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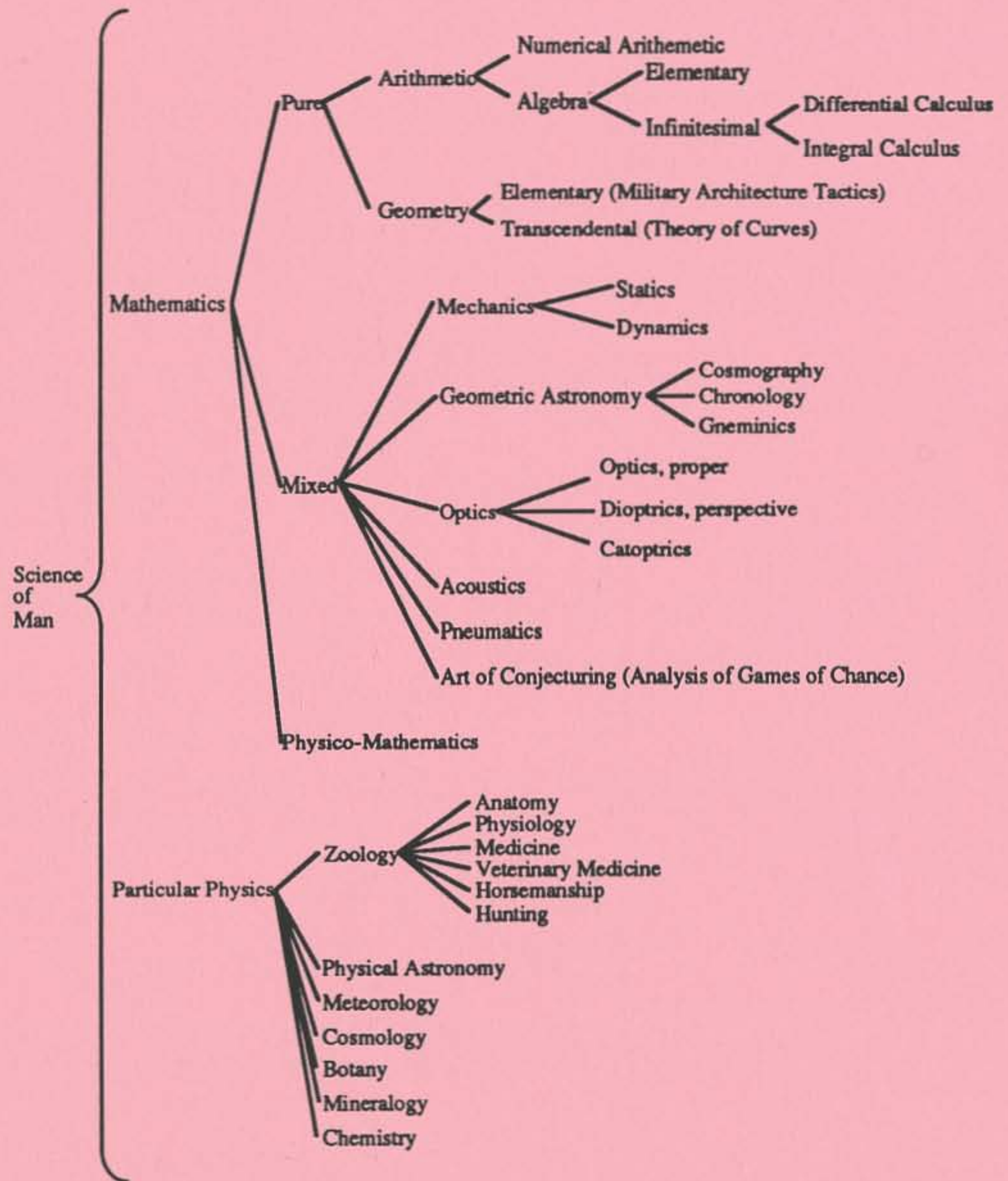

INVITATION TO AUTHORS

Essays, book reviews, syllabi and letters are welcome. Two copies, double spaced, should be sent to Alvin White, HUM. MATH. NET., Harvey Mudd College, Claremont, CA 91711. If possible, avoid footnotes and put references and bibliography at the end using a consistent style. If you use a word processor please send a diskette in addition to the typed paper. *The Journal* is assembled using Microsoft Word 4.0 and PageMaker 4.0 on a Macintosh. It is possible, however, to convert from other word processing systems. Clean typed copy can be scanned (but not dot matrix). Your essay should have a title, your name and address, and a brief summary. Your telephone number (not for publication) would be helpful. Essays and communications may be transmitted by electronic mail to the editor at AWHITE@HMC.EDU.
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NOTE TO LIBRARIANS

The Humanistic Math Network Journal #8, #9 and #10, ISSN# 1065-8297 are the successors to the Humanistic Math Network Newsletter, ISSN# 1047-627X.

