)}, \mathbf{W}^{(+)}) \, . \tag{11}$$

The general significance of the forms $F(*, *)$ is that they have the exchange symmetry ($\mathbf{W}^{(+)} \leftrightarrow \mathbf{W}^{(-)}$) and are bilinear except for the diffusion terms and pressure $p^{\star}$.

With these considerations, we express the MHD equations somewhat schematically in terms of the spectral coefficients X and Y with the simple notation

$$\dot{X}_i = \sum_{jk} c_{jki} X_j Y_k - \alpha X_i - \beta Y_i \, , \tag{12}$$

$$\dot{Y}_i = \sum_{jk} c_{jki} Y_j X_k - \alpha Y_i - \beta X_i \, . \tag{13}$$

These are merely the MHD equations (4) and (5). Given a basis and boundary conditions, the c_{ijk} terms would be fully determined by the MHD equations. This shows that the coupling coefficients, c_{ijk}, are not arbitrary or unknown quantities; they need to be worked out for each problem. We will later discuss properties of the coupling coefficients in a more general way. In these equations, (12) and (13), the scalars α and β are half the sum and difference of viscosity and resistivity respectively.

Notice that these equations have a non-linear term as well as a linear term; however, unlike in the Navier-Stokes equation, this non-linear term is "off diagonal," as XY rather than on the diagonal like X^2 or Y^2. Imagine there is a very high friction in a fluid; then the equations (12) and (13) are approximately linear. In this case, consideration of a closure or realizability is irrelevant. Now imagine the opposite limiting case where the frictions represented by α and β are very small. This is the case where the closure we will apply on the non-linear term is most important.

As we are interested in the evolution equation for the energy spectrum, we will introduce these equations now. To get an expression for energy we multiply the X equation (12) by X_i, and the Y equation (13) by Y_i:

$$\frac{1}{2}\frac{d(X_i)^2}{dt} = \sum_{jk} c_{jki} X_j Y_k X_i - \alpha (X_i)^2 - \beta Y_i X_i \; , \tag{14}$$

$$\frac{1}{2}\frac{d(Y_i)^2}{dt} = \sum_{jk} c_{jki} Y_j X_k Y_i - \alpha (Y_i)^2 - \beta X_i Y_i \; . \tag{15}$$

Herring [17] and Leslie [7] have used this notation for neutral fluid turbulence, i. e. when the magnetic field B is zero. This notation is simpler and more convenient to work with than the notation of $\mathbf{W}^{(\pm)}$.

0.1 An Alternate Representation of the Incompressible MHD Equations

In this section we discuss the simplifications of the MHD equations in the case of incompressible turbulent flow. We use some well known vector identities[8] to rewrite our expressions for $W^{(\pm)}$ in equations (6) and (7). We commute all the differentials with respect to dx_j, and then collect all the divergences (the features that will vanish in the incompressible case), then what is left is the divergence of a

---

[8] Recall that

$$\begin{aligned} \nabla \times (a \times b) &= a(\nabla \cdot b) + (b \cdot \nabla) a - b\,(\nabla \cdot a) - (a \cdot \nabla)\, b \\ [a \times (\nabla \times b)]_i &= a_j \frac{\partial}{\partial x_i} b_j - a_j \frac{\partial}{\partial x_j} b_i \end{aligned}$$

For pure Navier-Stokes turbulence, this gives us the convenient relation:

$$(v \cdot \nabla) v = (\nabla \times v) \times v + \frac{1}{2} \nabla (v \cdot v) \; .$$

We now have to see what this implies for our $\mathbf{W}^{(\pm)}$ variables.

"simple" stress tensor, analogous to the Reynolds stress tensor of the incompressible Navier-Stokes case.[9]

$$
\begin{aligned}
(\frac{\partial}{\partial t} \quad &- \quad \alpha \frac{\partial^2}{\partial x_j \partial x_j}) W_i^{(+)} - \beta \frac{\partial^2}{\partial x_j \partial x_j} W_i^{(-)} = -\frac{\partial}{\partial x_i} p^\star \\
&- \quad W_j^{(-)} \frac{\partial}{\partial x_j} W_i^{(+)} + \frac{1}{2} W_i^{(-)} \frac{\partial}{\partial x_j} W_j^{(+)} - \frac{1}{2} W_i^{(+)} \frac{\partial}{\partial x_j} W_j^{(-)} \ ,
\end{aligned} \tag{16}
$$

Here we have redefined our 'pressure' to include the all gradient terms, so that now:

$$
p^* \to p^* + \frac{1}{2} W_j^{(+)} W_j^{(-)} = p - \frac{1}{2} B^2 \ . \tag{17}
$$

Our argument to get rid of the pressure gradient will only apply for homogeneous unbounded MHD turbulence. For the incompressible case, $\nabla \cdot \mathbf{W}^{(\pm)} = 0$. To proceed then, we sum these two equations.

$$
\begin{aligned}
(\frac{\partial}{\partial t} \quad &- \quad \alpha \frac{\partial^2}{\partial x_j \partial x_j})(W_i^{(+)} + W_i^{(-)}) - \beta \frac{\partial^2}{\partial x_j \partial x_j}(W_i^{(+)} + W_i^{(-)}) = \\
&- \quad 2 \frac{\partial}{\partial x_i} p^* + \frac{\partial}{\partial x_i} W_j^{(+)} W_j^{(-)} - W_j^{(-)} \frac{\partial}{\partial x_j} W_i^{(+)} - W_j^{(+)} \frac{\partial}{\partial x_j} W_i^{(-)} \ .
\end{aligned} \tag{18}
$$

We take the divergence in order to analyze the pressure contribution, thus:

$$
-\nabla^2 p^* \quad = \quad \left(\frac{\partial W_j^{(-)}}{\partial x_i} \right) \left(\frac{\partial W_i^{(+)}}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} W_j^{(-)} W_i^{(+)} \ . \tag{19}
$$

The last equality follows from the condition of incompressibility. Next we will solve this Poisson equation for the pressure by putting it into wavenumber space. The satisfaction of this Poisson equation is a necessary and sufficient condition for a solenoidal velocity field to remain solenoidal ([9] 19). We will solve this Poisson equation for the simplest possible case, homogeneous unbounded turbulence. If

[9] Note that an equation for the evolution of $\mathbf{W}^{(+)}$ can be found by switching all $+$ and $-$ superscripts, since these equations are completely symmetric.

we wanted any particular geometry, we could just revisit this Poisson equation for the pressure, and solve it with some specific boundary conditions.

$$\mathcal{F}_k\left\{-\nabla^2 p^*\right\} = \mathcal{F}_k\left\{\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\left(W_j^{(-)}W_i^{(+)}\right)\right\} \; . \tag{20}$$

We then define a function $\hat{G}_l$ so that the divergence of $\hat{G}_l$ is equal to the right hand side of our Poisson equation:

$$\hat{G}_l = \mathcal{F}_k\left\{\frac{\partial}{\partial x_m}\left(W_m^{(-)}W_l^{(+)}\right)\right\} \; . \tag{21}$$

Now we solve for the term we want to substitute into our equation of motion in wavenumber space:

$$-ik_i p^* = \frac{k_i k_l}{k^2}\hat{G}_l \; . \tag{22}$$

Substituting we get the result:

$$(\frac{\partial}{\partial t} + \alpha k_j^2)\hat{W}_i^{(+)} + \beta k_j^2 \hat{W}_i^{(-)} = \frac{k_i k_l}{k^2}\hat{G}_l - \hat{G}_i = -(\delta_{il} - \frac{k_i k_l}{k^2})\hat{G}_l \tag{23}$$

We arrive at the final simple equation of motion:

$$\begin{aligned}(\frac{\partial}{\partial t} \quad &+ \quad \alpha k^2)\hat{W}_i^{(+)} + \beta k^2 \hat{W}_i^{(-)} = \\ &- \quad ik_m(\delta_{il} - \frac{k_i k_l}{k^2})\sum_{\mathbf{k}'} W_m^{(-)}\left(\mathbf{k}-\mathbf{k}', \mathrm{t}\right) W_l^{(+)}\left(\mathbf{k}', \mathbf{t}\right) \; .\end{aligned} \tag{24}$$

For pure Navier-Stokes turbulence this equation is standard; see Pope or Eq. (44) for the analogous relation. With this equation of motion (24) we have found a succinct form for the incompressible MHD turbulence equations. This is significant because we want to set the problem in analogy to the Navier-Stokes case so we can take advantage of results established there. This equation is particularly remarkable because of the extremely close analogy with the corresponding result for Navier-Stokes turbulence. I am unaware of a previous derivation. This equation of motion will be useful when we come to discuss the coupling coefficients used in the equations above; it shows the origin of those coupling coefficients. When we come to work on a direct numerical simulation for incompressible MHD turbulence, this is the equation we will work with. It is phrased in a particularly useful form to perform a pseudo-spectral calculation, which we will discuss in detail later.

If the $\bigtriangledown \cdot \mathbf{W}^{(\pm)}$ is initially zero, then our equation dictates that it will stay zero for all time. If we assume that initially $\mathbf{k} \cdot \mathbf{W}^{(\pm)} = 0$, this property of the equation is easy to show. We take the curl in Fourier Space of our final equation above (24), and ascertain that the solenoidal part has been successfully eliminated. Clearly the left hand side is zero since the $W^{(\pm)}$ are solenoidal:

$$\begin{aligned} 0 &= k_i i k_m (\delta_{il} - \frac{k_i k_l}{k^2}) \sum_{\mathbf{k}'} W_m^{(-)}(\mathbf{k}-\mathbf{k}', \mathbf{t}) \mathbf{W}_{\mathbf{l}}^{(+)}(\mathbf{k}', \mathbf{t}) & (25) \\ 0 &= i k_m k_l \sum_{\mathbf{k}'} W_m^{(-)}(\mathbf{k}-\mathbf{k}', \mathbf{t}) \mathbf{W}_{\mathbf{l}}^{(+)}(\mathbf{k}', \mathbf{t}) & (26) \\ &\quad - i k_m k_i \frac{k_i k_l}{k^2} \sum_{\mathbf{k}'} W_m^{(-)}(\mathbf{k}-\mathbf{k}', \mathbf{t}) \mathbf{W}_{\mathbf{l}}^{(+)}(\mathbf{k}', \mathbf{t}) \; . & (27) \end{aligned}$$

Since $k_i k_i = k^2$, we have the identity:

$$\begin{aligned} 0 &= i k_m k_l \sum_{\mathbf{k}'} W_m^{(-)}(\mathbf{k}-\mathbf{k}', \mathbf{t}) \mathbf{W}_{\mathbf{l}}^{(+)}(\mathbf{k}', \mathbf{t}) & (28) \\ &\quad - i k_m k_l \sum_{\mathbf{k}'} W_m^{(-)}(\mathbf{k}-\mathbf{k}', \mathbf{t}) \mathbf{W}_{\mathbf{l}}^{(+)}(\mathbf{k}', \mathbf{t}) \; , & (29) \end{aligned}$$

thus we have made the right hand side solenoidal.

In 1996 Leaf Turner worked out the equations for Navier-Stokes (neutral fluid) turbulence using a helicity representation to get rid of the solenoidal part of the equations, without separately solving the Poisson equation like we did above [18]. Turner's development gives more direct equations that are consequently easier to use with closures. They also have the advantage of side-stepping the multiple factors that result from the solution to the Poisson equation; these factors can further complicate the solutions to more sophisticated problems. Because our Eq. (24) is so close to the Navier-Stokes equation, Turner's development can be directly carried over to the field of MHD turbulence. We will assume that this is possible for our proof. It may be possible to write a pseudo-spectral code with Turner's equations, using either a FFT or a fast wavelet transform. However, for a direct numerical simulation we will use our simple equation above (24).

0.2 The Coupling Coefficients

We make some general restrictions on the $\mathbf{c}$ coefficients of equations (14) and (15). We want to conserve energy; to do this, we ignore frictional terms, because energy cannot be conserved when it is being dissipated by friction. We will find the different restrictions on the couple coefficients for both the cases of Navier-Stokes turbulence and for MHD turbulence.

