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Combinatorial Explanations of Known Harmonic Identities

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A combinatorial investigation of harmonic numbers

Final Report

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Abstract

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We seek to discover combinatorial explanations of known identities involving harmonic numbers. Harmonic numbers do not readily lend themselves to combinatorial interpretation, since they are sums of fractions, and combinatorial arguments involve counting whole objects. It turns out that we can transform these harmonic identities into new identities involving Stirling numbers, which are much more apt to combinatorial interpretation. We have proved four of these identities, the first two being special cases of the third.

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Chapter 1

Introduction

1.1 Harmonic Numbers

Harmonic numbers are defined to be partial sums of the harmonic series. Let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k} \qquad \text{for} \quad n \geq 1.$$

Since the harmonic series diverges, H_n gets arbitrarily large for big enough n. However, it diverges very slowly, with $H_{1000000} \approx 14.39$.

An example of how harmonic numbers appear in real life is if you try to stack up playing cards to overhand the edge of a table by as far as possible. If the cards are 2 inches long, then with n cards, the maximum amount any card can hang off the edge of the table is H_n.[3] So, for example, with 4 cards, the top card could extend past the table by just over 2 inches, since $H_4 = \frac{25}{12}$.

1.2 Stirling Numbers

Stirling numbers (of the first kind) arise in many situations. The Stirling number $\left[{n \atop k}\right]$ is the number of permutations of n elements with exactly k cycles. Equivalently they are the number of ways that n distinct people can sit at k identical circular tables in a room, with no empty tables. For example, $\left[{3 \atop 2}\right] = 3$ enumerates the permutations $(12)(3)$, $(13)(2)$, and $(1)(23)$. $\left[{n \atop k}\right]$ can be computed recursively. From their definition, we see for $n \geq 1$,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)! \tag{1.1}$$

Also for $k \geq 2$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{1.2}$$

Equation (1.2) can be explained combinatorially. On the left we are directly counting the number of permutations of n elements with exactly k cycles. On the right we count the same thing while conditioning on what happens to the element n. If n is to be in a cycle by itself, then the remaining $n-1$ elements can be arranged in $k-1$ cycles in $\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]$ possible ways. If n is not to be in a cycle by itself, then we first take the elements $[1 \ldots n-1]$ and arrange them into k cycles (there are $\left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]$ ways to do this.) Then element n can go to the right of any element, giving us $(n-1)\left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]$ total permutations where n is not in its own cycle. We note that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is defined to be 0 when $k > n$ or $n < 0$, and $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = 1$.

To represent permutations we use cycle notation. For example, in S_9 (the group of all permutations of 9 elements) the permutation $\pi = (143)(297)(5)(68)$ has 4 cycles and satisfies $\pi(1) = 4$, $\pi(4) = 3$, $\pi(3) = 1$, $\pi(2) = 9$, $\pi(9) = 7$, $\pi(7) = 2$, $\pi(5) = 5$, $\pi(6) = 8$, and $\pi(8) = 6$. Notice that we explicitly list the fixed point 5 in its own cycle. Notice that the cycle (143) is equivalent to (431) and (314). We shall adopt the convention of always writing our cycles with the smallest element first, and listing the cycles in increasing order according to the first element.

Definitions: Let T_n denote the subset of S_n consisting of all of the permutations in S_n with exactly 2 cycles. Thus $|T_n| = \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]$. So for example T_9 would include $\pi_1 = (18574)(2963)$, but not $\pi_2 = (195)(2487)(36)$ nor $\pi_3 = (123)(456)(7)(8)$. By our convention, we always list the cycle containing 1 first, and call this the *left cycle*. The remaining cycle is called the *right cycle*. Thus in T_n all permutations are of the form $(\alpha_1, \alpha_2, \ldots \alpha_j)(\alpha_{j+1}, \ldots \alpha_n)$ where $1 \leq j \leq n-1$, $\alpha_1 = 1$, and $\alpha_{j+1} =$

$\min\{\alpha_{j+1}, \dots \alpha_n\}$. We say the left cycle has length j and has *endpoint* α_j. In our earlier example, π_1 has left cycle (18574) with endpoint 4, and has right cycle (2963).

Alternate Interpretation: The Stirling numbers can also be defined as coefficients in the expansion of the factorial function [1]:

$$(x)_{n+1} = x(x+1)(x+2)\cdots(x+n) = \sum_{m=1}^{n+1} \begin{bmatrix} n+1 \\ m \end{bmatrix} x^m. \tag{1.3}$$

We will primarily think of Stirling numbers in terms of permutations, but it is useful to keep this more analytic definition in mind. To see that these definitions are equivalent, one can easily show that when $\left[{n \atop k}\right]$ is defined this way, it also satisfies (1.1), (1.2), and the same initial conditions. However, in the spirit of this thesis, we can also see this combinatorially.

1.3 *The relationship between* H_n *and* $\left[{n \atop 2}\right]$

At first it would seem that no combinatorial interpretation of harmonic numbers should exist since it can be shown [3] that for $n > 1$, H_n is not an integer. Nonetheless, since H_n can always be expressed as a rational number, the numerator and denominator might have combinatorial significance. A surprising connection between harmonic numbers and Stirling numbers can be found by examining the following identity, proved in [1], which will be crucial to the rest of this thesis.

Theorem 1.3.1 *For* $n \geq 1$,

$$H_n = \frac{1}{n!} \begin{bmatrix} n+1 \\ 2 \end{bmatrix}$$

Proof 1:(Induction)

We can always write H_n as a rational number with denominator $n!$. Let a_n be the numerator of the (typically, non-reduced) fraction so that $H_n = \frac{a_n}{n!}$. Note that $a_1 = 1$.