COVER

This diagram appears in Gary I. Brown's article, "Applied Mathematics Should Be Taught Mixed." It is d'Alembert's tree of knowledge, which appeared in his famous *Discours Preliminaire*. See pages 1-12.

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From the Editor

The Network is growing in a natural and healthy way. The early response to humanistic mathematics questioned how it is defined. Now that question is rarely asked. Recent books such as *The Art of Mathematics*, *The Nature and Power of Mathematics*, and *Patterns in Mathematics* explore some humanistic aspects. Hermann Weyl wrote: "We do not claim for mathematics the prerogative of a Queen of Science, there are other fields which are of the same or even higher importance in education. But mathematics sets the standard of objective truth for all intellectual endeavors; science and technology bear witness to its practical usefulness. Besides language and music, it is one of the primary manifestations of the free creative power of the human mind, and it is the universal organ for world-understanding through theoretical construction. Mathematics must therefore remain an essential element of the knowledge and abilities which we have to teach, of the culture we have to transmit, to the next generation".

At the January '95 mathematics meetings in San Francisco, the Humanistic Mathematics Network will be assigned a room in the evening for an hour and a half. (Check FOCUS and the meeting program.) The room will have a slide projector and a VCR and monitor, so you are invited to share slides and videos. You are also invited to share poetry—your own or someone else's. You will have an opportunity to make a short presentation about humanistic mathematics. JoAnne Growney has arranged an evening of poetry reading at the San Francisco meeting. Please let me know your plans so that informal scheduling can occur.

The essays by Shobha Gulati, Susan Byerly and Jo Anne Growney have some points of contact. The course Female Voices in Mathematics reviews the struggle and discrimination that women have faced to be educated and recognized in mathematics. A math-phobic young woman is awakened to the excitement of mathematics by reading Marcia Ascher's book *Ethnomathematics*. Jo Anne Growney quotes a passage from *Gone With the Wind* where Frank learns, to his dismay, that Scarlett O'Hara is competent at arithmetic and business. "Now he saw that she understood entirely too well and he felt the usual masculine indignation at the duplicity of women. Added to it was the usual masculine disillusionment in discovering that a woman has a brain."

In *Space, Time, Matter*, Weyl says: "Not only in geometry, but to a still more astonishing degree in physics, has it become more and more evident that as soon as we have succeeded in unravelling fully the natural laws which govern reality, we find them to be expressible by mathematical relations of surpassing simplicity and architectonic perfection. It seems to me to be one of the chief objects of mathematical instruction to develop the faculty of perceiving this simplicity and harmony, which we cannot fail to observe in the theoretical physics of the present day. It gives us deep satisfaction in our quest for knowledge."

There are now over 1100 on our mailing list from every continent. Humanistic mathematics is growing as an influential concept. I am teaching a seminar on humanistic mathematics next fall. The first Ph.D. in humanistic mathematics may be awarded within a few years. The next issue of our journal will be mailed near February 1995. We'll try to maintain a twice a year schedule.

Your essays, descriptions of teaching or learning experiences, opinions, criticisms, book reviews, and discussions of "humanistic mathematics" and its place in the world are welcome.

Alvin White

Applied Mathematics Should be Taught Mixed

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1. INTRODUCTION

Many would argue that "the power of mathematics" is derived from the great variety of problems that can be modeled and solved mathematically. According to *The Power of Mathematics, Applications to Management and Social Science*, by Whipkey, Whipkey and Conway, "the power of mathematics is derived from two sources. First, the same mathematical concept can be used to solve a multitude of problems from diverse academic fields. For example, the algebra of matrices can be applied to problems arising in mathematics, physics, chemistry, economics, sociology, psychology and statistics" (Introduction, p 3).