0.2.1 Navier-Stokes Turbulence

For turbulence in an electrically neutral fluid we have the equation $\frac{\partial}{\partial t} X_i = c_{jki} X_j X_k = c_{kji} X_k X_j$. This has a symmetry on the j and k indices. Thus we constrain the coupling coefficients:

$$c_{jki} = c_{kji} \ . \tag{30}$$

To conserve energy, we take the sum of all energies and set it equal to zero:

$$\sum_i \frac{\partial}{\partial t} X_i^2 = 0 \tag{31}$$

This gives us a cyclic condition:

$$c_{jki} + c_{kij} + c_{ijk} = 0 \tag{32}$$

These restrictions (30) and (32) on these coupling coefficients can be shown to be precisely the same as those derived by Zwanzig using the Louiville equation in his discussion of nonlinear dynamics of collective modes ([19] 247).

A major open problem in turbulence theory is the question of what to do with a mean field. If we require the symmetry $U_i = U_{-i}$, where U_i represents a second order moment of X_i. This is a consequence of the Random Phase Approximation (RPA) discussed later in this thesis, then the mean field evolution can be described by:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle X_i \rangle = \sum_j c_{jji} U_j \tag{33}$$

To maintain zero mean field, we thus require that:

$$c_{jji} = -c_{-j-ji} \; . \tag{34}$$

This last constraint will be used in our discussion of realizability.

0.2.2 MHD Turbulence

In the MHD case, we have the energy equation $\frac{1}{2}\frac{\partial}{\partial t}X_i^2 = c_{jki}X_jY_kX_i$. By setting the left hand side of the energy evolution equation to zero when the frictional terms are zero (conservation of energy and also conservation of cross helicity)[10] we can discover one constraint on the **c** parameters:

$$\sum_i \frac{\partial}{\partial t} X_i^2 = \sum c_{jki} X_i X_j Y_k = 0 \; . \tag{35}$$

From this conservation equation we can see a symmetry on the i and j indices. Thus we conclude that:

$$c_{ikj} = c_{-i-k-j}. \tag{36}$$

[10]Cross helicity is defined as $u \cdot B = (u+B)^2 - (u-B)^2$, so conserving $X^2 - Y^2$ provides conservation of cross helicity. Total energy is defined as $u^2 - B^2 = (u+B)^2 + (u-B)^2$, so conserving $X^2 + Y^2$ provides conservation of cross helicity. Since both X and Y are separately conserved by setting this sum equal to zero, energy and cross helicity are both conserved.

We can say analogous things about the mean field in MHD turbulence. When we apply the Random Phase Approximation, we find the symmetry $U_j^z = U_{-j}^z$, where U_j^z defined later in Part V of this thesis. Then the evolution of the mean field is of the form:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle X_i\rangle = \sum_j c_{jji}U_j^z \; . \tag{37}$$

To maintain a zero mean field this means that:

$$c_{jji} = -c_{-j-ji} \; . \tag{38}$$

Thus we find that neutral fluid turbulence gives us three constraints on the coupling coefficients, and MHD turbulence gives us two, because the nonlinear term in the MHD case is off diagonal, as we discussed.

0.3 The Eddy Damping Function

We choose an Eddy Damping function to obey the simple conditions:

$$\Theta_{jki} = \Theta_{ikj} \; , \tag{39}$$

$$\Theta_{jki} = \Theta_{kji} \; , \tag{40}$$

$$\Theta_{jki} \geq 0 \; . \tag{41}$$

This will be used in our proof in Part V of this thesis, as well as the computations that we do.

Part III

The Eddy Damped Quasi-Normal Markovian Closure

Here we take the time to discuss the usage of the statistical model of turbulence called the Eddy Damped Quasi-Normal Closure. The EDQNM closure is a standard scheme used to close the Navier-Stokes equations for turbulence; it has been shown to give useful results in comparison to direct numerical simulations. The Eddy-Damped Quasi-Normal Markovian Closure can be solved in a less computationally expensive way than a direct numerical simulation (DNS) for a given ensemble of initial conditions. It also provides us with an intuitive statistical model to understand the dissipative process. These equations deal with homogeneous turbulence. The Navier-Stokes equation for pure incompressible Navier-Stokes turbulence is:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \tag{42}$$

where u is the velocity field, and the $u_i u_j$ is the Reynolds stress. We can write this in wavenumber space:

$$\left(\frac{d}{dt} + \nu k^2\right)\hat{u}_i(k,t) = -ik_i\hat{p} - ik_l \sum_{k'} \hat{u}_i(k',t)\hat{u}_l(k-k',t) \tag{43}$$

([9] 214). Working out the pressure gives:

$$\left(\frac{d}{dt} + \nu k^2\right)\hat{u}_i(k,t) = -ik_l(\delta_{il} - \frac{k_i k_l}{k^2}) \sum_{k'} \hat{u}_j(k',t)\hat{u}_l(k-k',t)\ . \tag{44}$$

See Eq. (25) for the MHD case. We will often represent the RHS of this equation as simply uu; this is in order to show the basics of the statistical closure without cluttering up our arguments with convolutions. Since in the standard EDQNM closure the mean velocity is assumed to be zero, or $\langle u \rangle = 0$, we multiply the equation (44) by the velocity again to get an energy equation:

$$(\frac{\partial}{\partial t} + \nu(k^2 + k'^2))\langle u(k)u(k')\rangle = \langle uuu \rangle. \tag{45}$$

The brackets indicate averages over an ensemble of initial conditions that all have the constraints of the specified initial moments ([10] 214); the rules governing these averages will be discussed further when we propose a modified EDQNM closure. To obtain an equation for the third order moments on the right hand side of this equation we multiply again by u to get ([10] 215):

$$(\frac{\partial}{\partial t} + \nu(k^2 + p^2 + q^2))\langle u(k)u(p)u(q)\rangle = \langle uuuu\rangle. \tag{46}$$

Already the closure problem is evident, because each time we find a new equation, it contains a higher order moment, and so we always have more variables than equations.

The quasi-normal theory involves closing the equations by assuming that u has a distribution that is close to Gaussian. We will talk more about the Gaussian approximation and the Markovian approximation later in this manuscript. If we let X denote a phase position at (x, y, z, t), and $g(X)$ be a given function of X, then given N arbitrary numbers a_i and N values of X_i, the combination $\sum a_i g(X_i)$ is a Gaussian random variable if the sum is distributed according to a Gaussian distribution. Then we have the following theorem:

> Iff $g(x)$ is a Gaussian random variable with zero mean, the odd moments (or cumulants) are zero and the even moments can be expressed in terms of the second moments.

So all cumulants beyond the second are zero, e.g.

$$\begin{aligned}\langle g(X_1)g(X_2)g(X_3)g(X_4)\rangle &= \langle g(X_1)g(X_2)\rangle\langle g(X_3)g(X_4)\rangle \\ &+\langle g(X_1)g(X_3)\rangle\langle g(X_2)g(X_4)\rangle + \langle g(X_1)g(X_4)\rangle\langle g(X_2)g(X_3)\rangle.\end{aligned} \tag{47}$$

([10] p216). The quasi-normal assumption uses the Gaussian model to guess at moment relations and on that basis derives non-Gaussian results. Thus we will assume not only that the second moments are non-zero, but that the third moment is non-zero and can be expressed in terms of the second moment as well.

After applying the quasi-normal assumption we need two more adjustments to the equations in order to make the theory realizable for Navier Stokes turbulence: the Eddy Damping and Markovian aspects of the closure. Eddy Damping involves multiplying by an Eddy Damping function, Θ, that has dimensions of the inverse of time and is a characteristic eddy damping rate of the third order moments. There are many more sophisticated forms of Θ used for specific cases, but they are all symmetric under permutations of k, p, and q. The choice of the Eddy Damping function is more difficult for non-isotropic situations. In such cases, the form of the Eddy Damping function is still an open question ([10]). This is a topic explored in this thesis. Making the equation Markovian means that time memory is neglected. This step consists of assuming that an exponential term varies with a characteristic time much smaller than the characteristic evolution time of the double correlations. Eddy damping and Markovianization together guarantee realizability. In the literature, these two steps equate to the following substitution for isotropic homogeneous turbulence. If we let $E(k,t)$ be the energy of modes with wavevector k at time t, the equation:

$$\begin{aligned}[\frac{\partial}{\partial t} \quad &+ \quad 2\nu k^2]E(k,t) = \frac{1}{2}\int_\triangle\int dpdq \int_0^t dse^{-\nu(k^2+p^2+q^2)(t-s)}kp^{-1}q^{-1} \\ &\times \quad \{2k^2 a(k,p,q)E(p,s)E(q,s) - E(k,s)[p^2 b(k,p,q)E(q,s) + q^2 b(k,p,q)E(p,s)]\}\end{aligned} \tag{48}$$

becomes the equation

$$\begin{aligned}[\frac{\partial}{\partial t} \quad &+ \quad 2\nu k^2]E(k,t) = \frac{1}{2}\int_\triangle\int dpdq\Theta(k,p,q;t)kp^{-1}q^{-1} \\ &\times \quad \{2k^2 a(k,p,q)E(p,t)E(q,t) - E(k,t)[p^2 b(k,p,q)E(q,t) + q^2 b(k,p,q)E(p,t)]\}\end{aligned} \tag{49}$$

where

$$\Theta(k,p,q;t) = \int_0^t exp[-\int_s^t \{\gamma(k,r)+\gamma(p,r)+\gamma(q,r)\}dr]ds \tag{50}$$

and where γ is an "eddy viscosity" coefficient included to account for the effects of nonlinear scrambling ([6] 312). The time derivative of energy is dependent only on current values of the energy spectrum ([20]). A virtue of Markovianization is that it preserves realizability of the solution. Notice that equations (49) and (50) are the full equations, not the simplified equations we will be dealing with in this manuscript. How equations such as (44) and (45) are obtained will be dealt with in the following section.

Part IV

Realizability

It is useful to define what we mean by realizability, before we begin to analyze the EDQNM closure. For a statistical closure to be realizable, the most basic physical properties must still be preserved after the approximations are applied. The concern is that some terms that made these variables positive were ignored by our EDQNM assumptions. In the case of Navier-Stokes turbulence, this means simply that all components of the energy spectrum must remain positive as the system evolves under the EDQNM closure. The realizability conditions for MHD are more involved because two energy quantities must be shown to remain positive. Additionally the Schwarz Inequality must be preserved; this issue is not presented by the problem of Navier-Stokes turbulence. Recall that the Schwarz Inequality is a basic principle of linear algebra that says:

$$X \cdot Y = |X||Y|cos(\theta) \leq |XY| \tag{51}$$

Realizability is a necessary condition before verifying whether a closure scheme is useful. Once realizability is established, it remains to determine that a closure gives accurate results in comparison with a direct numerical simulation.

0.1 Statistical Issues

Consider a turbulent flow. Picture a grating perpendicular to the flow of motion, resulting in isotropic turbulence somewhere downstream. Each time that the grating is placed in the flow under specific initial conditions, the resulting turbulence has many complicated movements. Each time these complicated movements are slightly different. Orszag tells us that while the details of fully developed turbulent motions are extremely sensitive to triggering disturbances, the average properties are not ([6] 239). We are interested in an average motion over many different trial runs. This statistical averaging is beneficial to our understanding, because as we have seen, the Navier-Stokes equations involve non-linear terms that couple wavelength modes to each other. To solve this equations we need to resolve a large number of scales. Because of all of these reasons, the statistical approach gives us a simpler picture of the dynamics of the fluid flow.

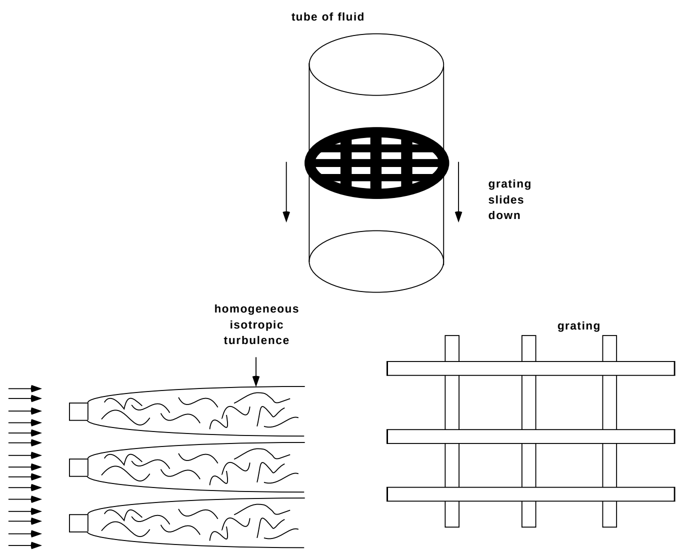

Figure 6: An experiment that would give isotropic homogeneous turbulence (suggested by [6] p256).