Since

$$H_{n+1} = H_n + \frac{1}{n+1},$$

it follows that

$$\begin{aligned}\frac{a_{n+1}}{(n+1)!} &= \frac{a_n}{n!} + \frac{1}{n+1} \\ &= \frac{(n+1)a_n + n!}{(n+1)!}\end{aligned}$$

Consequently, for $n \geq 1$,

$$a_{n+1} = (n+1)a_n + n! \tag{1.4}$$

When $n = 1$,

$$H_1 = 1 = \frac{\left[{2 \atop 2}\right]}{1!}.$$

Inductively, we assume that $a_n = \left[{n+1 \atop 2}\right]$ and try to show that $a_{n+1} = \left[{n+2 \atop 2}\right]$. From Equation (1.2), we know that

$$\begin{bmatrix} n+2 \\ 2 \end{bmatrix} = (n+1)\begin{bmatrix} n+1 \\ 2 \end{bmatrix} + \begin{bmatrix} n+1 \\ 1 \end{bmatrix}$$

and from Equation (1.1) and our induction hypothesis,

$$= (n+1)a_n + n!$$

and finally from Equation (1.4),

$$= a_{n+1}.$$

So $a_n = \left[{n+1 \atop 2}\right]$, and therefore,

$$H_n = \frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix}$$

□

Proof 2: We interpret $\left[{n \atop 2}\right]$ using our alternate definition of Stirling numbers from Equation (1.3). We see that $\left[{n+1 \atop 2}\right]$ is the coefficient of the x^2 term of $x(x+1)(x+$

$2) \cdots (x+n)$. But the coefficient of the x^2 term will simply be the sum all possible products of 1 through n with exactly one term missing in each product. In other words,

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix} = \frac{n!}{1} + \frac{n!}{2} + \ldots + \frac{n!}{n} = n!(\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}) = n!H_n.$$

□

Theorem 1.3.1 can also be proved combinatorially. To do this, we further explore the structure of T_{n+1}, the set of permutations of $n+1$ elements with exactly 2 cycles.

Lemma 1.3.2 *For $2 \leq r \leq n+1$, the number of permutations in T_{n+1} with r as the smallest element in the right cycle is*

$$\frac{n!}{r-1}.$$

The permutations we are counting are all of the form $(1, \ldots)(r, \ldots)$ where elements 1 through $r-1$ all appear in the left cycle, and elements $r+1$ through $n+1$ can go in either cycle. Proofs 1 and 2 give two ways to count this.

Proof 1: Begin with an empty array with $n+1$ positions, numbered 1 through $n+1$, and put element 1 in the first position. Now we will choose the $r-1$ spots where elements 2 through r will go. There are n remaining spots available, so we have $\binom{n}{r-1}$ ways to do this. Once those spots are chosen, put element r in the rightmost chosen spot. This spot indicates the beginning of the right cycle. Now there are $(r-2)!$ ways to arrange the remaining elements 2 through $r-1$ in the other chosen spots. Note that this guarantees that every element smaller than r is in the left cycle. Now we have $n-r+1$ elements remaining to place (elements $r+1$ through $n+1$) in the remaining $n-r+1$ spots, and there are $(n-r+1)!$ possible ways to do this. Now we have a permutation of the form $(1\ldots)(r\ldots)$, and every element less than r must be in the left cycle. Therefore r is the smallest element in the right cycle, as desired. The total number of permutations like this is

$$\binom{n}{r-1}(r-2)!(n-r+1)! = \frac{n!}{(r-1)!(n-r+1)!}(r-2)!(n-r+1)!$$
$$= \frac{n!}{r-1}.$$

□

Proof 2 of Lemma 1.3.2: Begin with $(1)(r)$. Insert the elements 2 through $r-1$ into the left cycle, but keeping 1 as the first element. There are $(r-2)!$ ways to do this. Now we insert elements $r+1$ through $n+1$, one at a time, in such a way that 1 and r remain first in their cycles. The element $r+1$ can go to the right of any of the r elements. Next, $r+2$ can go to the right of any of $r+1$ elements. Continuing in this way, the number of ways to insert elements $r+1$ through $n+1$ is $r(r+1)(r+2)\cdots n = \frac{n!}{(r-1)!}$. We have thus created a permutation in T_{n+1} with r as the smallest element in the right cycle, and there are

$$(r-2)!\frac{n!}{(r-1)!} = \frac{n!}{r-1}$$

such permutations. □

Now we can easily use Lemma 1.3.2 to show that $\left[{n+1 \atop 2}\right] = n!H_n$.

Proof 3 of Theorem 1.3.1:(Combinatorial) Since every permutation in T_n must have some smallest number r in the right cycle, and r can range from 2 to $n+1$, we get

$$\left[{n+1 \atop 2}\right] = \sum_{r=2}^{n+1} \frac{n!}{r-1} = n! \sum_{k=1}^{n} \frac{1}{k} = n!H_n.$$

□

Chapter 2

Identity 1

Now with a grasp of harmonic and Stirling numbers, we seek combinatorial explanations of other harmonic identities.

Identity 1

$$\sum_{j=1}^{n-1} H_j = nH_n - n.$$

First we show how this identity can easily be proven algebraically.

Algebraic Proof: The sum on the left is just

$$\begin{array}{lllll}
1 & & & & \\
+1 & +\frac{1}{2} & & & \\
+1 & +\frac{1}{2} & +\frac{1}{3} & & \\
 & \vdots & & \ddots & \\
+1 & +\frac{1}{2} & +\frac{1}{3} & \dots & +\frac{1}{n-1}
\end{array}$$

Adding column by column gives us

$$\sum_{j=1}^{n-1} H_j = \frac{n-1}{1} + \frac{n-2}{2} + \frac{n-3}{3} + \cdots + \frac{1}{n-1}$$

now we add and subtract n in the form $\frac{1}{1} + \frac{2}{2} + \cdots + \frac{n-1}{n-1} + \frac{n}{n} - n$,

$$\begin{aligned}
&= \frac{n}{1} + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n-1} + \frac{n}{n} - n \\
&= nH_n - n
\end{aligned}$$

$\square$

To prove Identity 1 combinatorially, we introduce the notion of inclusive and exclusive permutations.

2.1 Inclusive and Exclusive permutations

Definition: A permutation $\pi \in T_n$ is *inclusive* if elements $1, 2, \ldots, k$ are all in the left cycle of π, where k is the endpoint of π. Equivalently, using our notation of starting each cycle with the least element $\pi = (1, \ldots k)(r, \ldots)$ is inclusive iff $k < r$. Otherwise, π is called *exclusive*. Note that if π has endpoint $k = 1$ or $k = 2$ then it must be inclusive.

For example, in T_9, $\pi_1 = (136274)(598)$ and $\pi_2 = (1)(25487639)$ are both inclusive, whereas $\pi_3 = (18426)(3759)$ is exclusive since the left cycle does not contain all numbers from 1 to 6.