The hidden message coming from such a derivation seems to be that (pure) mathematics is this self-contained, autonomous abstract subject completely devoid of any human value until it is "applied" to a given problem or discipline. As a young student I first felt this hidden message when encountering "modeling" diagrams such as the one in figure 1*:

One was supposed to transform a "real world problem" into a "mathematical model" and solve the resulting problem mathematically. Then, one was supposed to interpret the mathematical solutions in terms of the real world problem. It always seemed that the "real world problem" was completely separated from the "mathematical problem". As an aspiring mathematician, I hated to deal with the "real world problem" and only wanted to deal with the "mathematical model."

The metaphor emanating from the term "applied" seems to reinforce this separation between (pure) mathematics and applied mathematics. Is is possible to change metaphors and reverse this hidden message? Historically, the term "applied" was not used in the literature until about 170 years ago ("applied" appeared in J. Gergonne's journal called 'Annales de mathématiques pures et appliquées" that was first published in 1810). Prior to this, mathematics was divided into "pure mathematics" and "*mixed* mathematics." The purpose of this paper is to outline historically the meaning of the term "mixed mathematics" and then to choose aspects of its meaning that can be modified to present a different vision of how to teach applied mathematics. I will first argue that there is a dichotomy today between "pure" and "applied" mathematics. Then I will provide some historical analysis of mixed mathematics in England in the early seventeenth century and in France in the eighteenth century. This will be followed by a brief examination of two nineteenth century practitioners of mixed mathematics, Gaspard Monge and William Whewell. Finally, I will provide some ideas for a possible new model for the teaching of applied mathematics that is partially based upon a recent article in the *Bulletin of the American Mathematical Society* by Arthur Jaffe and Frank Quinn. I an not arguing that mixed mathematics is historically "better" than applied mathematics, but that one can adapt some of the main ideas from mixed mathematics to our modern evolving view of applied mathematics. I hope that

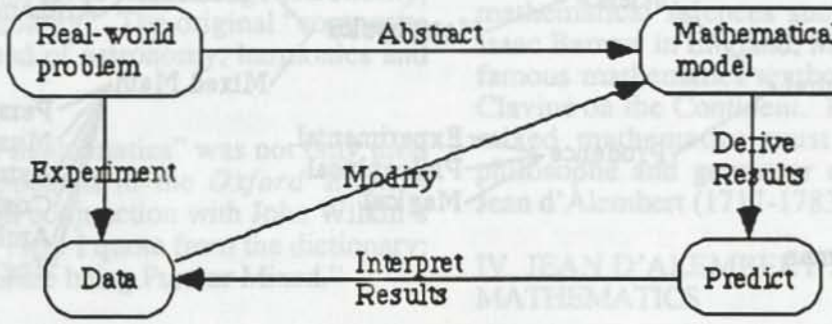


Figure 1

the changing of metaphors will result in reversing this hidden message of applied mathematics.

II. THE EXISTENCE OF A DICHOTOMY BETWEEN "PURE" AND "APPLIED"

Before discussing "mixed mathematics," it perhaps is necessary to demonstrate the existence of a dichotomy between "pure" and "applied" mathematics today. The first thing I did was to examine how the terms "pure" and "applied" were used in everyday English language. According to Funk and Wagnalls Dictionary,

> *Pure*— Free from mixture or contact with that which weakens or impairs or pollutes; considered apart from practical application, opposed to applied.
> *Applied*— To bring into contact with something; to devote or put to a particular use; as to apply steam to navigation or money to payment or debts.

Secondly, I looked at many general source books such as encyclopedias. According to the *World Book Encyclopedia*,

> The work of mathematicians may be divided into *pure mathematics* and *applied mathematics*. Pure mathematics seeks to advance mathematical knowledge for its own sake rather than for any immediate practical use...applied mathematics seeks to develop mathematical technique for use in science and other fields.

Also from the *Mathematics Dictionary* edited by James and James, I found the following:

> *applied mathematics*: A branch of mathematics concerned with the study of physical, biological and sociological worlds... In a restricted sense, the term refers to the use of mathematical principles as *tools* in the fields of physics, chemistry, engineering, biology, and social studies.

From Avner Friedman, current SIAM President, testifying to the House of Representatives on NSF funding, and discussing the *goals* of the mathematical sciences,

> to lead the development and transfer of applications of mathematics to problems in science, technology and industry.

All of these sources seem to reinforce the idea that to do "applied mathematics" one must apply theory A from "pure mathematics" to problem B from "the real world." This separation did not seem to exist prior to the French Revolution (It is difficult to find evidence of this separation in non-European cultures).