Part V

A Proof of Realizability for MHD Turbulence with no Mean Fields

Here I present a proof developed by Leaf Turner and me during the summer of 1999 at Los Alamos National Laboratory. This proof has been presented in two talks given to meetings of the American Physical Society, (Pratt and Turner 1999 [12]) and (Pratt and Turner 2000 [21]), and the International Sherwood Fusion Theory Conference (Turner and Pratt 2001 [22]). We prove that for turbulence when mean fields are zero the EDQNM closure applied to the set of MHD equations give realizable results. This is an important first step to proving that EDQNM is a useful closure when dealing with magnetohydrodynamics.

We begin with the energy equations (14) and (15). Working out the equation for the third order term we get:

$$\begin{aligned}\frac{\partial}{\partial t}(X_j Y_k X_i) = \sum_{j,k} c_{lmj}\langle X_l Y_m Y_k X_i\rangle + c_{lmk}\langle X_j Y_l X_m X_i\rangle + c_{lmi}\langle X_j Y_k X_l Y_m\rangle \qquad (52)\\ -3\alpha X_j Y_k X_i - \beta(Y_k X_i Y_j + X_k X_j X_i + X_j Y_k Y_i)\ .\end{aligned}$$

At this point we need to modify the standard EDQNM closure for neutral fluids. We will Markovianize, so we substitute:

$$\begin{aligned}X_j Y_k X_i &\approx \frac{\partial}{\partial t}(X_j Y_k X_i) + 3\alpha X_j Y_k X_i \qquad (53)\\ &+ \beta(Y_j Y_k X_i + X_j X_k X_i + X_j Y_k Y_i)\ .\end{aligned}$$

The quantity we are interested in is the energy. We define the "energies" as the averaged of products of X and Y:

$$\begin{aligned}U_i^x &= \langle X_i X_i\rangle \qquad (54)\\ U_i^y &= \langle Y_i Y_i\rangle \qquad (55)\\ U_i^z &= \langle Y_i X_i\rangle\ . \qquad (56)\end{aligned}$$

This tensor is diagonal because we have used the random phase approximation (RPA), which we will discuss later. These energies are the spectral "energies" for $(\mathbf{u}+\mathbf{B})^2$ and $(\mathbf{u}-\mathbf{B})^2$ and $(\mathbf{u}+\mathbf{B})\cdot(\mathbf{u}-\mathbf{B})$.

Further expanding this sum we see

$$
\begin{aligned}
\langle X_j Y_k X_i\rangle &= \sum_{j,k} \Theta_{jki}(c_{llj}U^z_l U^z_k \delta_{ki} + c_{ikj}U^y_k U^x_i + c_{kij}U^z_k U^z_i \\
&\quad + c_{llk}U^z_l U^x_i \delta_{ij} + c_{jik}U^z_j U^x_i + c_{ijk}U^x_j U^z_i \\
&\quad + c_{lli}U^z_j U^z_l \delta_{jk} + c_{jki}U^x_j U^y_k + c_{kji}U^z_k U^z_j)\ . \qquad (57)
\end{aligned}
$$

If we take an ensemble average of the energy evolution equation (14) and (15) over all initial conditions we can write out the basic energy equation:

$$
\frac{1}{2}\dot{U^x_i} = \sum_{j,k} c_{jki}\langle X_j Y_k X_i\rangle - \alpha U^x_i - \beta U^z_i\ . \qquad (58)
$$

Substituting the triple product, (57), into the right hand side of the averaged energy evolution equation, (58):

$$
\begin{aligned}
\dot{U^x_i} &= \sum_{j,k} 2\Theta_{jii}c_{jii}c_{kkj}U^z_k U^z_i + 2\Theta_{jji}c_{jji}c_{kki}U^z_k U^z_j \\
&\quad +2\Theta_{iki}c_{iki}c_{jjk}U^z_j U^x_i \qquad (59) \\
&\quad +2\Theta_{jki}c_{jki}c_{ikj}U^y_k U^x_i + 2\Theta_{jki}c_{jki}c_{kij}U^z_k U^z_i \\
&\quad +2\Theta_{jki}c_{jki}c_{kji}U^z_k U^z_j + 2\Theta_{jki}c^2_{jki}U^y_k U^x_j \\
&\quad +2\Theta_{jki}c_{jki}c_{jik}U^z_j U^x_i + 2\Theta_{jki}c_{jki}c_{ijk}U^x_j U^z_i \\
&\quad -2\alpha U^x_i - 2\beta U^z_i\ , \qquad (60)
\end{aligned}
$$

$$
\begin{aligned}
\dot{U^y_i} &= \sum_{j,k} 2\Theta_{jii}c_{jii}c_{kkj}U^z_k U^z_i + 2\Theta_{jji}c_{jji}c_{kki}U^z_k U^z_j \\
&\quad +2\Theta_{iki}c_{iki}c_{jjk}U^z_j U^y_i \qquad (61) \\
&\quad +2\Theta_{jki}c_{jki}c_{ikj}U^x_k U^y_i + 2\Theta_{jki}c_{jki}c_{kij}U^z_k U^z_i \\
&\quad +2\Theta_{jki}c_{jki}c_{kji}U^z_k U^z_j + 2\Theta_{jki}c^2_{jki}U^x_k U^y_j \\
&\quad +2\Theta_{jki}c_{jki}c_{jik}U^z_j U^y_i + 2\Theta_{jki}c_{jki}c_{ijk}U^y_j U^z_i \\
&\quad -2\alpha U^y_i - 2\beta U^z_i\ . \qquad (62)
\end{aligned}
$$

And the equation for the z spectral energy:

$$
\begin{aligned}
\dot{U}_i^z &= \sum_{j,k}(2\Theta_{jji}c_{jji}c_{kki}U_k^zU_j^z + 2\Theta_{iki}c_{iki}c_{jjk}U_j^zU_i^z + 2\Theta_{jki}c_{jki}^2U_k^zU_j^z \\
&\quad +2\Theta_{jki}c_{jki}c_{kji}U_k^xU_j^y + 2\Theta_{jki}c_{jki}c_{jik}U_i^zU_j^z \\
&\quad +\Theta_{jki}c_{jki}c_{ijk}(U_i^xU_j^y + U_j^xU_i^y) + \Theta_{jki}c_{jki}c_{ikj}U_i^z(U_k^x + U_k^y) \\
&\quad +\Theta_{jki}c_{jki}c_{kij}U_k^z(U_i^x + U_i^y) + \Theta_{jii}c_{jii}c_{kkj}U_k^z(U_i^x + U_i^y)) \\
&\quad -2\alpha U_i^z - \beta(U_i^y + U_i^x)\ . \qquad (63)
\end{aligned}
$$

These equations are simply the result of applying the EDQNM closure to the coupled symmetric MHD equations. These equations are large and somewhat complicated. But we can see that the quadratic terms form the bulk of these equations, and the linear frictional terms are also present. It can easily be shown that the viscosity terms do not affect the realizability of the two energies. If the U_i^x or U_i^y energy starts at zero, the α terms will be zero. The β terms cannot affect realizability for the energies because if one of the energies starts off zero, the U_i^z term has to be zero also. Viscous terms do not affect the realizability; for this reason, the nonlinear terms will hold most of our attention.

The proof that follows shows a correspondence between the Schwarz Inequality condition and that of positive energies. Thus if one of these conditions is preserved by the closure, the other condition will also be preserved, and hence the closure will be realizable. We prove first that when the Schwarz Inequality is assumed, then the energy components stay positive if initially positive. Then we proceed to show that when the energies are assumed positive, the Schwarz Inequality is assured.

0.1 Schwarz Inequality Implies Positive Energies

We would like to determine whether the U_i^x and U_i^y energies are positive after the EDQNM closure is applied. Clearly the differential equations for the U_i^x and U_i^y components are identical in form, so proving all U_i^x remain positive implies that the U_i^y remain positive. We approach this question by assuming all components of U^x and U^y components are initially positive, and we let the first component to become zero be U_i^x. We then investigate whether the derivative of U_i^x can be negative when U_i^x is zero; if the derivative is positive, then $U_i^x \geq 0$ for all time.

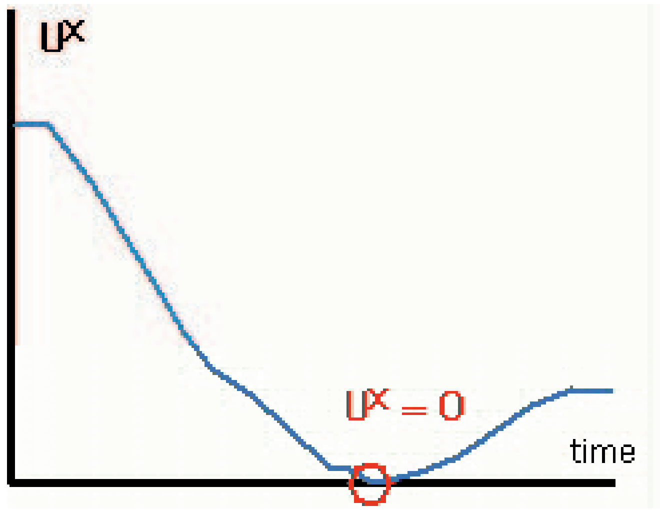

Figure 7: Positive Energy: If U_i^x derivative is always positive when U_i^x is zero, U_i^x cannot be negative.

From the EDQNM closure above we have an equation for $\dot{U_i^x}$, namely (60). We will attack this proof by setting up a contradiction. We assume that $U_i^x = 0$ at some point circled in Fig. 7. This requires that $U_i^z = 0$ because the Schwarz Inequality $(U_i^z)^2 \leq U_i^x U_i^y$. This argument thus assumes Schwarz. Eliminating all of the linear terms from Eq. (60), leaves the more manageable

$$
\begin{aligned}
\dot{U_i^x} &= \sum_{j,k}(2\Theta_{jji}c_{jji}c_{kki}U_k^z U_j^z + 2\Theta_{jki}c_{jki}c_{kji}U_k^z U_j^z \\
&+ \Theta_{jki}c_{jki}^2 U_k^y U_j^x + \Theta_{jki}c_{kji}^2 U_j^y U_k^x) \ . \qquad (64)
\end{aligned}
$$

For ease of notation, we define two ratios that we will use throughout our proofs:

$$
\begin{aligned}
\gamma_i &= \frac{\sqrt{U_i^y U_i^x}}{U_i^z} = (cos\theta_i)^{-1} \ , \qquad (65) \\
\lambda &= \frac{1}{\gamma_j \gamma_k} \ . \qquad (66)
\end{aligned}
$$

where $-1 \leq \lambda \leq 1$ is a constant (no summing is being done in the definition of this ratio). Immediately we can simplify this to a square, a definitely positive term, and an additional term:

$$
\begin{aligned}
\dot{U_i^x} &= \sum_{j,k}(\Theta_{jki}\left(c_{jki}\sqrt{U_j^x U_k^y} + \lambda c_{kji}\sqrt{U_k^x U_j^y}\right)^2 \\
&\quad + \Theta_{jki}(1-\lambda^2)c_{kji}^2 U_k^x U_j^y + 2\Theta_{jji}c_{jji}c_{kki}U_k^z U_j^z) \; .
\end{aligned} \tag{67}
$$

Thus the positivity of $\dot{U_i^x}$ depends on this last term $\Theta_{jji}c_{jji}c_{kki}U_k^z U_j^z$. We assume that there are no mean fields, and thus we can use the symmetry equation for the coupling coefficients. We also assume a similar symmetry on the Eddy Damping function, which was not evident before this proof:

$$
\Theta_{jji} = \Theta_{-j-ji} \; . \tag{68}
$$

Without these symmetries, it can be shown that U^x and U^y energies can be negative; they are a physically natural set of symmetries that makes this problem physically realizable but do not determine these coefficients uniquely. This symmetry on the **c** parameters also give the useful condition that whenever **j** or **k** is zero, we have a term $c_{00i} = -c_{00i} = 0$. Therefore, zero **j** and **k** terms need not be included in consideration of our sum. By applying the symmetries on the coupling coefficients and the Eddy Damping function as well as the definition of the eddy damping function as a positive function, we can factor out everything but the U^z terms, so that we are left with:

$$
\begin{aligned}
\sum_{j,k}\Theta_{jji}c_{jji}c_{kki}U_k^z U_j^z &= \sum_{j,k\geq 0}\Theta_{jji}c_{jji}c_{kki}(U_{-k}^z U_{-j}^z - U_k^z U_{-j}^z - U_j^z U_{-k}^z + U_k^z U_j^z) \\
&= \sum_{j,k\geq 0}\Theta_{jji}c_{jji}c_{kki}(U_{-j}^z - U_j^z)(U_{-k}^z - U_k^z) \; .
\end{aligned} \tag{69}
$$

Without removing the ability to generate magnetic fields from the problem, we choose the symmetry condition $U_j^z = U_{-j}^z$. Using this condition, the problematic term is clearly zero. If this condition is neglected, then the energy can be shown in some cases to become negative. Removing this zero term from our equation (68), U_i^x can be written as the sum of squares

$$
\begin{aligned}
\dot{U_i^x} = \sum_{j,k} \quad \Theta_{jki} \quad & (c_{jki}\sqrt{U_j^x U_k^y} + \lambda c_{kji}\sqrt{U_k^x U_j^y})^2 \\
+ \quad & \Theta_{jki}(1-\lambda^2)c_{kji}^2 U_k^x U_j^y \geq 0 \; .
\end{aligned} \tag{70}
$$

Note that even if the U_j^z or U_k^z terms were as large as possible within the limits of the Schwarz Inequality, the derivative of U^x would still be positive. If U^x were to approach zero, then the derivative would be positive; this implies that U^x and U^z would never reach zero, and we have a contradiction. This contradiction suggests that both the positive energies and Schwarz Inequality will not be violated at the same time. This result cannot be seen as clearly from the Schwarz Inequality proof that follows, because in that case we divide by U_i^x.