Lemma 2.1.1 *The number of exclusive permutations in T_n with endpoint k is*

$$\frac{(n-2)!}{(k-2)!}\begin{bmatrix} k-1 \\ 2 \end{bmatrix}.$$

Proof. We begin by placing the elements 1 through $k-1$ into exactly two cycles. There are $\left[{k-1 \atop 2}\right]$ ways to do this. Next, insert the element k to be the endpoint of the left cycle. Next we insert elements $k+1$ through n, one at a time as follows. Insert $k+1$ to the right of any element except k. There are $k-1$ ways to do this. Next insert $k+2$, to the right of any element but k. There are k ways to do this. Continue inserting in this manner until all of the elements from $k+1$ to n have been inserted. This insertion process can be accomplished in $(k-1)(k)\ldots(n-2) = \frac{(n-2)!}{(k-2)!}$ ways, resulting in an exclusive permutation with endpoint k. Hence the number of exclusive permutations in T_n with endpoint k is

$$\frac{(n-2)!}{(k-2)!}\begin{bmatrix} k-1 \\ 2 \end{bmatrix}.$$

$\square$

We shall present several proofs of the following lemma.

Lemma 2.1.2 *The number of inclusive permutations in T_n is $(n-1)!$.*

Proof 1. (Algebraic) We first count permutations with endpoint 1, i.e., where element 1 is the only element in the left cycle. There are $(n-2)!$ permutations of this type, since the remaining $n-1$ elements can be ordered in the right cycle $(n-2)!$ different ways.

Now we count the number of inclusive permutations where 1 is not alone. Such a permutation π will have an endpoint k, where $2 \leq k \leq n-1$. Let r be the smallest element in the right cycle. Since π is inclusive, $r \geq k+1 \geq 3$. So π is of the form $(1 \ldots k)(r \ldots)$. We claim that the number of permutations with this combination of k and r is

$$(r-3)!(r-1)(r)(r+1)\cdots(n-2) = \frac{(n-2)!}{r-2}.$$

To see this, we first insert the numbers 1 through $r-1$ into the left cycle with the restriction that 1 is first and k is last. There are $(r-3)!$ ways to do this. Then put r into the right cycle (1 way), then $r+1$ to the right of any element except k ($r-1$ ways), then $r+2$ to the right of any element except k (r ways), and so on through the nth element ($n-2$ ways).

Thus for $k > 1$, the total number of inclusive permutations π with endpoint k is

$$\begin{aligned}
\sum_{r=k+1}^{n} \frac{(n-2)!}{r-2} &= (n-2)! \sum_{r=k+1}^{n} \frac{1}{r-2} \\
&= (n-2)! \sum_{j=k-1}^{n-2} \frac{1}{j} \\
&= (n-2)!(H_{n-2} - H_{k-2}).
\end{aligned}$$

When we include the case where 1 is alone, the total number of inclusive permutations is

$$(n-2)! + \sum_{k=2}^{n-1}(n-2)!(H_{n-2} - H_{k-2}) = (n-2)!\left(1 + H_{n-2}\sum_{k=2}^{n-1} 1 - \sum_{k=2}^{n-1} H_{k-2}\right).$$

Using Identity 1 for the second summation, this simplifies to

$$
\begin{aligned}
(n-2)!\Big(1 + H_{n-2}(n-2) - (n-2)(H_{n-2}-1)\Big) \qquad (1.2) \\
&= (n-2)!(n-1) \\
&= (n-1)!
\end{aligned}
$$

as desired. Of course this proof is slightly unsatisfying, since it relies on Identity 1, which we are trying to prove combinatorially. □

Proof 2. (Semi-Combinatorial) We will again separate the case where 1 is alone, knowing that there are $(n-2)!$ of these permutations. It remains to show that there are $(n-2)(n-2)!$ remaining inclusive permutations. We shall now construct all of the possible inclusive permutations with exactly m elements in the right cycle, where $1 \leq m \leq n-2$. First, we put the elements 1 and 2 into the left cycle. There is only one way to do this. Next, we arrange the elements 3 through $m+2$ in the right cycle. There are $(m-1)!$ ways to do this. We note that our final permutation will contain m elements, though not necessarily 3 through $m+2$, but in the same relative order as these elements.

Now we begin a special insertion process to introduce the remaining elements $m+3$ through n into a cycle. We begin by choosing a number t so that $2 \leq t \leq m+3$. There are $m+2$ ways to do this. We then increase by 1 every element in the permutation that is greater than or equal to t. Then we insert the number t to the right of 1. See Figure 2.1 for an example with $m=4$, and $t=5$.

This gives us an inclusive permutation of the elements 1 through $m+3$. If $t=2$, the left cycle now has endpoint 3; otherwise the endpoint is still 2. Notice that the number of elements in the right cycle has remained constant at m. So each choice of t takes us from an inclusive permutation in T_{m+2} with m elements in the right cycle

$$\begin{array}{c} (12)\,\underbrace{(3645)}_{m \text{ locations}} \\ t=5 \;\Big\downarrow \\ (153)\,\underbrace{(3746)}_{m \text{ locations}} \end{array}$$

Figure 2.1: Building an inclusive permutation.

to an inclusive permutation in T_{m+3} with m elements in the right cycle. Notice that this process is easily reversed; for any inclusive permutation in T_{m+3}, the value of t is the number to the right of 1, which leads to a unique inclusive permutation in T_{m+2}. If we now repeat this process, this time with $m+3$ choices for t, we generate an inclusive permutation in T_{m+4}. We continue in this fashion until we have an inclusive permutation in T_n.