III. "MIXED MATHEMATICS IN EARLY SEVENTEENTH CENTURY ENGLAND

The term "mixed mathematics" occurred frequently in many so called "trees of knowledge" in the Seventeenth and Eighteenth centuries.** One of the first prominent "trees of knowledge" comes from Francis Bacon's *Of the Proficience and Advancement of Learnings* (1605). In Bacon's tree of knowledge (See Figure 2), philosophy was divided into three categories (Divine Theology, Natural, and Human). Natural Philosophy was

Bacon's Tree of Knowledge (Human Learning)

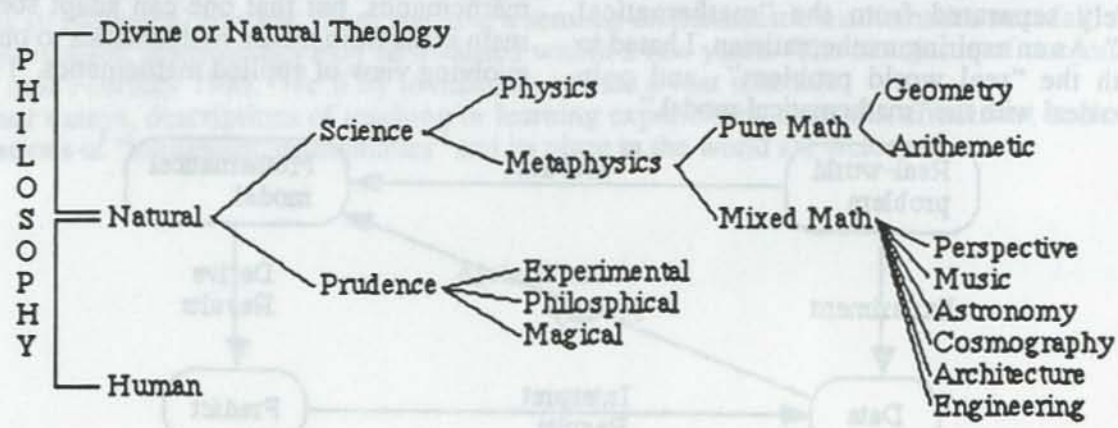


Figure 2

divided into Science and Prudence; Science was divided into Physics and Metaphysics. Finally mathematics was classified under Metaphysics (See Figure 2).

According to Bacon, mathematics belonged to metaphysics because its aim was to inquire *about fixed and constant causes not indefinite causes.* He said "all other forms are the most abstracted and separated from matter and therefore most proper to Metaphysics."

After placing mathematics in the category of metaphysics Bacon subdivided mathematics into "pure" and "mixed". He said "to pure mathematics belong those sciences which handle Quantity entirely severed from matter and from the Axioms of Natural Philosophy. These are Geometry and Arithmetic...Mixed Mathematics has for its subject

According to d'Alembert, "quantity, the object of mathematics, could be considered either alone and independent of real and abstract things from which one gained knowledge of it"

some axioms and parts of natural philosophy, and consider quantity *determined as it is auxiliary and incident unto them. For many parts of nature can neither be inverted with sufficient subtlety nor demonstrated with sufficient perspicuity* without the aid and intervening of mathematics."

Bacon added to the Ancient Greek's list of the "composite sciences" the branches of *architecture* (important in the 16th Century Renaissance), *Engineering* and *cosmography* (science that describes the universe including astronomy, geography and geology). The original "composite sciences" consisted of astronomy, harmonics and optics.

The term "mixed mathematics" was not only used by Bacon but appeared in the *Oxford English Dictionary 1648* in conjunction with John Wilkin's "Math Magick". Here I quote from the dictionary: "The Mathematics are being Pure or Mixed."