The Schwarz inequality always implies that the energy U_i^x is positive in the MHD equations with EDQNM closure, assuming no mean fields. The positivity of U_i^x implies the positivity of U_i^y. Therefore, EDQNM closure satisfies the realizability condition that these energies are positive.

0.2 Positive Energies Imply the Schwarz Inequality is Preserved

We want to show that the Schwarz Inequality, $\mathbf{X}\cdot\mathbf{Y} \leq \sqrt{\mathbf{X^2}}\sqrt{\mathbf{Y^2}}$, or in our notation $(U_i^z)^2 \leq U_i^x U_i^y$, is maintained by EDQNM closure. We will make a similar proof by contradiction argument to that in the last section. To that end, we assume that all U^z satisfy the inequality initially and U_i^z is the first to reach $(U_i^z)^2 = U_i^x U_i^y$ (see Fig. 8). At this point we look at the derivative. When U_i^z is positive, we want a negative derivative in order to keep U_i^z within the bounds of the inequality; when U_i^z is negative, we want a positive derivative. By using the square of U_i^z we can look at both the positive and negative cases at once.

From the working out the EDQNM closure, we know the equations for each component (60), (62), (63). We want to characterize the derivative of U_i^z in the case that U_i^z reaches the limit of the Schwarz Inequality. Examining the Inequality that we want to prove, $(U_i^z)^2 \leq U_i^x U_i^y$, we define B_i:

$$B_i = \frac{(U_i^z)^2}{U_i^x U_i^y} \leq 1 \; . \tag{71}$$

B_i is always positive because we proved in the previous section that U_i^x and U_i^y are positive. We are also assuming that $0 \leq B_j \leq 1$ and $0 \leq B_k \leq 1$ as an initial

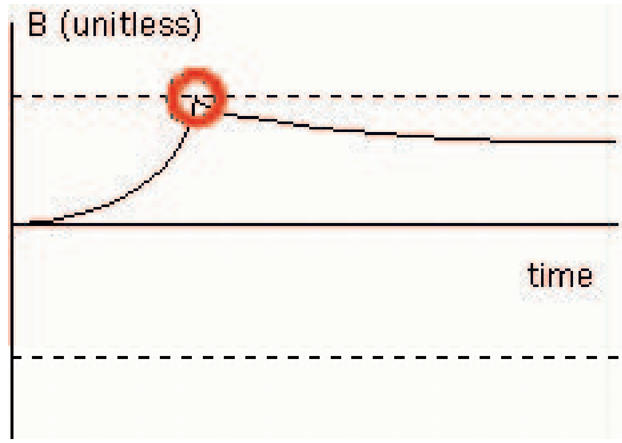

Figure 8: If every time a U^z component reaches the maximum (either positive or negative), it's derivative turns it around, then U^z cannot violate the Schwarz Inequality.

condition. Thus our conclusion will be conditional on these assumptions. Differentiating B_i at the point $B_i = 1$, we get an expression symmetric in X and Y:

$$\dot{B}_i = (\frac{\dot{U}_i^z}{U_i^z} - \frac{\dot{U}_i^x}{U_i^x}) + (\frac{\dot{U}_i^z}{U_i^z} - \frac{\dot{U}_i^y}{U_i^y}) \leq 0\ . \tag{72}$$

We now look at $\dot{B}_i$ when $B_i = 1$. The combination of $B_i = 1$ and (72) (as well as our assumption $B_j \leq 1$ and $B_k \leq 1$) will lead to the maintenance of the Schwarz Inequality. We prove that every time a U^z component approaches the limit of the Schwarz Inequality it turns around, staying within the Inequality.

Any terms containing $U_i^x U_i^y - (U_i^z)^2$ can be eliminated, because we are looking at the case where $B_i = 1$. The terms multiplied by both α and β naturally cancel out, since U_i^z has reached the limit of the Schwarz inequality in the i direction. For B_i, we have the clean result:

$$
\begin{aligned}
\frac{1}{2}\dot{B}_i = & - \sum_{jk} U_k^x U_j^y \Theta_{jki} \left(-\frac{2c_{jki}c_{kji}}{U_i^z} + \frac{c_{jki}^2}{U_i^y} + \frac{c_{kji}^2}{U_i^x}\right) \\
& - \sum_{jk} U_k^z U_j^z \Theta_{jki} \left(-\frac{2c_{jki}^2}{U_i^z} + \frac{c_{jki}c_{kji}}{U_i^y} + \frac{c_{jki}c_{kji}}{U_i^x}\right) \\
& - \sum_{jk} U_k^z U_j^z c_{jji} c_{kki} \Theta_{jji} \left(-\frac{2}{U_i^z} + \frac{1}{U_i^y} + \frac{1}{U_i^x}\right) . \qquad (73)
\end{aligned}
$$

We examine each term in this equation (73) separately to determine whether $\dot{B}_i$ can ever be positive. First we will look at the third line of the above equation, and individually prove that it is negative. Then we will look at the first and second lines of the equation, simplifying them until we can show that they are also negative contributions to $\dot{B}_i$.

0.2.1 *The Last Term*

We will first examine the last term in our equation for $\dot{B}_i$, (73). Call this last term LT:

$$
LT = -\sum_{j,k} U_k^z U_j^z c_{jji} c_{kki} \Theta_{jji} \left(-\frac{2}{U_i^z} + \frac{1}{U_i^y} + \frac{1}{U_i^x}\right) . \qquad (74)
$$

Now since we have assumed that there are no mean fields, $\sum_k c_{kki} U_k^z = 0$. This term is in LT, and so $LT = 0$.

0.2.2 *The First and Middle Terms*

Now we are left to examine the first term and the middle term of our equation for $\dot{B}_i$, (73), FT and MT respectively:

$$
\begin{aligned}
FT &= U_k^x U_j^y \Theta_{jki} \left(\frac{2c_{jki}c_{kji}}{U_i^z} - \frac{c_{jki}^2}{U_i^y} - \frac{c_{kji}^2}{U_i^x}\right) , \qquad (75) \\
MT &= U_k^z U_j^z \Theta_{jki} \left(\frac{2c_{jki}^2}{U_i^z} - \frac{c_{jki}c_{kji}}{U_i^y} - \frac{c_{jki}c_{kji}}{U_i^x}\right) . \qquad (76)
\end{aligned}
$$

To make the algebra easier to follow, we will use the dummy variables **r** and s instead of fractions. Define the dummy variables

$$r = \frac{1}{\sqrt{U_i^y}} \;, \tag{77}$$

$$s = \frac{1}{\sqrt{U_i^x}} \;. \tag{78}$$

We again will employ the scalar γ_i that we defined in Eq. (66); now that we have assumed the Schwarz Inequality, this ratio is simply $\gamma_i = \pm 1$, because U_i^z can be either negative or positive. Then we can see that:

$$\gamma_i r s = \frac{1}{U_i^z} \;. \tag{79}$$

We also define $e_{jk} = \gamma_j \gamma_k$ such that $e_{jk}^2 = 1$. Now if we write FT in this fractionless notation we can see:

$$\begin{aligned} FT &= -U_k^x U_j^y \Theta_{jki} (c_{kji}^2 s^2 - 2\gamma_i c_{jki} c_{kji} r s + c_{jki}^2 r^2) & (80) \\ &= -U_k^x U_j^y \Theta_{jki} (c_{kji} s - \gamma_i c_{jki} r)^2 \leq 0 & (81) \end{aligned}$$

Since we are assuming U^x and U^y are positive for all time, FT individually can never make $\dot{B}_i$ become positive. However, because MT contains U^z terms, this term individually can be positive. We analyze FT and MT together, in the hope that their sum can never be positive

$$\begin{aligned} FT + MT &= U_k^x U_j^y \Theta_{jki} \Big(\frac{2 c_{jki} c_{kji}}{U_i^z} - \frac{c_{jki}^2}{U_i^y} - \frac{c_{kji}^2}{U_i^x} \Big) \\ &+ U_k^z U_j^z \Theta_{jki} \Big(\frac{2 c_{jki}^2}{U_i^z} - \frac{c_{jki} c_{kji}}{U_i^y} - \frac{c_{jki} c_{kji}}{U_i^x} \Big) \;. \end{aligned} \tag{82}$$

In our fractionless notation, this sum is

$$\begin{aligned} FT + MT &= U_k^x U_j^y \Theta_{jki} (-2\gamma_i c_{jki} c_{kji} r s - c_{jki}^2 r^2 - c_{kji}^2 s^2) \\ &+ U_k^z U_j^z \Theta_{jki} (-2\gamma_i c_{jki}^2 r s - c_{jki} c_{kji} r^2 - c_{jki} c_{kji} s^2) \;. \end{aligned} \tag{83}$$

The worst case scenario occurs when MT is large and positive. Perhaps if MT is large enough, $FT + MT$ will become positive. We examine this case, assuming that the Schwarz Inequality holds at some initial time, and that the U_i^z component is the first component to reach the limits dictated by the Inequality. Based on our assumptions, the Schwarz Inequality still holds in the j and k directions. We want MT to be as large and positive as possible, and in this case $e_{jk} = \gamma_j \gamma_k = 1$

$$MT_{max} \leq \sqrt{U_k^x U_k^y U_j^x U_j^y}\Theta_{jki}(2\gamma_i c_{jki}^2 rs - c_{jki}c_{kji}r^2 - c_{jki}c_{kji}s^2)\ . \tag{84}$$

Now our sum $FT + MT$ can be bounded using this inequality (84):

$$\begin{aligned} FT + MT_{max} \ \leq\ & -U_k^x U_j^y \Theta_{jki}(-2\gamma_i c_{jki}c_{kji}rs + c_{jki}^2 r^2 + c_{kji}^2 s^2) \\ & + e_{jk}\sqrt{U_k^x U_k^y U_j^x U_j^y}\Theta_{jki}(-2\gamma_i c_{jki}^2 rs + c_{jki}c_{kji}r^2 \\ & + c_{jki}c_{kji}s^2). \end{aligned} \tag{85}$$

When we re-group by **c** parameters, one square appears:

$$\begin{aligned} FT + MT_{max} \ \leq\ & -c_{jki}^2(r\sqrt{U_k^x U_j^y} - \gamma_i s\sqrt{U_j^x U_k^y})^2 \\ & + e_{jk}c_{jki}c_{kji}(r^2\sqrt{U_k^x U_k^y U_j^x U_j^y} - 2\gamma_i rs U_k^x U_j^y \\ & s^2\sqrt{U_k^x U_k^y U_j^x U_j^y})\ . \end{aligned} \tag{86}$$

This inequality is more tractable because everything is expressed in terms of U^x and U^y . The first part of (87) is a negative square, so we have only the remaining last expression in (87) to analize. We will take a moment to examine just this troublesome last term in (87).