Thus for $1 \leq m \leq n-2$ the number of inclusive permutations of T_n with exactly m elements in the right cycle is

$$(m-1)!(m+2)(m+3)\dots(n-1) = \frac{(n-1)!}{m(m+1)}. \tag{2.1}$$

Hence the total number of inclusive permutations of T_n with at least 2 elements in the left cycle (i.e., at most $n-2$ elements in the right cycle) is

$$\begin{aligned} \sum_{m=1}^{n-2} \frac{(n-1)!}{m(m+1)} &= (n-1)!\big(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-2)(n-1)}\big) \\ &= (n-1)!\big((1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + \cdots + (\frac{1}{n-2} - \frac{1}{n-1})\big) \\ &= (n-1)!\big(1 - \frac{1}{n-1}\big) \\ &= (n-1)!\big(\frac{n-2}{n-1}\big) \\ &= (n-2)(n-2)! \end{aligned}$$

Adding this to the $(n-2)!$ permutations where 1 is alone, we again obtain a total of $(n-1)!$ inclusive permutations in T_n. $\square$

Proof 3. (Induction) Let A_n be the set of inclusive permutations in T_n. We want to show that $|A_n| = (n-1)!$. T_2 has only one inclusive permutation, $\pi = (1)(2)$, $|A_2| = 1 = (2-1)!$. Now we inductively assume $|A_n| = (n-1)!$ and try to show that $|A_{n+1}| = n!$. We will do this using a 1 to n mapping from A_n to A_{n+1} defined as follows: Let $\pi \in A_n$. If $\pi(1) \neq 1$, i.e. π does not have element 1 alone in the left cycle, then use the insertion method described in Proof 2, giving us n choices for how to get to A_{n+1}. That is, we choose a number t ($2 \leq t \leq n+1$) to go to the right of 1 and increase by 1 all other elements that were at least t. If $\pi(1) = 1$, i.e. 1 is alone in the left cycle, then we either insert 2 into the left cycle and increase every element in the right cycle by 1, or we insert $n+1$ to the right of any element in the right cycle. In either case, this gives us n choices to get to A_{n+1}. So every $\pi \in A_n$ maps to n elements in A_{n+1}, therefore

$$|A_{n+1}| = n|A_n| = n(n-1)! = n!$$

as desired. $\square$

Proof 4. (Bijective) As before, let A_n be the set of inclusive permutations in T_n. We want to show that $|A_n| = (n-1)!$. Let B_n be the set of integer sequences $\beta_3, \beta_4, \ldots, \beta_n$ such that $1 \leq \beta_i \leq i-1$ for $3 \leq i \leq n$. Clearly $|B_n| = (n-1)!$. If we can find a one-to-one mapping from B_n onto A_n then $|A_n| = |B_n| = (n-1)!$. We define the mapping $f : B_n \to A_n$ as follows:

For $n \geq 3$, let $\beta \in B_n$. Thus $\beta = \beta_3\beta_4\beta_5 \ldots \beta_n$ where $1 \leq \beta_i \leq i-1$ for $3 \leq i \leq n$. We shall interpret β as instructions on how to build some $\pi_n \in A_n$ starting with the only permutation in A_2, which is $\pi_2 = (1)(2)$. We will use two rules to build a permutation in A_n. We apply Rule 1 until $\beta_t = 1$, for some t. If $\beta_i \neq 1$ for all i, then let $t = n + 1$. Then we apply Rule 2 until we are done. Begin with $i = 3$, and stop when $i = n$. We illustrate these rules with $\beta = 2243141$, that is the sequence $\beta_3 = 2, \beta_4 = 2, \beta_5 = 4, \beta_6 = 3, \beta_7 = 1, \beta_8 = 4, \beta_9 = 1$, and $t = 7$.

Rule 1. For $3 \leq i < t$, we create π_i by inserting the number i to the right of β_i in π_{i-1}. In our example, beginning with $\pi_2 = (1)(2)$, this rule generates $\pi_3 = (1)(23), \pi_4 = (1)(243), \pi_5 = (1)(2453), \pi_6 = (1)(24536)$. Notice that the permutations generated by Rule 1 are inclusive since only 1 appears in the left cycle.

Rule 2 will increase the size of the left cycle. Essentially, it "makes room" for the number $\beta_i + 1$ to be inserted to the right of the number 1.

Rule 2. For $t \leq i \leq n$, we create π_i by first increasing by one all numbers in π_{i-1} that are greater than β_i. Then we insert $\beta_i + 1$ to the right of 1. In our example, Rule 2 generates inclusive permutations. $\pi_7 = (12)(35647), \pi_8 = (152)(36748), \pi_9 = (1263)(47859)$. Notice that Rule 2 always generates an inclusive permutation and that given an inclusive permutation $\pi \in T_n$ there is a unique sequence $\beta_3, \ldots, \beta_n$ that generates it. Specifically, if $\pi = \pi_n$ has l elements in its left cycle, reverse Rule 2 exactly $l - 1$ times, according to the element to the right of 1. After that we reverse Rule 1 according to the location of the largest element.

We have defined a bijection between A_n and B_n. Therefore, $|A_n| = |B_n| = (n-1)!$, and we are done. $\square$

Proof 5. (Combinatorial) We directly count the number of inclusive permutations, conditioning on the smallest element in the right cycle. We count the number of inclusive permutations in T_n where the smallest number in the right cycle is r, where $2 \leq r \leq n$. If $r = 2$, then the left cycle must only contain 1, and there are $(n-2)!$ such permutations. Now suppose $r \geq 3$. First we arrange elements 1 to $r - 1$ in the left cycle, with the requirement that 1 be listed first. There are $(r-2)!$ ways to do this. Let k denote the endpoint of the left cycle. Now place r in the right cycle. Note that $2 \leq k \leq r - 1$, and the permutation thus far created is inclusive.

At this point, we have created the inclusive permutation $(1, \ldots k)(r)$ where the left cycle contains the numbers 1 through $r-1$ in some order. Next insert element $r+1$ to the right of any element except k, with $r-1$ choices. Then insert element $r+2$ to the right of any element except k, with r choices, and so on until element n has been inserted with $n-2$ choices. After each insertion, the endpoint is still k, and the permutation remains inclusive. Thus for any $r \geq 3$ the total number of inclusive permutations of T_n with r being the smallest element of the right cycle is

$$(r-2)!(r-1)(r)(r+1)\ldots(n-2) = (n-2)!$$

as was the case for $r = 2$. Since r can be chosen $n-1$ ways, the number of inclusive permutations of T_n is $(n-1)(n-2)! = (n-1)!$ $\square$

2.2 Proof of Identity 1

Identity 1

$$\sum_{j=1}^{n-1} H_j = nH_n - n.$$

Combinatorial Proof of Identity 1: We must first convert Identity 1 to Stirling numbers using Theorem 1.3.1, which gives us

$$\begin{aligned}\sum_{j=1}^{n-1} \frac{1}{j!} \begin{bmatrix} j+1 \\ 2 \end{bmatrix} &= \frac{n}{n!} \begin{bmatrix} n+1 \\ 2 \end{bmatrix} - n \\ &= \frac{1}{(n-1)!} \begin{bmatrix} n+1 \\ 2 \end{bmatrix} - n\end{aligned}$$