The reference to John Wilkins pertains to the former Puritan clergyman and popularizer of Bacon's approach to science. Wilkins was one of the founders of the Royal Society (1660), the first scientific organization whose purpose was the promotion and advancement of science and its applications to the improvement of society. The book "Math Magick" refers to the longer title "Math Magick: The Wonders that may be Performed by Mechanical Geometry." Mechanical Geometry was described as "one of the most easy, useful and yet most neglected parts of mathematics. It is a liberal art like astronomy and music." Topics included a discussion of pulleys, cranes, bows and catapults used in levers and wedges; the possibilities of various kinds of machines, such as submarines and carriages propelled by sails; and "secret and speedier ways of attacking forts by approaches and galleries." (inventions in fortification)

The purpose of "Math Magick" was to familiarize the average person with the basic and long-accepted principles of mechanics. Wilkins begins with a defense of mechanics as a liberal art that was more like astronomy and music than the so called "illiberal sciences" which involved some physical activity such as manufacturing or trade. The basic subject of mechanics was the relationship between weight and power. Weight was no longer to be considered a "natural quality, whereby condensed bodies do of them selves tend downwards," but "an affection which might be measured." Quantity, the subject of seventeenth century mathematics, could therefore pertain to qualities of physical objects such as weight and power. If mechanics were to become a mathematical science then one could not separate the physical from the theoretical. The scientist must use the proper mixture of theoretical reasoning, direct observation and experimentation.

During the seventeenth century there were other advocates of a careful handling of the mixed mathematical sciences such as John Wallis and Isaac Barrow in England, Marin Mersenne and the famous mathematics textbook writer Christopher Clavius on the Continent. However, any study of mixed mathematics must include the famous philosophe and geometer of the Enlightenment, Jean d'Alembert (1717-1783).

IV. JEAN D'ALEMBERT'S VIEWS OF MIXED MATHEMATICS

D'Alembert was a key advocate of extending the

kind of thinking involved in geometry to decision making in society. As a philosophe he considered himself as a literary man who exercised public responsibility and whose function was to criticize, analyze, and redefine intellectual norms and institutional practice. He became one of the leaders among a group of philosophes that called themselves rationalists.

D'Alembert thought that rational thinking involved the processes of *analysis* and *methodical doubt*. Analysis was the process of starting with an idea and breaking it into simpler ideas until one reaches simple ideas (atoms). Simple ideas are self-evident truths based upon sensory receptions. One would know from *experience* that so and so is a simple idea. Then, once the simple idea (first principles) was found by analysis, one tried to construct a deductive chain of ideas that built back to the original idea. This was the process of synthesis found in geometry.

Notice that this form of analysis is similar to that form used in algebra. Consider examples 1 and 2 below:

Example I: Solve $2x+3=2$
Assume there is a solution x.
analysis → $2x=-1$
$x=-1/2$

synthesis → $x=-1/2$
$2x=-1$
$2x+3=2$

Example 2: Solve $2x^2+3=2$
Assume there is a solution x.
analysis → $2x^2=-1$
Impossible

synthesis → Hence one of the elements in the chain is false

In Example 1, one assumes the equation is true and finds "simpler" statements that follow from the assumption. Eventually, one will arrive at a "simplest" statement that cannot be broken down any further. Then one tries to build a chain of statements from this "simplest" statement to the original statement. Such a procedure will constitute a proof that there is a solution to $2x+3=2$. In proofs involving mixed mathematics (which might include example 2 as a trivial example), one must factor in experience when analyzing each step in the chain. This is apparent in example 2 where an assumption of $2x^2+3=2$ having a real solution will lead to a positive real number equalling a negative real number. The chain has been severed and one must re-examine the problem.

D'Alembert was convinced that man had knowledge on only a few links that joined that gigantic chain of each discipline together. Hence, the philosophe had to look at arguments with a healthy dosage of *methodical doubt* for errors in any chain. The philosophe especially had to search for abuses of theory as might occur in the second example above. Furthermore, the philosophe should use methodical doubt then "indicating the paths which have deviated from the truth, so that it facilitates the search for the path which conducts to it." This chain always had to factor in common experience (the use of the sense perceptions), i.e. *one should never separate the pure from the applied.*