The Bound on the Last Parenthesis

The last possibility that $\dot{B}_i$ can be positive is the last term of the sum (87). Let's call the last remaining problem RP:

$$RP = c_{jki}c_{kji}(r^2\sqrt{U_k^x U_k^y U_j^x U_j^y} - 2rs\gamma_i U_k^x U_j^y + s^2\sqrt{U_k^x U_k^y U_j^x U_j^y})\ . \tag{87}$$

Since **r** and **s** are only functions of **i** , and since the terms that precede RP are symmetric in **j** and **k** we can split the middle term and switch **j** and **k** on one of the middle terms to get:

$$RP = c_{jki}c_{kji}(r^2\sqrt{U_k^xU_k^yU_j^xU_j^y} - rs\gamma_iU_k^xU_j^y - rs\gamma_iU_j^xU_k^y + s^2\sqrt{U_k^xU_k^yU_j^xU_j^y})\ . \tag{88}$$

This can be factored so that:

$$RP = c_{jki}(r\sqrt{U_k^xU_j^y} - s\gamma_i\sqrt{U_j^xU_k^y})c_{kji}(r\sqrt{U_j^xU_k^y} - s\gamma_i\sqrt{U_k^xU_j^y})\ . \tag{89}$$

Now that we have this last from equation (87) in a more tractible form, we re-insert it and again attempt to bound the $FT + MT$.

Bounding the First and Middle Terms

With this representation (89), our original inequality (87) can now be written as a square:

$$\begin{aligned} FT + MT &\leq -c_{jki}^2(r\sqrt{U_k^xU_j^y} - \gamma_is\sqrt{U_j^xU_k^y})^2 \\ &\quad -e_{jk}c_{jki}(r\sqrt{U_k^xU_j^y} - \gamma_is\sqrt{U_j^xU_k^y})c_{kji}(r\sqrt{U_j^xU_k^y} - \gamma_is\sqrt{U_k^xU_j^y}) \qquad (90) \\ &\leq -\frac{1}{2}(c_{jki}(r\sqrt{U_k^xU_j^y} - \gamma_is\sqrt{U_j^xU_k^y}) \qquad (91) \\ &\quad +e_{jk}c_{kji}(r\sqrt{U_j^xU_k^y} - \gamma_is\sqrt{U_k^xU_j^y}))^2\ . \end{aligned}$$

Regardless of the $\pm$, this sum is always negative. To see more clearly the importance of this result, we can use this sum $FT + MT$ to bound $\dot{B}_i$ thus

$$\begin{aligned} \dot{B}_i &\leq -\frac{1}{2}\{c_{jki}(\frac{1}{\sqrt{U_i^y}}\sqrt{U_k^xU_j^y} - \gamma_i\frac{1}{\sqrt{U_i^x}}\sqrt{U_j^xU_k^y}) \qquad (92) \\ &\quad +e_{jk}c_{kji}(\frac{1}{\sqrt{U_i^y}}\sqrt{U_j^xU_k^y} - \gamma_i\frac{1}{\sqrt{U_i^x}}\sqrt{U_k^xU_j^y})\}^2 \\ &\leq 0\ . \end{aligned}$$

From this expression it is readily apparent that $\dot{B}_i$ is always negative when $B_i = 1$. Thus it cannot reach 1, and our proof by contradiction is complete. The EDQNM approximation does not destroy the Schwarz Inequality.

Thus we conclude our proof. We have found that after the EDQNM closure is applied, a positive energy implies that the Schwarz Inequality is satisfied and vice versa. Only an extremely pathological flow would violate both of these conditions at once; therefore we may assume that the EDQNM closure of MHD equations gives a realizable result when there are no mean fields present. The important step toward this conclusion is assuming a basic symmetry on the $\mathbf{c}$ parameters and the eddy damping function.

Part VI

Hypothesis

In many physical cases where we find MHD mean fields are an important part of the dynamics. Examples are the dynamics of the sun, as we discussed in the introduction to this thesis, and the inside of planets. The proof of Pratt and Turner, discussed in the previous section, is proven to give realizable results when we constrain the initial energy so that $U_j^z = U_{-j}^z$, and have the symmetries discussed on the coupling coefficients and eddy damping constant. However, when we do not have the initial symmetry on the energies, a mean field can grow. The results are shown in Fig. 9 This mean field first violates the Schwarz Inequality and then causes some of our "energy" terms to become negative.

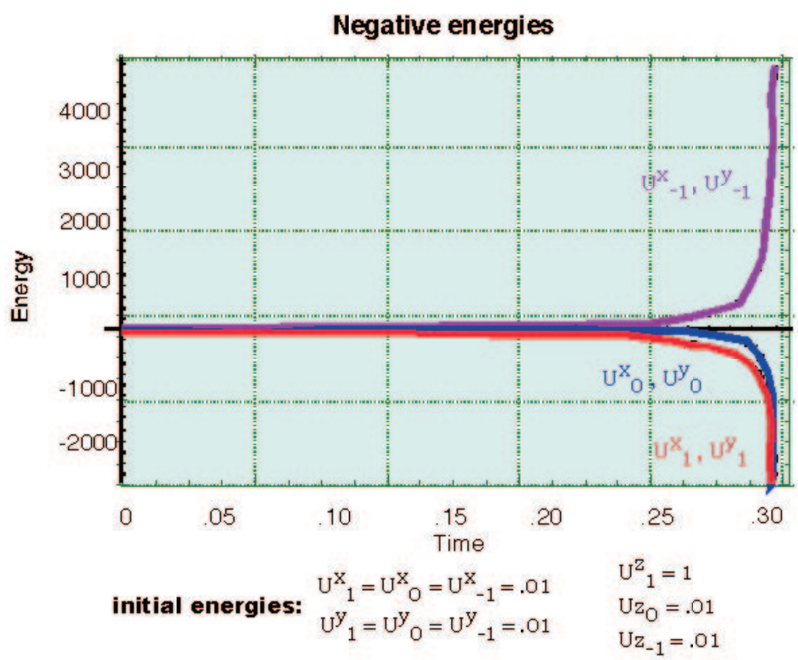

Figure 9: When a mean field grows, realizability fails.

This result is clearly a failing of the closure, which was not designed to deal with mean fields. We would like to modify the original closure scheme, so that the closure can handle the development of mean fields.

Part VII

Mean Fields and Fluctuations

The question of the role of mean fields is one of long standing. One of the first simple models for dealing with mean fields, the turbulent viscosity hypothesis, was introduced in 1877 by Boussinesq. He proposed that the Reynolds stress is proportional to the mean rate of strain, which is expressed in terms of mean velocity gradients.

$$-\langle u_i u_j \rangle + \frac{2}{3} k \delta_{ij} = \nu_T \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) , \tag{93}$$

where ν_T is the eddy viscosity ([9] 93). For sinple shear flows, this model is pretty good. The turbulent viscosity hypothesis was the beginning of the Reynolds Averaged Navier-Stokes (RANS) models ([9] 358). After over a century of research, the problem of how to best treat mean fields remains a open problem. In this section we would like to propose a way of extending the EDQNM closure to apply to mean fields in both Navier-Stokes turbulence and MHD turbulence.

To see how mean fields factor into the EDQNM theory, we want to represent the MHD equations in terms of mean and fluctuating fields and then develop equations for each of these terms based on the model. We take a Reynolds decomposition of $\mathbf{X}$ and $\mathbf{Y}$ in terms of a mean field term denoted by $\langle\rangle$ and a fluctuated (primed) term:

$$\begin{aligned} \mathbf{X} &= \langle \mathbf{X} \rangle + \mathbf{X}' \\ \mathbf{Y} &= \langle \mathbf{Y} \rangle + \mathbf{Y}' \end{aligned} \tag{94}$$

In these equations the brackets denote an average taken[11].
We will first examine the case of turbulence, where the tensor $\langle X'_j X'_k \rangle$ is diagonal. These tensors are diagonal because we have used the random phase approximation

[11] This average may be an ensemble average or special space or time averages. The requirements are that the average satisfy "Reynolds' Rules" ([23] 21):

$$\begin{aligned} \langle\langle X \rangle\rangle &= \langle X \rangle \\ \langle X + Y \rangle &= \langle X \rangle + \langle Y \rangle \\ \langle \frac{dF}{dt} \rangle &= \frac{d\langle F \rangle}{dt} . \end{aligned}$$

When we say average in this manuscript, we mean ensemble average.

(RPA) ([24] p335). For further discussion of the use of RPA in turbulent flows, the reader should refer to [25]. The turbulent kinetic energy is the sum of the U's which are diagonal elements of this tensor:

$$\langle X_j' Y_k' \rangle = U_j^z \delta_{jk} \tag{95}$$

$$\langle X_j' X_k' \rangle = U_j^x \delta_{jk} \tag{96}$$

$$\langle Y_j' Y_k' \rangle = U_j^y \delta_{jk}. \tag{97}$$

where δ_{jk} is the delta function, where $\delta_{jk} = 1$ when $j = k$ and $\delta_{jk} = 0$ otherwise.

We begin with the coupled symmetric MHD equations (12) and (13). Taking an average on the **X** equation gives:

$$\begin{aligned} \langle \dot{X}_i \rangle &= \sum_{j,k} c_{jki} \langle X_j Y_k \rangle - \alpha \langle X_i \rangle - \beta \langle Y_i \rangle \\ &= \sum_{j,k} c_{jki} (\langle X_j \rangle \langle Y_k \rangle + \langle X_j' Y_k' \rangle) - \alpha \langle X_i \rangle - \beta \langle Y_i \rangle. \end{aligned} \tag{98}$$

The $X_j' Y_k'$ feature is, in the Navier-Stokes case, referred to as the Reynolds stresses which we discussed above in the turbulent viscosity hypothesis (93).

To get a differential equation in the fluctuating variable we subtract this mean equation (99) from the original equation $\dot{X}_i' = \dot{X}_i - \langle \dot{X}_i \rangle$. From this we obtain the equation:

$$\begin{aligned} \dot{X}_i' &= \sum_{j,k} c_{jki} (X_j Y_k - \langle X_j \rangle \langle Y_k \rangle - \langle X_j' Y_k' \rangle) - \alpha X_i' - \beta Y_i' \\ &= \sum_{j,k} c_{jki} (\langle X_j \rangle Y_k' + X_j' \langle Y_k \rangle + X_j' Y_k') - c_{jji} U_j^z - \alpha X_i' - \beta Y_i' \ . \end{aligned} \tag{99}$$

In the spirit of the EDQNM closure, we are interested in the energy of the system, or $X_i'^2$. Multiplying the differential equation for X_i' by X_i' we get the desired equation:

$$\begin{aligned} \frac{1}{2} \frac{d(X_i')^2}{dt} &= \sum_{j,k} c_{jki} (\langle X_j \rangle Y_k' X_i' + X_j' \langle Y_k \rangle X_i' + X_j' Y_k' X_i') \\ &- \sum_{j,k} c_{jji} U_j^z X_i' - \alpha (X_i')^2 - \beta (X_i' Y_i') \ . \end{aligned} \tag{100}$$

Taking the average of equation (101), we have:

$$\frac{1}{2}\frac{d}{dt}U_i^x = \sum_j c_{jii}\langle X_j\rangle U_i^z + \sum_{j,k} c_{jki}\langle X_j'Y_k'X_i'\rangle - \alpha U_i^x - \beta U_i^z \; . \tag{101}$$

Notice that the right side here has a linear contribution from the mean field, $c_{jii}\langle X_j\rangle$. This equation is fully determined except for the triple correlation. We find an equation for this term using the chain rule:

$$\frac{d}{dt}\langle X_j'Y_k'X_i'\rangle = \langle X_j'Y_k'\frac{d}{dt}X_i' + X_j'X_i'\frac{d}{dt}Y_k' + Y_k'X_i'\frac{d}{dt}X_j'\rangle \tag{102}$$

Substituting equation (100) into this rule we get the equation for the third moments:

$$\begin{aligned}
\frac{d}{dt}\langle X_j'Y_k'X_i'\rangle &= c_{abj}(\langle X_a\rangle\langle Y_b'Y_k'X_i'\rangle + \langle Y_b\rangle\langle X_a'Y_k'X_i'\rangle + \langle X_a'Y_b'Y_k'X_i'\rangle) \\
&- c_{aaj}U_a^zU_i^z\delta_{ik} - \alpha\langle X_j'Y_k'X_i'\rangle - \beta\langle Y_j'Y_k'X_i'\rangle \\
&+ c_{abk}(\langle Y_a\rangle\langle X_b'X_j'X_i'\rangle + \langle X_b\rangle\langle Y_a'X_j'X_i'\rangle + \langle Y_a'X_b'X_j'X_i'\rangle) \\
&- c_{aak}U_a^zU_i^x\delta_{ij} - \alpha\langle Y_k'X_j'X_i'\rangle - \beta\langle X_k'X_j'X_i'\rangle \\
&+ c_{abi}(\langle X_a\rangle\langle Y_b'X_j'Y_k'\rangle + \langle Y_b\rangle\langle X_a'X_j'Y_k'\rangle + \langle X_a'Y_b'X_j'Y_k'\rangle) \\
&- c_{aai}U_a^zU_j^z\delta_{jk} - \alpha\langle X_i'X_j'Y_k'\rangle - \beta\langle Y_i'X_j'Y_k'\rangle.
\end{aligned} \tag{103}$$