Multiplying both sides by $(n-1)!$, we have

$$\sum_{j=1}^{n-1} \frac{(n-1)!}{j!} \begin{bmatrix} j+1 \\ 2 \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} - n!$$

or equivalently, after re-indexing,

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! + \sum_{j=1}^{n-2} \frac{(n-2)!}{j!} \begin{bmatrix} j+1 \\ 2 \end{bmatrix}$$

Letting $k = j + 2$, this is equivalent to

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! + \sum_{k=3}^{n} \frac{(n-2)!}{(k-2)!} \begin{bmatrix} k-1 \\ 2 \end{bmatrix} \tag{2.2}$$

The left side of this equation directly counts all permutations in T_n. If we can show that the right side of the equation counts the same thing, then we are done. But by Lemma 2.1.1, for $3 \leq k \leq n$, $\frac{(n-2)!}{(k-2)!} \begin{bmatrix} k-1 \\ 2 \end{bmatrix}$ counts the number of exclusive permutations in T_n with endpoint k. Thus the total number number of exclusive permutations is

$$\sum_{k=3}^{n} \frac{(n-2)!}{(k-2)!} \begin{bmatrix} k-1 \\ 2 \end{bmatrix}.$$

And by Lemma 2.1.2, $(n-1)!$ counts the number of inclusive permutations in T_n. Since every permutation in T_n is either inclusive or exclusive, we have counted them all, and the identity is proved. $\square$

2.3 A Bonus Identity

As a byproduct of our analysis of inclusive permutations we offer a combinatorial proof of the following identity, typically proved by induction or telescoping sums.

$$\sum_{m=1}^{n-1} \frac{1}{m(m+1)} = 1 - \frac{1}{n}.$$

To prove this combinatorially, we first create integer quantities by multiplying both sides by $n!$. That is, our identity is equivalent to

$$\sum_{m=1}^{n-1} \frac{n!}{m(m+1)} = n! - (n-1)!$$

This is explained by asking the following question:

Question: How many inclusive permutations are there in T_{n+1} with at least 2 elements in the left cycle.

Answer 1: $n! - (n-1)!$, since the total number of inclusive permutations in T_{n+1} is $n!$, and the number of inclusive permutations in T_{n+1} with exactly 1 element in the left cycle is $(n-1)!$.

Answer 2: $\sum_{m=1}^{n-1} \frac{n!}{m(m+1)}$. We know from Proof 2 of Lemma 2.1.2 that for $1 \leq m \leq n-1$, the number of inclusive permutations in T_{n+1} with exactly m elements in the right cycle is $\frac{n!}{m(m+1)}$. And having at most $n-1$ elements is equivalent to having at least 2 elements in the left cycle. $\square$

Chapter 3

Identity 2

Now that we have proved Identity 1, we wish to prove the following similar identity:

Identity 2

$$\sum_{k=0}^{n-1} kH_k = \frac{n(n-1)}{2} H_n - \frac{n(n-1)}{4}$$

3.1 Proof of Identity 2

Combinatorial Proof Using Theorem 1.3.1 to convert Identity 2 to Stirling numbers yields the following:

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = \frac{(n-1)!}{2} + 2 \sum_{k=3}^{n} \frac{(n-3)!}{(k-3)!} \begin{bmatrix} k-1 \\ 2 \end{bmatrix} \tag{3.1}$$

The left side of this equation counts the number of permutations in T_n. From Lemmas 3.2.1, 3.2.3, and 3.2.4, we shall see that the right side of this equation also counts the number of permutations in T_n, by breaking T_n into three cases.

3.2 A new endpoint

In proving Identity 1, we looked at the endpoint of the left cycle, which we called k. Now we will also look at the endpoint of the right cycle, which we will call s. So using our cycle notation, our permutations take the form $\pi = (1 \ldots k)(r \ldots s)$. We keep the convention of listing the smallest element in the right cycle, r, first. We say π is *left-dominant* if $k > s$ and π is *right-dominant* if $k < s$.

Lemma 3.2.1 *For $3 \leq k \leq n$, the number of left-dominant permutations in T_n with left endpoint k is*

$$\begin{bmatrix} k-1 \\ 2 \end{bmatrix} \frac{(n-3)!}{(k-3)!}.$$

Proof. Given k, we can create the desired permutations as follows: First put elements 1 through $k-1$ into two cycles. There are $\left[{k-1 \atop 2}\right]$ ways to do this. Call the last element in the right cycle s. Next insert k at the end of the left cycle. Since $k > s$, the permutation generated so far is left-dominant. To remain left-dominant with left endpoint k, we insert the remaining elements $k+1$ through n one at a time to the right of any element except k or s. Since there are $(k-2)(k-1)\cdots(n-3) = \frac{(n-3)!}{(k-3)!}$ ways to do this, the number of left-dominant permutations is $\left[{k-1 \atop 2}\right]\frac{(n-3)!}{(k-3)!}$. □

Since a left-dominant permutation must have left endpoint $k \geq 3$, we have as an immediate corollary

Corollary 3.2.2 *The number of left-dominant permutations in T_n is*

$$\sum_{k=3}^{n} \begin{bmatrix} k-1 \\ 2 \end{bmatrix} \frac{(n-3)!}{(k-3)!}.$$

□

Lemma 3.2.3 *The number of right-dominant permutations in T_n with right endpoint s and at least 2 elements in the right cycle is*

$$\begin{bmatrix} s-1 \\ 2 \end{bmatrix} \frac{(n-3)!}{(s-3)!}.$$

Proof Given s, begin by putting elements 1 through $s-1$ into two cycles. There are $\left[{s-1 \atop 2}\right]$ ways to do this. Call the endpoint of the left cycle k. Insert s at the end of the right cycle. Now the right cycle has at least two elements. Finally, insert the remaining elements $s+1$ through n one at a time to the right of any element except k or s. There are $\frac{(n-3)!}{(s-3)!}$ ways to do this.