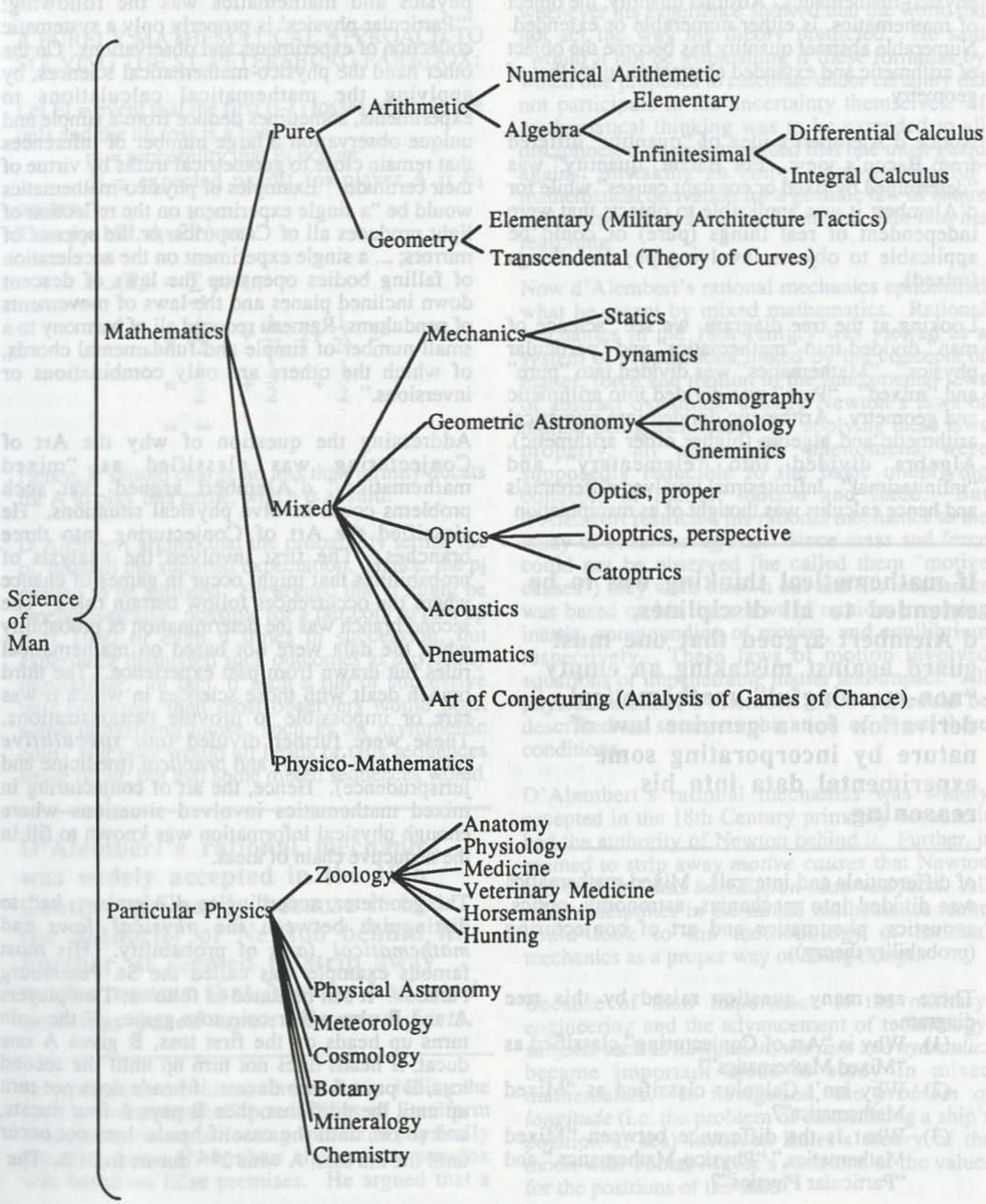
Now d'Alembert was involved in the 1750s with Diderot in the Encyclopedia project. They wanted to classify all of knowledge and d'Alembert was especially interested in classifying mathematics. After all, there were many advances in mathematics

For Bacon "quantity" was "determined by fixed or constant causes" while for d'Alembert was applicable to objects that were independent of real things (pure) or could be applicable to objects involving physical things (mixed).

since the last time someone had tried to do such a project. We find d'Alembert's classification of mathematics in his famous *Discours Preliminaire*. (See d'Alembert's tree of knowledge).

According to d'Alembert, "quantity, the object of mathematics, could be considered either alone and independent of real and abstract things from which one gained knowledge of it, or it could be considered in their efforts and investigated according to real or supposed causes; this reflection leads to the division of mathematics into

Figure 3:
d'Alembert's Tree of Knowledge



pure mathematics, mixed mathematics, and physico-mathematics. Abstract quantity, the object of mathematics, is either numerable or extended. Numerable abstract quantity has become the object of arithmetic and extended abstract quantity that of geometry.