Rearranging this so that all third order terms are on the LHS we get:

$$\begin{aligned}
(\frac{d}{dt} &+ 3\alpha)\langle X_j'Y_k'X_i'\rangle + \beta(\langle X_k'X_j'X_i'\rangle + \langle Y_j'Y_k'X_i'\rangle + \langle Y_i'X_j'Y_k'\rangle) \\
&= -c_{aaj}U_a^zU_i^z\delta_{ik} - c_{aak}U_a^zU_i^x\delta_{ij} - c_{aai}U_a^zU_j^z\delta_{jk} \\
&\quad +\langle X_a\rangle(c_{abj}\langle Y_b'Y_k'X_i'\rangle + c_{bak}\langle Y_b'X_j'X_i'\rangle + c_{abi}\langle Y_b'X_j'Y_k'\rangle) \\
&\quad +\langle Y_a\rangle(c_{baj}\langle X_b'Y_k'X_i'\rangle + c_{abk}\langle X_b'X_j'X_i'\rangle + c_{bai}\langle X_b'X_j'Y_k'\rangle) \\
&\quad +c_{abj}\langle X_a'Y_b'Y_k'X_i'\rangle + c_{abk}\langle Y_a'X_b'X_j'X_i'\rangle + c_{abi}\langle X_a'Y_b'X_j'Y_k'\rangle \; .
\end{aligned} \tag{104}$$

Some things to notice about this equation are that the RHS consists of pairs of second order terms, third order terms multiplied by a mean field term, and fourth order terms. The idea for modifying the EDQNM Closure is to apply the quasi-normal approximation to the third order equation (105) in order to express the fourth order terms in of second order ones. We must then solve the system of ordinary differential equations for the third order (105), the energy (101), and the mean fields (99) simultaneously. We will solve these equations in the **Simulation** section of this thesis.

Part VIII

Simulations

0.1 Some History of Direct Numerical Simulations

Analytical solutions to the turbulent Navier-Stokes equations do not exist. Complete description of turbulent flow, in which the flow variables velocity, density, and pressure are known as functions of space and time can only be obtained from numerical solutions. These numerical solutions come from direct numerical simulations (DNS).

The range of spatial scales in turbulent flows increases rapidly with the Reynolds number. There are two approaches to handling this: a statistical closure, or the large eddy simulation (LES). LES methods directly compute the large energy-containing scales. These simulations involve filtering out the larger wave numbers, using a filter like the Fast Fourier Transform or a wavelet. The range of scales that need to be accurately represented in a calculation is dictated by the physics. The grid determines the scales that are represented, while the accuracy of the scales is determined by the numerical method. The grid scale must be fine enough to resolve the Kolmogorov dissipation length scale $\eta = (\frac{\nu^3}{\epsilon})^{\frac{1}{4}}$. The largest scale resolved must be the largest scales of the turbulent flow; for wall-bounded flows, this is the size of the apparatus.

DNS calculations require a wide range of time scales. Sometimes time advancements are done implicitly, and sometimes explicitly depending on the situation. Implicit time advancements are attractive when discrete equations represent frequencies far higher than those required by the physics. They are commonly used with compressible wall bounded flows for the viscous terms. Explicit time advancements generally give a more reliable picture of the turbulence, and so more commonly used in these simulations. Explicit codes also have the advantage that they are easier to implement. For instance, Runge Kutta type codes are sufficient.

The range of scales in turbulent flows increases rapidly with the Reynolds number. DNS computational complexity increases as Re^6 ([9] p348-349). Currently DNS can get to Reynolds numbers of 10^2 but we need Reynolds numbers of 10^{12} for our Sun. So we need 10^{60} more computer power for a DNS of our Sun. We need alternatives to the DNS to solve important problems; one area of research that provides an alternative to the DNS is that of the statistical closures, including the EDQNM closure discussed in this thesis. A comparison between DNS results and our closure at low Reynolds numbers is an important step toward demonstrating the accuracy of our closure.

In 1972, Orszag and Patterson performed a 32^3 computation of isotropic turbulence at a Reynolds number of 35. These calculations, although simple, showed how spectral methods could be used to perform large scale computations of three dimensional turbulence. In 1981 Rogallo combined a transformation of the governing equations with an extension of the Orszag-Patterson algorithm to compute homogenous turbulence subjected to mean strain. Subsequent DNS have essentially used Rogallo's algorithm. We discuss the methods of Orszag and Patterson's code in detail, because it is very close to the one we wish to implement.

0.1.1 Orszag and Patterson's Algorithm

Orszag and Patterson simulate three dimensional homogeneous, isotropic turbulence through pseudo spectral methods. They simulate the equation for incompressible Navier-Stokes turbulence (44) ([9] 214), with an initial energy spectrum that obeys the continuity equation $k_i u_i (k, t) = 0$.

The details of their calculation are as follows. Orszag and Patterson apply periodic boundary conditions in all three spatial directions, with periodicity interval 2π in each direction. To filter out sufficiently small scales, they use a spherical cutoff, that applies the condition $u (k, t) \equiv 0$ for all $|k| \geq \bar{K}$ where $\bar{K}$ is a maximum wavenumber. Orszag notes that this truncation is the most natural for isotropic turbulence. Orszag and Patterson use Crank-Nicholson time differencing for the viscous terms and leapfrog time differencing otherwise ([20]). Although the calculation size is extremely small, the simulation that Orszag and Patterson perform gives results sufficient for comparison with the results of the TFM and DIA models.

0.2 A Pseudo-Spectral Direct Numerical Simulation

A spectral method involves advancing the Fourier modes in small time steps according to the Navier-Stokes equations in wavenumber space. For this kind of calculation you would use the incompressible equation (44). Summing over the N^3 wavenumbers requires order of N^6 operations. In order to avoid this large cost, pseudo-spectral methods evaluate the nonlinear terms in the Navier-Stokes equations differently. The velocity field is transformed into physical space; the nonlinear terms are formed and then they are transformed back into wavenumber space. Pseudo-spectral methods require order of $N^3 logN$ operations. The main numerical issues in pseudo-spectral codes are the time stepping strategy, and control of the aliasing errors ([9] 344-345). Fig. 10 shows pictorially how the pseudo-spectral method is implemented.

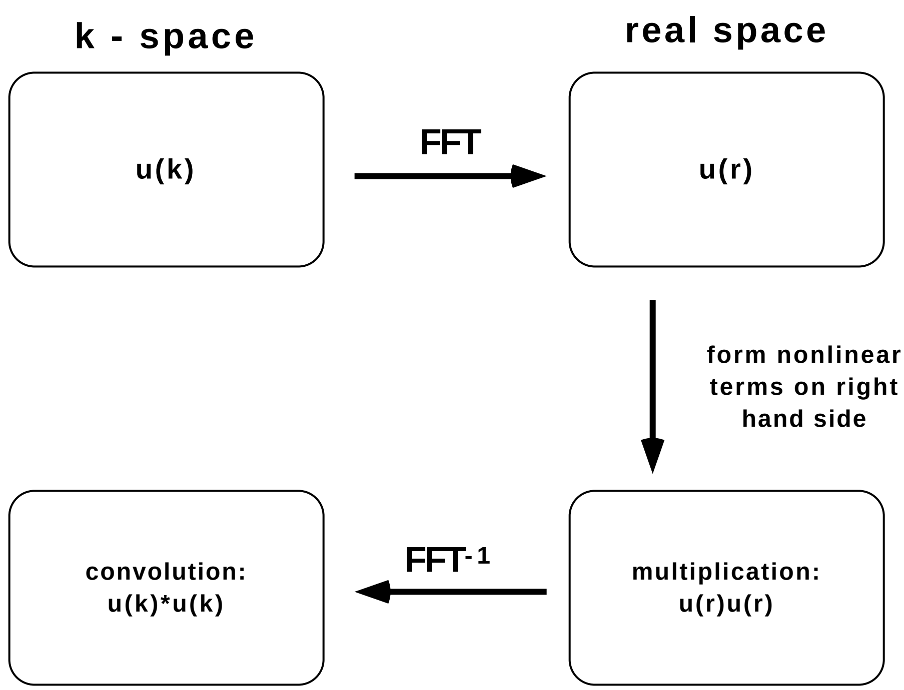

Figure 10: The Pseudo-Spectral Method

We have written a pseudo-spectral code using the equations for neutral fluids(44) and electrically conducting fluids (24). For the Fast Fourier Transform part of the psuedo-spectral code we used the subroutine *fourn* from Numerical Recipes ([26] 518).

We take advantage of the convolution theorem to form our nonlinear terms. The right hand of our equations are a convolution in wavenumber space, so using the convolution theorem:

$$g * h \Leftrightarrow G(f)H(f)\ , \tag{105}$$

we can simply multiply these terms in real space. In other words, the Fourier Transform of the convolution is just the product of the individual Fourier Transforms ([26] 492).

For our time stepping we use the modified midpoint rule:

$$y'(x_n) = \frac{y(x_{n+1}) - y(x_{n-1})}{2h} + O(h^2) \tag{106}$$

so that the formula for the next step in y is given by:

$$y_{n+1} = y_{n-1} + 2hf(x_n, y_n) \tag{107}$$

([27] 342). The benefit of our code is in its simplicity. By our choice of finite differences and simple FFT subroutines, the way that the code works is easy to see.

0.3 Pseudo-Spectral results

In this section we present and discuss the results of our pseudo-spectral code. These results were calculated for a cube of side 16. We use a step size of .001 and we take several thousand steps, until the energy levels out. All of these results were calculated from the equations we derived in Section II-1 “An Alternate Representation of the Incompressible MHD Equations,” and hence apply for incompressible turbulence with periodic boundary conditions.

Our pseudo-spectral calculations have good accuracy if the initial conditions are not too large. This is expected, since a high initial energy would indicated a high Reynolds number, and DNS become increasingly expensive and numerically complex at high Reynolds numbers.

0.3.1 Neutral Fluid results

In the graph in Fig 11, we see the evolution of the velocity squared in a turbulent flow with viscosity. As expected, the energy is dissipated by molecular viscosity until it reaches zero. We have graphed the energy in the x, y, and z directions of the cube, and because all of them are given the same initial energy, the graphs evolve identically.

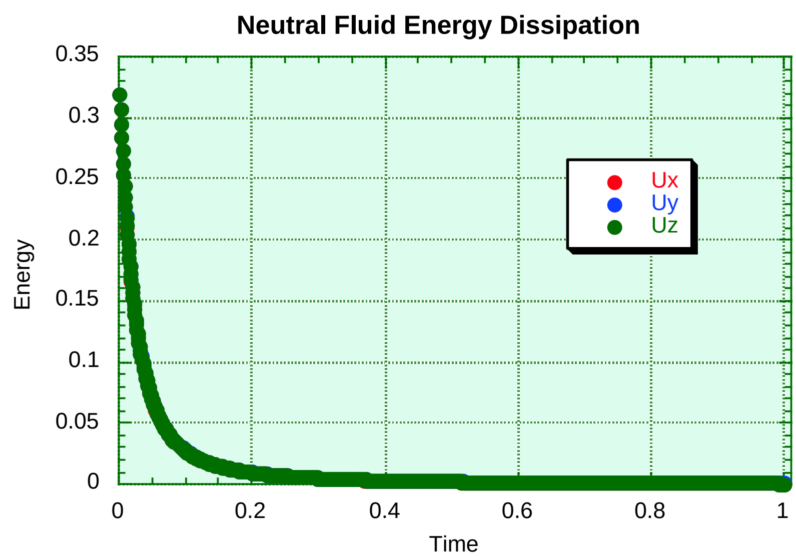

Figure 11: Dissipation of kinetic energy from a pseudo-spectral Navier-Stokes calculation

0.3.2 MHD results

In the graph in Fig. 12 we see that for MHD turbulence, the energy from the velocity is dissipating in much the same way as it did above. However, the initially zero magnetic field is growing to a non-zero term, which will later also decay. The details of this transient magnetic field and its decay are not obvious and therefore likely to be quite interesting for further work. This is what we would expect from MHD.