Since a right-dominant permutation must have $s \geq 2$, in total the number of right-dominant permutations in T_n with at least two elements in the right cycle is

$$\sum_{s=3}^{n} \begin{bmatrix} s-1 \\ 2 \end{bmatrix} \frac{(n-3)!}{(s-3)!}.$$

□

The combinatorial proof of Identity 2 is completed by enumerating the only remaining case.

Lemma 3.2.4 *The number of right-dominant permutations in T_n with exactly one element in the right cycle is*

$$\frac{(n-1)!}{2}.$$

Proof 1 We are counting permutations of the form $(1 \ldots k)(s)$ with $k < s$. We can see the number of permutations with only one element in the right cycle is $(n-1)!$, since we have $(n-1)$ choices for the lone element in the right cycle, and $(n-2)!$ ways to fill in the left cycle. Exactly half of them have $k < s$, since either $(1, \alpha_1, \ldots, \alpha_{n-3}, \alpha_{n-2})(\alpha_{n-1})$ or $(1, \alpha_1, \ldots, \alpha_{n-3}, \alpha_{n-1})(\alpha_{n-2})$ satisfies $k < s$. □

Proof 2 We condition on the value of s. For any given s, we have $s-2$ choices for k, since $k < s$ and $k \neq 1$. Now the left cycle can be completed in $(n-3)!$ ways. Since $3 \leq s \leq n$, we get a total of

$$\sum_{s=3}^{n} (s-2)(n-3)! = \frac{(n-1)(n-2)}{2}(n-3)! = \frac{(n-1)!}{2}.$$

□

Chapter 4

Identity 3

Now that we have proved Identities 1 and 2, we wish to prove the following more general Identity:

Identity 3

$$\sum_{j=m}^{n-1} \binom{j}{m} H_j = \binom{n}{m+1}(H_n - \frac{1}{m+1})$$

We must first convert Identity 3 to Stirling numbers using Theorem 1.3.1, which gives us

$$\sum_{j=m}^{n-1} \binom{j}{m} \frac{1}{j!} \begin{bmatrix} j+1 \\ 2 \end{bmatrix} = \binom{n}{m+1}(\frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} - \frac{1}{m+1})$$

$$\sum_{j=m}^{n-1} \frac{j!}{m!(j-m)!} \frac{1}{j!} \begin{bmatrix} j+1 \\ 2 \end{bmatrix} = \frac{n!}{(m+1)!(n-(m+1))!}(\frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix} - \frac{1}{m+1}).$$

When we multiply both sides by $(m+1)!(n-(m+1))!$ we get

$$(m+1)\sum_{j=m}^{n-1} \frac{(n-(m+1))!}{(j-m)!} \begin{bmatrix} j+1 \\ 2 \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} - \frac{n!}{m+1},$$

or equivalently, after re-indexing, letting $t = m+1$, $k = j+2$, and replacing n with $n-1$,

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = \frac{(n-1)!}{t} + t\sum_{k=t+1}^{n} \begin{bmatrix} k-1 \\ 2 \end{bmatrix} \frac{(n-1-t)!}{(k-1-t)!} \tag{4.1}$$

Notice that when $t = 1$ and $t = 2$, Equation (4.1) simplifies to Stirling Identities 2.2 and 3.1 respectively. In fact, when $t = n-1$, Equation (4.1) simplifies to Equation (1.2) with $k = 2$.

Every permutation $\pi \in T_n$ must be of the form $\pi = (1\alpha_2\alpha_3 \ldots \alpha_j)(\alpha_{j+1} \ldots \alpha_n)$ for some j, with $1 \leq j \leq n-1$ where $\alpha_{j+1} = \min\{\alpha_{j+1}, \ldots, \alpha_n\}$. For $1 \leq t \leq n-1$ we define $M_t(\pi) = \max\{\alpha_n, \alpha_{n-1}, \ldots, \alpha_{n+1-t}\}$. That is, $M_t(\pi)$ is the largest of the last t elements of π. To explain this combinatorially, we first interpret $\frac{(n-1)!}{t}$.

Lemma 4.0.5 *The number of permutations $\pi \in T_n$ that have $M_t(\pi) = \alpha_n$ and $\pi(\alpha_n) = \alpha_n$ is $\frac{(n-1)!}{t}$.*

We are counting permutations of the form $\pi = (1\alpha_2 \ldots \alpha_{n-1})(\alpha_n)$ where $\alpha_n = \max\{\alpha_n, \alpha_{n-1}, \ldots, \alpha_{n+1-t}\}$.

Proof 1 We count permutations of the form $(1\alpha_2 \ldots \alpha_{n-1})(\alpha_n)$ where $\alpha_n = \max\{\alpha_n, \alpha_{n-1}, \ldots, \alpha$ m, and sum over all possible values of m. Such a permutation must have $\alpha_n = m$. Next we assign $\alpha_{n-1}, \ldots, \alpha_{n-(t-1)}$ from the set $\{2, \ldots, m-t\}$. First we choose $t-1$ elements from $\{2 \ldots m-1\}$ and arrange them in positions $n+1-t$ through $n-1$. There are $\binom{m-2}{t-1}(t-1)!$ ways to do this. Now we arrange the remaining elements in the first $n-t$ positions, making sure to put the 1 first. There are $(n-1-t)!$ ways to do this. Since $t+1 \leq m \leq n$ we get the total number of permutations to be

$$\begin{aligned}
\sum_{m=t+1}^{n} \binom{m-2}{t-1}(t-1)!(n-1-t)! &= (t-1)!(n-1-t)! \sum_{p=t}^{n-1} \binom{p-1}{t-1} \\
&= (t-1)!(n-1-t)! \binom{n-1}{t} \\
&= \frac{(n-1)!}{t}
\end{aligned}$$

The summation in the next to last equation can be seen combinatorially as follows: To count the number of size t subsets of $\{1, \ldots, n-1\}$, we condition on p, the largest element of the subset. $\square$

Proof 2 Among all $(n-1)!$ permutations of the form $(1\alpha_2 \ldots \alpha_{n-1})(\alpha_n)$ precisely $\frac{1}{t}$th of them will have $M_t(\pi)$ in the last position. We count this directly. First choose which elements from $\{2 \ldots n\}$ will fill the last t positions. There are $\binom{n-1}{t}$ ways to do this. Place the largest of these t elements at the end. Now we have $(t-1)!$ ways to insert the remaining chosen elements in positions $n-1$ through $n-(t-1)$. We can now place the remaining elements in the first $n-t$ positions, making sure to put the 1 first, in $(n-1-t)!$ ways. This gives us the total number of permutations as

$$\binom{n-1}{t}(t-1)!(n-1-t)! = \frac{(n-1)!}{t!(n-1-t)!}(t-1)!(n-1-t)! = \frac{(n-1)!}{t}.$$