Notice d'Alembert's idea of "quantity" differed from Bacon's view. For Bacon "quantity" was "determined by fixed or constant causes" while for d'Alembert it was applicable to objects that were independent of real things (pure) or could be applicable to objects involving physical things (mixed).

Looking at the tree diagram, we see "science of man" divided into "mathematics" and "particular physics". "Mathematics" was divided into "pure" and "mixed". "Pure" was divided into arithmetic and geometry. Arithmetic divided into numerical arithmetic and algebra (higher order arithmetic). Algebra divided into "elementary" and "infinitesimal". Infinitesimal involved differentials and hence calculus was thought of as manipulation

If mathematical thinking was to be extended to all disciplines, d'Alembert argued that one must guard against mistaking an empty "non-meaningful" mathematical derivation for a genuine law of nature by incorporating some experimental data into his reasoning.

of differentials and integrals. Mixed mathematics was divided into mechanics, astronomy, optics, acoustics, pheumatics and art of conjecturing (probability theory).

There are many question raised by this tree diagram.

(1) Why is "Art of Conjecturing" classified as "Mixed Mathematics"?

(2) Why isn't Calculus classified as "Mixed Mathematics"?

(3) What is the difference between "Mixed Mathematics," "Physico-Mathematics," and "Particular Physics"?

According to d'Alembert, the difference between physics and mathematics was the following: "'Particular physics' is properly only a systematic collection of experiments and observations. On the other hand the physico-mathematical sciences, by applying the mathematical calculations to experiments, sometimes deduce from a simple and unique observation a large number of inferences that remain close to geometrical truths by virtue of their certitude." Examples of physico-mathematics would be "a single experiment on the reflection of light produces all of Catoptrics, or the science of mirrors; ... a single experiment on the acceleration of falling bodies opens up the laws of descent down inclined planes and the laws of movements of pendulums; Rameau reduced all of harmony to a small number of simple and fundamental chords, of which the others are only combinations or inversions."

Addressing the question of why the Art of Conjecturing was classified as "mixed mathematics," d'Alembert argued that such problems could involve physical situations. He classified the Art of Conjecturing into three branches. The first involved the analysis of probabilities that might occur in games of chance where the occurrences follow certain rules. The second branch was the determination of probability when the data were not based on mathematical rules but drawn from past experience. The third branch dealt with those sciences in which it was rare or impossible to provide demonstrations. These were further divided into *speculative* (physics and history) and *practical* (medicine and jurisprudence). Hence, the art of conjecturing in mixed mathematics involved situations where enough physical information was known to fill in the deductive chain of ideas.

The geometer, according to d'Alembert, had to distinguish between the *physical laws* and *mathematical laws* of probability. His most famous example was called the St. Petersburg Paradox. It can be stated as follows: Two players A and B play a fair coin toss game. If the coin turns up heads on the first toss, B gives A one ducat, if heads does not turn up until the second toss, B pays A two ducats, if heads does not turn up until the third toss, then B pays A four ducats, and so on, until the case if heads does not occur until the nth toss, A wins 2^{n-1} ducats from B. The

question is how much should A pay B to play the game.

A PURELY MATHEMATICAL APPROACH TO SOLVING THE ST. PETERSBURG PARADOX:

Let E_i = event that the first (i-1) tosses of a coin are tails and the ith toss is a head.

p_i = probability that E_i occurs.

d_i = expected payoff for player A if E_i occurs.

Then $p_i=1/2^i$, $d_i=2^{i-1}$ and

$$E\cup(E_i) = \sum_{i=1}^{\infty} p_i d_i$$

$$= \frac{1}{2}\cdot 1 + \frac{1}{2^2}\cdot 2 + \frac{1}{2^3}\cdot 2^2$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= \infty$$

Hence, B should charge no less than infinity ducats for A to play this game, which is absurd.

D'Alembert argued that the probability terms, the p_i, inflated the summed expectation. Hence, the p_i needed to be adjusted. He argued that it might be "metaphysically" possible for a fair coin to turn up tails 1000 or even n times in succession, but *experience* dismissed such outcomes as *physically impossible*. In fact, if such a run of successive tails occurred, then most observers would posit some underlying cause, such as an asymmetric coin. Not only would all heads or tails sequences never occur, but that some mixed sequences would be repeated two or three times. By including the mainly "metaphysical" possibilities of a uniform sequence on an equal footing with more physically plausible ones, d'Alembert claimed the paradox was based on false premises. He argued that a reasonable man can make judgments that disagree with the conventional mathematical theory, whenever his experience showed that the world did not act the way the theory predicted. He said "would it not be astonishing if these formulas by which one proposes to calculate under certainty did not participate in the uncertainty themselves. If mathematical thinking was to be extended to all disciplines, d'Alembert argued that one must guard against mistaking an empty "non-meaningful" mathematical derivation for a genuine law of nature by incorporating some experimental data into his reasoning.