It would be interesting to see non-zero steady state solutions for the magnetic field but these imagined steady solutions are probably rather special, requiring special conditions. These ruminations beg questions about stability of the imagined steady solution. One consideration is the magnetic field reversals in the earth, in which the magnetic field is steady then abruptly changes. Investigation of all of these points will have to be reserved for later.

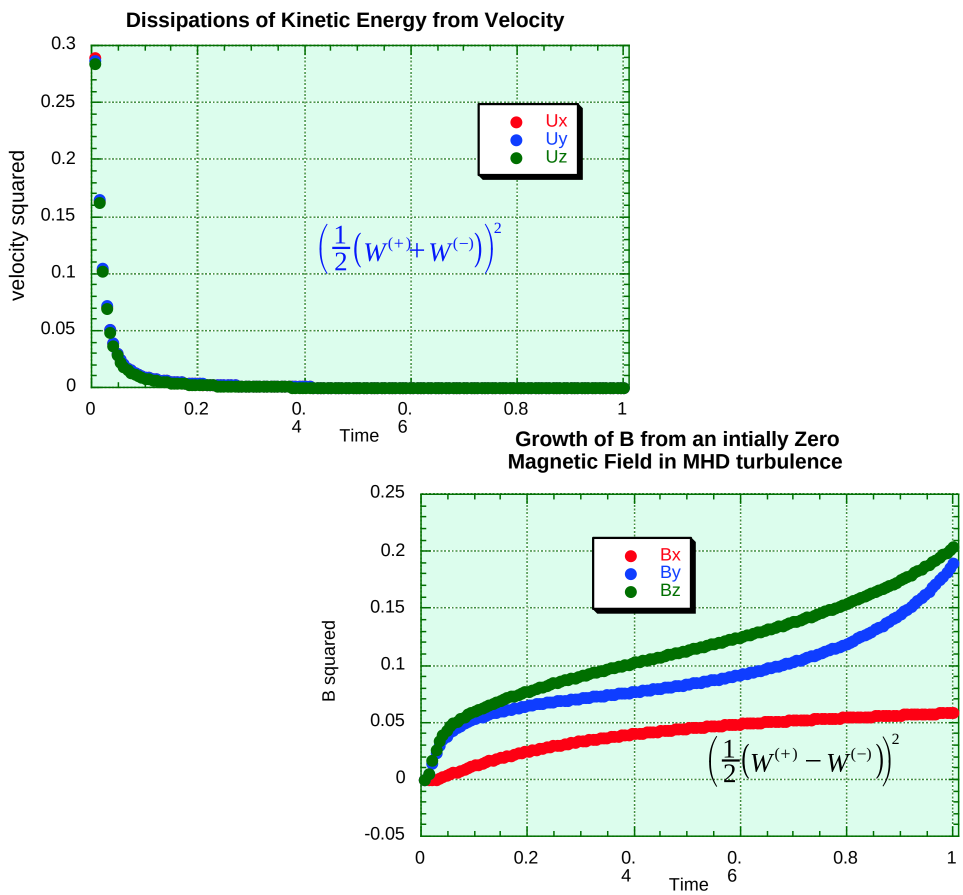

Figure 12: Dissipation of kinetic energy from a pseudo-spectral MHD calculation

In the Fig. 13, we see graphs of the energies that come from the Elsasser variables. We will compare these graphs to our closure results. Notice that the U^x and U^y energies are positive as they should be, and the U^z is bounded by the Schwarz Inequality:

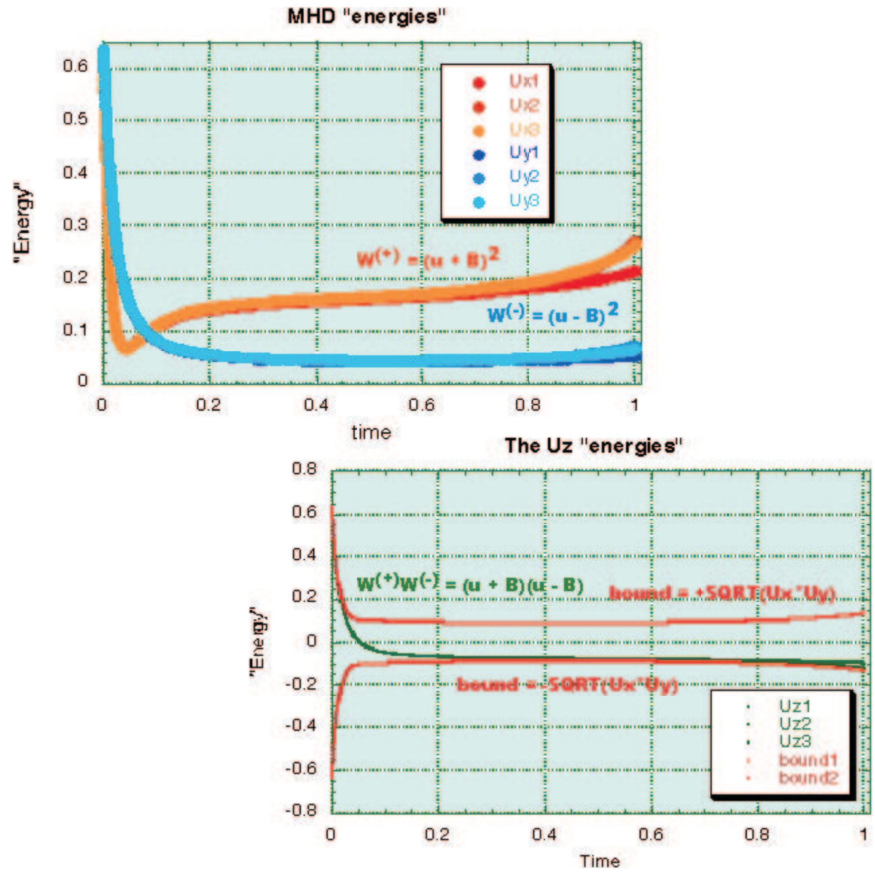

Figure 13: U^x, U^y, and U^z energies from a pseudo-spectral MHD calculation

0.4 A Closure Calculation

To calculate the dissipation of spectral energy with our closure, we use Mathematica to solve our large system of ordinary diffential equations. These differential equations need an initial mean field, as well as an initial fluctuating energy.

0.4.1 Neutral Fluids

In Fig. 14, we see the evolution of the velocity squared in fluid viscosity. In the top graph we have the dissipation of the energy when there is no mean field present. The energy in the x, y, and z direction are initially the same, and so the energy evolves identically. However in the bottom graph, we see the evolution of energy given the same initial conditions except for a non zero mean field in the x, y, and z directions. This mean field affects the speed with which energy is dissipated in the system. Because the mean field has a different speed in each direction, the energy in each direction dissipates with a different speed.

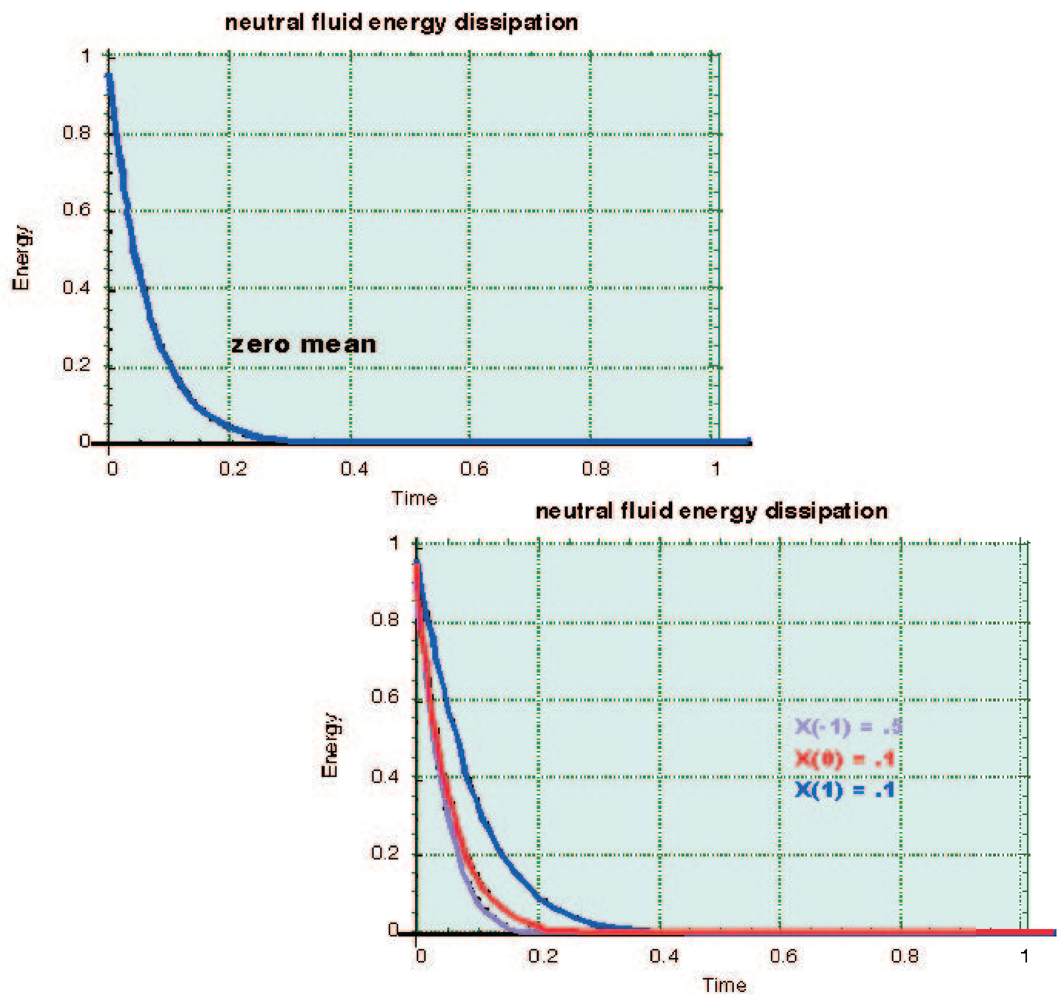

Figure 14: Dissipation of kinetic energy from a neutral fluid closure calculation

0.4.2 MHD

In Fig. 15 we again see U^x and U^y are positive. Their graphs are the same shape as those we calculated from the pseudo-spectral code, and these energies are both positive. The U^z component is within the bound, and also the same shape as the same result from the pseudo-spectral code (see Fig. 13).

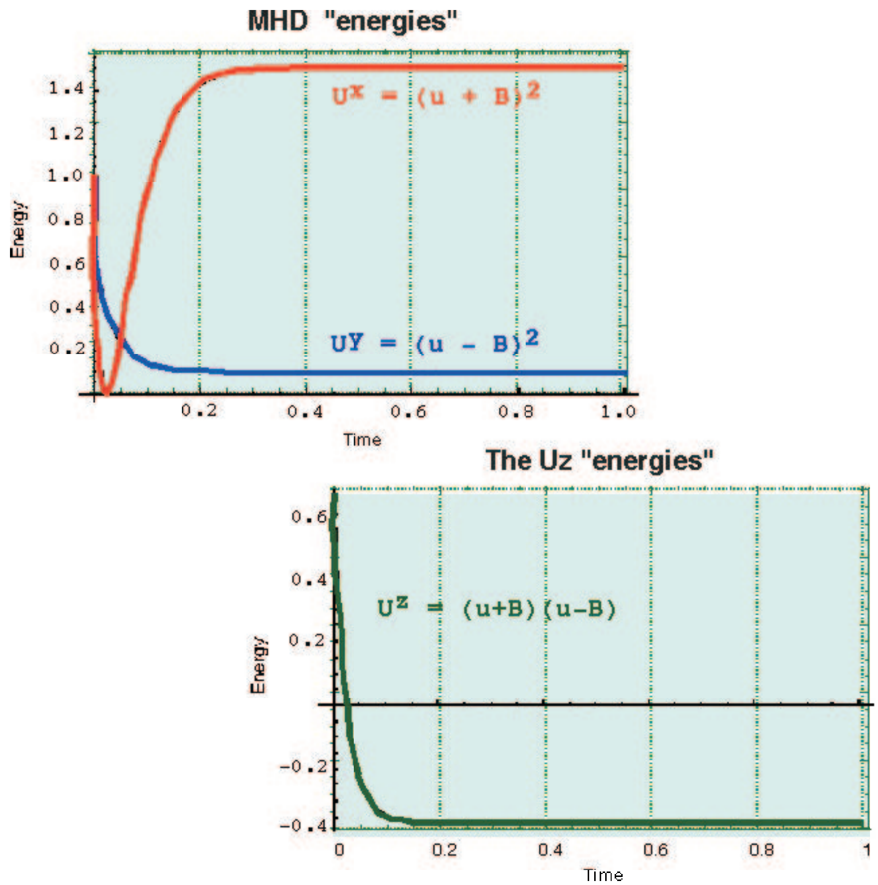

Figure 15: Dissipation of kinetic energy from a MHD closure calculation

0.5 Conclusions

The mean field theory seems to work if the mean fields are not too large. The computations we have carried out run into trouble when the mean field is too large. Currently it is not clear why a large mean field gives us problems. Maybe this is purely an issue of computational accuracy. We plan to examine this area further in the future.