□

Lemma 4.0.6 *The number of permutations $\pi \in T_n$ that have $M_t(\pi) = k$ and $\pi(k) \neq k$ is*

$$t\begin{bmatrix} k-1 \\ 2 \end{bmatrix}\frac{(n-1-t)!}{(k-1-t)!}.$$

Proof We are counting permutations where $k = \max\{\alpha_n, \alpha_{n-1}, \ldots, \alpha_{n+1-t}\}$ and k is not alone in the right cycle. Note that since $\alpha_1 = 1$, $t+1 \leq k \leq n$. We begin by putting elements $\{1 \ldots k-1\}$ into two cycles. Then insert k to the right of any of the last t elements. There are $\left[{k-1 \atop 2}\right] t$ ways to do this. The right cycle contains at least one element less than k. So that k remains the largest among the last t elements, we insert elements of $\{k+1 \ldots n\}$ one at a time to the right of any but the last t elements. There are $(k-t)\cdots(n-1-t) = \frac{(n-1-t)!}{(k-1-t)!}$ ways to do this. The total is

$$t\begin{bmatrix} k-1 \\ 2 \end{bmatrix}\frac{(n-1-t)!}{(k-1-t)!}.$$

□

4.1 Proof of Identity 3

The left side of Equation 4.1 directly counts the number of permutations in T_n. If we can show that the right side of the equation counts the same thing, then we are done. But from Lemmas 4.0.5 and 4.0.6, and knowing that $t+1 \leq M_t(\pi) \leq n$, it is clear that the right side also counts exactly all permutations in T_n. $\square$

4.2 Other interpretations of $\frac{(n-1)!}{t}$

In this section we include other interpretations of $\frac{(n-1)!}{t}$. These came from earlier attempts to prove (4.1) combinatorially.

Lemma 4.2.1 *The number of inclusive permutations in T_n with at least t elements in the right cycle is $\frac{(n-1)!}{t}$, for $1 \leq t \leq n-1$.*

Proof. We build on Proof 4 of Lemma 2.1.2, but now we let B_n be the set of integer sequences of length $n-2$ such that the number in the i'th position must be chosen from $[1 \ldots i+1]$ whenever $i > t-1$, and from $[2 \ldots i+1]$ whenever $i \leq t-1$. So we have 1 choice for the first position, 2 choices for the second position, all the way up to $t-1$ choices for the $t-1$'th position. Then for the t'th position we have $t+1$ choices, and for the $t+1$'th position we have $t+2$ choices, all the way up to $n-1$ choices for the $n-2$'nd position. So there are $1 \cdot 2 \cdots (t-2) \cdot (t-1) \cdot (t+1) \cdot (t+2) \cdots (n-1) = \frac{(n-1)!}{t}$ elements in B_n, so $|B_n| = \frac{(n-1)!}{t}$.

Now using the rules from Proof 4 of Lemma 2.1.2, we generate permutations from elements β in B_n. We begin by using Rule 1, which adds elements to the right cycle. And since there can't be any 1's in β until the t'th position, we must use Rule 1 at least $t-1$ times, giving us at least t elements in the right cycle, which is what we want. $\square$

Lemma 4.2.2 *The number of permutations (inclusive or exclusive) in T_n with exactly t elements in the right cycle is $\frac{(n-1)!}{t}$.*

Proof 1. Begin by putting the element 1 in the left cycle. Now choose t of the remaining $n-1$ elements to go into the right cycle, and arrange them in one of the $(t-1)!$ possible ways. Now arrange the remaining $n-1-t$ element in the left cycle in one of the $(n-1-t)!$ possible ways. The total number of permutations with exactly t elements in the right cycle is therefore

$$\binom{n-1}{t}(t-1)!(n-1-t)! = \frac{(n-1)!}{t!(n-1-t)!}(t-1)!(n-1-t)! = \frac{(n-1)!}{t}$$

□

Proof 2. List all n elements in any order, requiring only that the 1 comes first. There are $(n-1)!$ such listings. The first $n-t$ elements listed form the left cycle of a permutation. The remaining t elements form the right cycle only when the smallest is listed first. This will happen precisely 1 out of every t times. So the number of these arrangements that form valid permutations is exactly $\frac{(n-1)!}{t}$. It is easy to see that any permutation with t elements in the right cycle can be generated one such valid arrangement. □

Corollary 4.2.3 *The number of permutations in T_n with $n-t$ elements in the left cycle is* $\frac{(n-1)!}{t}$.

Lemma 4.2.4 *The number of permutations (inclusive or exclusive) in T_n with smallest element in the right cycle equal to $t+1$ is* $\frac{(n-1)!}{t}$.

The proof of this follows immediately from Lemma 1.3.2.

4.3 Alternate Proof of Identity 1

Now with a new understanding of $\frac{(n-1)!}{t}$, we briefly return to Identity 1. By re-indexing equation 2.2, we get

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! + \sum_{k=1}^{n-2} \frac{(n-2)!}{k!} \begin{bmatrix} k+1 \\ 2 \end{bmatrix} \tag{4.2}$$

By viewing $(n-1)!$ as $\frac{(n-1)!}{1}$, we see from Lemma 4.2.4 that $(n-1)!$ counts the number of permutations with $r = 2$. If the sum counts all permutations with $r > 2$, then we have again proven Identity 1.

Theorem 4.3.1 *The number of permutations in T_n with minimal element in the right cycle greater than 2 is*

$$\sum_{k=1}^{n-2} \frac{(n-2)!}{k!} \begin{bmatrix} k+1 \\ 2 \end{bmatrix}$$

Proof:

Since the smallest element in the right cycle is greater than 2, 2 must be somewhere in the left cycle. Given any permutation in T_n with $r > 2$, we will define k to be the number of elements to the right of 2 in the left cycle, plus the size of the right cycle. Note that if 2 immediately follows 1, then $k = n - 2$. And if 2 is the last element in the left cycle while the right cycle contains only one element, then $k = 1$.

We shall construct all permutations with a given k. We first determine the elements in the left cycle between elements 1 and 2. There will be $n - k - 2$ such elements. Since these elements can be anything except 1 or 2, for the first element we have $(n-2)$ choices. Then for the second element, we have $(n-3)$ choices. And so on until for the $(n-k-2)$nd element we have $(k+1)$ choices. So in determining the possible arrangements of the elements between the 1 and the 2, we have $(n-2)(n-3)\cdots(k+1) = \frac{(n-2)!}{k!}$. We put these $n-k-2$ elements followed by 2 into a string, α.

Now we must determine the rest of the permutation. To do this, we take all of the elements not in α, for a total of $k+1$ elements, (this includes element 1) and split them into two cycles. There are $\left[{k+1 \atop 2}\right]$ ways to do this. Now we insert the string α to the right of 1. The result is a permutation with exactly k elements to the right of 2, and minimal element in the right cycle is greater than 2. The total number of such permutations is

$$\frac{(n-2)!}{k!}\begin{bmatrix} k+1 \\ 2 \end{bmatrix}.$$

And since $1 \leq k \leq n-2$, the total number of permutations in T_n with minimal element in the right cycle greater than 2 is

$$\sum_{k=1}^{n-2} \frac{(n-2)!}{k!}\begin{bmatrix} k+1 \\ 2 \end{bmatrix}$$

□

Chapter 5

Identity 4

The last identity we prove looks dissimilar to the previous three, but actually lends itself to a similar combinatorial argument.

Identity 4

$$\sum_{j=m}^{n-1} \binom{j}{m} \frac{1}{n-j} = \binom{n}{m}(H_n - H_m)$$

Using Theorem 1.3.1 to convert Identity 4 to Stirling numbers yields the following:

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = \begin{bmatrix} m \\ 2 \end{bmatrix} \frac{(n-1)!}{(m-1)!} + \sum_{t=m}^{n-1} \binom{t-1}{m-1} \frac{(m-1)!(n-m)!}{(n-t)} \tag{5.1}$$

5.1 Proof of Identity 4

Lemma 5.1.1 *The number of permutations in T_n that do not have elements $1, 2, \ldots m$ all in the left cycle is*

$$\begin{bmatrix} m \\ 2 \end{bmatrix} \frac{(n-1)!}{(m-1)!}.$$

Proof First split the elements of $\{1 \ldots m\}$ into two cycles, which can be done $\left[\begin{smallmatrix} m \\ 2 \end{smallmatrix}\right]$ ways. Next insert the remaining elements one at a time to the right of any existing element. As usual, element k, $m+1 \le k \le n$ has $k-1$ choices. Hence the number of ways to insert these elements is

$$m(m+1)\cdots(n-1) = \frac{(n-1)!}{(m-1)!}$$

and the lemma follows. $\square$

Lemma 5.1.2 *For $m \leq t \leq n-1$, the number of permutations in T_n with exactly t elements in the left cycle that do have elements $1, \ldots m$ all in the left cycle is*

$$\binom{t-1}{m-1}\frac{(m-1)!(n-m)!}{(n-t)}.$$

Proof We know the left cycle has exactly t elements, and therefore the right cycle must have $n-t$ elements. First place 1 at the front of the left cycle. Now choose $m-1$ of the remaining $t-1$ spots in the left cycle, and fill these spots with elements from $\{2 \ldots m\}$. There are $\binom{t-1}{m-1}(m-1)!$ ways to do this. See Figure 5.1 for an example where $n = 9$, $t = 6$, and $m = 4$.

1 through m are in the left cycle

$$\pi = \underbrace{(1\ \_\ \underline{2}\ \_\ \underline{4}\ \underline{3})}_{t-1 \text{ locations}}\underbrace{(\_\ \_\ \_)}_{n-t \text{ locations}}$$

Figure 5.1: Inserting elements 2 through m.

Now there are $(n-m)!$ ways to place elements $m+1$ through n in the remaining spots, but only $\frac{1}{n-t}$th of them will put the smallest element in the right cycle at the front of the right cycle. Hence, elements $m+1$ through n can be inserted in $\frac{(n-m)!}{n-t}$ ways. Our total is

$$\binom{t-1}{m-1}\frac{(m-1)!(n-m)!}{(n-t)}.$$

□

Proof of Identity 4

The left side of Equation 4 counts the number of permutations in T_n. But from Lemmas 5.1.1 and 5.1.2 and $m \leq t \leq n-1$, we see that that the right side of this equation also counts the number of permutations in T_n. □

Chapter 6

The Future

6.1 Generalizing from $\left[{n \atop 2}\right]$ to $\left[{n \atop k}\right]$

Identities 1 through 4 all involve $\left[{n \atop 2}\right]$. Perhaps they have more general forms involving $\left[{n \atop k}\right]$. These general forms would likely have combinatorial proofs involving permutations with k cycles. General identities may not have direct harmonic applications, but they might lead to a better understanding of the $\left[{n \atop 2}\right]$ versions of the identities, which could help better understand harmonic numbers.

6.2 Generalized Harmonic Numbers

We explore two generalizations of harmonic numbers. The first, $H_n^{(k)}$ is a sum of harmonic numbers, as follows

$$H_n^{(k)} = \sum_{i=1}^{n} H_i^{(k-1)}$$

with $H_n^{(1)} = H_n$. The identity below (see [2]) suggests that there must be a combinatorial interpretation of these numbers as well.

Identity 5 $H_n^{(k)} = \binom{n+k-1}{k-1}(H_{n+k-1} - H_{k-1})$

There may also be identities that can be proven combinatorially using a second generalization of harmonic numbers, $H_{n,k}$, where

$$H_{n,k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} = \sum_{j=1}^{n} \frac{1}{j^k}.$$

Unlike normal harmonic numbers, $H_{n,k}$ converges [3] as n goes to infinity for $k > 1$. This limit is called the Riemann-Zeta function, which is well known in mathematics, but is beyond the scope of this work.

6.3 Conclusion

We have found combinatorial explanations of several harmonic identities. Hopefully this work can lead to a better understanding of harmonic numbers. A side benefit may also be some new results involving permutations, particularly new classifications of permutations into inclusive and exclusive. It may also prove interesting to use this new knowledge of permutations to create entirely new Stirling identities and use them to discover new harmonic identities.

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