The original adaption of the EDQNM closure to MHD gave us singularities when mean fields developed. Our current closure scheme can handle mean fields, and seems to give reasonable results. The comparison between our pseudo-spectral DNS and our closure calculations is good. In the future, we hope that this closure scheme could be used at high Reynolds numbers to simulate energy decay in important problems like simulations of the Sun.

Part IX

Appendix A: Some Concepts from Statistical Physics

This Appendix is designed to review some basic mathematics that are often used in Statistical Physics.

.1 *Cumulants*

Cumulants were introduced by Thorvald Nicolai Thiele in 1903. In older texts they are sometimes called the semi-invariants of Thiele.
The moment generating function, in the sense that the coefficients of its Taylor expansion in k are the moments μ, is defined by:

$$G(k) = \sum_{m=0}^{\infty} \frac{(ik)^m}{m!} \mu_m \tag{108}$$

The same function also serves to generate the cumulants κ_m, which are defined by

$$\ln G(k) = \sum_{m=1}^{\infty} \frac{(ik)^m}{m!} \kappa_m \tag{109}$$

The cumulants are combinations of the moments, e.g.

$$\begin{aligned}
\kappa_1 &= \mu_1 && (110)\\
\kappa_2 &= \mu_2 - \mu_1^2 = \sigma^2 && (111)\\
\kappa_3 &= \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3 && (112)\\
\kappa_4 &= \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4 && (113)\\
\kappa_5 &= \mu_5 - 5\mu_1\mu_4 + 20\mu_1^2\mu_3 - 10\mu_2\mu_3 && (114)\\
&+ 30\mu_1\mu_2^2 - 60\mu_1^3\mu_2 + 24\mu_1^5 && (115)
\end{aligned}$$

where μ_i is the ith moment of X. We define the moments in terms of the expected value E of the random variable X.

$$\mu_i = \sum_{k \geq 0} k^m Pr(X = k) = \langle X^m \rangle \tag{116}$$

.2 The Gaussian Distribution

The general form of the Gaussian or normal distribution is

$$P(x) = (2\pi\sigma^2)^{\frac{1}{2}} \exp -\frac{(x-\mu_1)^2}{2\sigma^2} \tag{117}$$

The use of cumulants is particularly suited for this distribution, for

$$\kappa_1 = \mu_1 \tag{118}$$

$$\kappa_2 = \sigma^2 \tag{119}$$

$$\kappa_3 = \kappa_4 = ... = 0 \tag{120}$$

(Von Kampen 24-25).

Part X

Appendix B: Quick Reference of the Physical Equations

This Appendix is designed to list some of the basic equations of fluid mechanics and magnetism. Recall that the Reynolds decomposition is given by:

$$U_j = \langle U_j \rangle + u_j \ . \tag{121}$$

And the mean substantial derivative is defined:

$$\frac{\bar{D}}{\bar{D}t} = \frac{\partial}{\partial t} + \langle U \rangle \cdot \nabla \ . \tag{122}$$

Using these definitions we can write the Reynolds Equation:

$$\frac{\bar{D} \langle U_j \rangle}{\bar{D}t} = \nu \nabla^2 \langle U_j \rangle - \frac{\partial \langle u_i u_j \rangle}{\partial x_j} - \frac{1}{\rho} \frac{\partial \langle p \rangle}{\partial x_j} \ . \tag{123}$$

([9] 85). The term $\langle u_i u_j \rangle$ is the Reynolds Stress. These equations and the Navier Stokes equations are the same except for this Reynolds Stress term. Although the Navier-Stokes equation is seen exensively in this paper, we thought it would be convenient to re-iterate it here, for comparison with the Reynolds Equations.

$$\frac{DU_j}{Dt} = \frac{\nu}{\rho} \nabla^2 U_j - \frac{1}{\rho} \frac{\partial p}{\partial x_j} \ . \tag{124}$$

Bibliography

[1] Leaf Turner. Macroscopic structures of inhomogeneous, navier stokes turbulence. *Physics Of Fluids, Volume 11, Number 8*, 1999.

[2] E.N. Parker. *Cosmical Magnetic Fields; Their Origin and Their Activity*. Oxford: Clarendon, 1979.

[3] H.K. Moffatt. *Six Lectures on General Fluid Dynamics and Two on Hydromagnetic Dynamo Theory. Fluid Dynamics 1973, Les Houches Summer School of Theoretical Physics. edited by R. Balian and J. L. Peube*. Gordon and Breach, London, 1977.

[4] Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics. Mainly Electromagnetism and Matter.*, volume 2. United States of America; California Institute of Technology, 1964.

[5] E.N. Parker. The physics of the sun and the gateway to the stars. *Physics Today*, 53(6), June 2000.

[6] S.A. Orszag. Lectures on the statistical theory of turbulence. *Fluid Dynamics 1973, Les Houches Summer School of Theoretical Physics, edited by R. Balian and J. L. Peube.*, pages 235–374, 1977.

[7] D.C. Leslie. *Developments in the Theory of Turbulence.* Oxford; Clarendon Press, 1973.

[8] Uriel Frisch. *Turbulence; The legacy of A.N. Kolmogorov.* New York; Cambridge University Press, 1995.

[9] Stephen B. Pope. *Turbulent Flows*. Cambridge University Press, 2000.

[10] Marcel Lesieur. *Turbulence in Fluids: Stochastic and Numerical Modelling.* Boston: Martinus Nijhoff Publishers, 1997.

[11] G.K. Batchelor. *The theory of homogeneous turbulence. (2nd edn.).* Cambridge University Press, 1971.

[12] Jane Pratt and Leaf Turner. Realizability in an edqnm closure for mhd turbulence possessing cross-helicity. *52nd Annual Meeting of the APS Division of Fluid Dynamics.*, November 1999.

[13] J.A. Jacobs., editor. *Geomagnetism. vol 2.* London: Academic Press, 1987.

[14] Stephen Childress and Andrew D. Gilbert. *Stretch, Twist, Fold: The Fast Dynamo.* Berlin: Springer-Verlag, 1995.

[15] L.D. Landau and E.M. Lifshitz. *Electrodynamics of Continuous Media. Volume 8 of the Course of Theoretical Physics.* Oxford: Pergamon Press, 1960.

[16] H.K. Moffatt. *Magnetic Field Generation in Electrically Conducting Fluids.* Cambridge University Press, Cambridge, 1978.

[17] J. R. Herring. Self-consistent-field approach to turbulence theory. *Phys. Fluids*, 3:2219, 1965.

[18] Leaf Turner. Helicity decomposition of evolution of incompressible turbulence. i. arbitrary homogeneous, anisotropic, helical case. *LA-UR-96-618: Los Alamos National Laboratory*, 1996.

[19] Robert Zwanzig. Nonlinear dynamics of collective modes. *Statistical Mechanics: New Concepts, New Problems, New Applications. ed. Rice, Stuart A. and Karl F. Freed and John C. Light*, pages 241–256, 1972.

[20] S.A. Orszag and G.S. Patterson Jr. Numerical simulations of turbulence. *Statistical Models and Turbulence.*, 12:127–147, 1972.

[21] Jane Pratt and Leaf Turner. Realizability in an edqnm closure for mhd turbulence possessing cross-helicity. *42nd Annual Meeting of the Divison of Plasma Physics of the American Physical Society*, October 2000.

[22] Leaf Turner and Jane Pratt. Realizability in a closure for mhd turbulence. *2001 International Sherwood Fusion Theory Conference. Santa Fe, New Mexico*, April 2-4 2001.

[23] A.M. Soward. *Stellar and Planetary Magnetism. The Fluid Mechanics of Astrophysics and Geophysics.*, volume 2. Gordon and Breach Science Publishers, New York, 1983.

[24] Jean Mathieu and Julian Scott. *An Introduction to Turbulent Flow.* Cambridge; Cambridge University Press, 2000.

[25] Leaf Turner. Using helicity to characterize homogeneous and inhomogeneous turbulent dynamics. *Journal of Fluid Mechanics, Volume 408, pp. 205-238*, 2000.

[26] William H. Press and Saul A. Teukolsky and William T. Vetterling and Brian P. Flannery. *Numerical Recipes in Fortran 77; The Art of Scientific Computing.* Cambridge University Press, second edition, 1992.

[27] Germund Dahlquist and Ake Bjrck. *Numerical Methods. translated by Ned Anderson*. Prentice-Hall, Inc., 1974.

[28] Bruce J. Berne. *Statistical Mechanics. Part B: Time-Dependent Processes.* New York: Plenum Press, 1977.

[29] Bengt Fornberg. *A Practical Guide to Pseudospectral Methods.* New York; Cambridge University Press, 1996.

[30] Ronald L. Gram and Donald E. Knuth and Oren Patashnik. *Concrete Mathematics; A Foundation for Computer Science.* Reading, Massachusetts; Addison-Wesley Publishing Company, 1989.

[31] J. R. Herring and J. C. McWilliams. Comparison of dns of 2d turbulence with 2-point closure: the effects of intermittency. *Journal of Fluid Mechanics*, 153:229–242, 1985.

[32] Philip Holmes, John L. Lumley, and Gal Berkooz. *Turbulence, Coherent Structures, Dynamical Systems and Symmetry.* Cambridge: Cambridge University Press, 1996.

[33] Maurice Holt. *Numerical Methods in Fluid Dynamics.* Berlin: Springer-Verlag, 1984.

[34] J.A. Jacobs., editor. *Geomagnetism. vol 1.* London: Academic Press, 1987.

[35] F. Krause and K.H. Rädler. *Mean-Field Magnetohydrodynamics and Dynamo Theory.* Oxford: Pergamon Press, 1980.

[36] M. Lesieur and O. Metais. Spectral large-eddy simulations of isotropic and stably-stratified turbulence. *Annual Review of Fluid Mechanics*, 28:45–82, 1996.

[37] A. S. Monin and A. M. Yaglom. *Statistical Fluid Mechanics: Mechanics of Turbulence. ed. J. L. Lumley*, volume 1. The MIT Press, Cambridge, 1971.

[38] A. S. Monin and A. M. Yaglom. *Statistical Fluid Mechanics: Mechanics of Turbulence. ed. J. L. Lumley*, volume 2. The MIT Press, Cambridge, 1975.

[39] Parviz Moin and Krishnan Mahesh. Direct numerical simulation: A tool in turbulence research. *Annual Review of Fluid Mechanics.*, 30:539 – 578, 1998.

[40] A. Pouquet, U. Frisch, and J. Leorat. Journal of fluid mechanics. *Journal of Fluid Mechanics*, 77:321–354, 1976.

[41] R.S. Rogallo and P. Moin. Numerical simulation of turbulent flows. *Annu. Rev. Fluid Mech.*, 16:99–137, 1984.

[42] Thomas D. Taylor. *Computational Methods for Fluid Flow.* Springer-Verlag, New York, 1983.

[43] H. Tennekes and J.L. Lumley. *A First Course in Turbulence.* The MIT Press, Cambridge, 1994.

[44] Leaf Turner. Helicity decomposition of evolution of incompressible turbulence. ii. inhomogeneous case - free slip channel. *LA-UR-96-3257: Los Alamos National Laboratory*, 1996.

[45] N.G. Van Kampen. *Stochastic Processes in Physics and Chemistry*. New York; North-Holland Publishing Company, 1981.