D'Alembert's rational mechanics was widely accepted in the 18th Century primarily because it had the authority of Newton behind it. Further, it seemed to strip away *motive causes* that Newton had used and was *based upon observed facts*.

Now d'Alembert's rational mechanics epitomized what he meant by mixed mathematics. Rational mechanics in the 18th century was viewed as a system of propositions linked by the concepts of matter, force and motion to the fundamental laws of abstract dynamics, usually Newton's laws of motion being mentioned. By applying these laws properly, all observable phenomena were supposedly reducible to the basic underlying concepts of matter, motion and force. But d'Alembert restricted his rational mechanics to the study of *observed effects*. Since mass and force could not be observed (he called them "motive causes") they were thrown out and his mechanics was based on his three laws of motion, involving inertia, compounding of motion, and equilibrium respectively. These laws of motion involved equations of impenetrable matter and impact. All physical motion, d'Alembert believed, could be described as some combination of these three conditions.

D'Alembert's rational mechanics was widely accepted in the 18th Century primarily because it had the authority of Newton behind it. Further, it seemed to strip away *motive causes* that Newton had used and was *based upon observed facts*. All other disciplines in the mixed mathematics realm could look to the methodology of rational mechanics as a proper way of doing things.

Because of their importance to the military, engineering and the advancement of technology, subjects such as *navigation*, *warfare* and *hydraulics* became important areas to study in mixed mathematics. In navigation, the *problem of longitude* (i.e. the problem of determining a ship's position at sea) involved Euler's theory of the moon with Tobias Mayer's revisions of the values for the positions of the stars.

In *warfare*, the problem of projectile motion involved the so called quadratic theory of ballistics in combination with experimental data on the velocity of projectiles shot from a cannon.

In *hydraulics*, the problem of building waterworks and ships involved the theory of architecture and hydrodynamics with experimental results. In all of these cases, mixed mathematics was involved because all quantities used magnitudes subsisting in material bodies and interwoven everywhere with physical considerations. All subjects modeled their approach to mixed mathematics after that of rational mechanics.

V. SOME PROPONENTS OF THE TEACHING OF MIXED MATHEMATICS DURING THE NINETEENTH CENTURY

(a) Gaspard Monge

French geometers such as Laplace and Condorcet continued their advocation for mixed mathematics in the spirit of Jean d'Alembert. Laplace reiterated d'Alembert's idea that mixed mathematics involved the search for first principles and the building of a deductive chain when he said "natural phenomena are mathematical results of a small number of invariable laws." However, the major advocate for the teaching of mixed mathematics in the spirit of d'Alembert was Gaspard Monge, known today as the Father of Descriptive Geometry.

Monge was a major advocate of using physical drawings in doing geometry and an advocate of incorporating mathematics and science in an education intended to be broadly based. He was a major advocate of integrating science and mathematics into a humanistic education. Monge did not just teach mathematics, he molded future French citizens. His descriptive geometry lent itself beautifully to both the theoretical and practical sides of the curriculum because it consisted of a purely rational theory which could be translated into concrete graphic relations. This involved learning the essential skills of mechanical drawing by passing constantly from the abstract to the concrete and back again. Monge expressed this idea of mixing mathematics best in the preface of his famous *Gèomètrie descriptive*: "The second object of descriptive geometry is to deduce from the exact description of bodies all which necessarily follows from their forms and respective positions. In this sense it is a means of investigating truth; it perpetually offers examples of passing from the known to the unknown; and since it is always applied to objects with the most elementary shapes, it is necessary to introduce it into the plan of national education and make use of this geometry of the representation and determination of the elements of machines by which man, controlling the forces of nature, reserves for himself, so to speak, no other labor in his work but that of his intelligence."

In contrast to Monge's position of integrating descriptive geometry into a humanistic eduction, we have Cauchy's position that mathematics should be treated educationally as a separate, specialized discipline. According to Cauchy, "let us then admit that there are truths other than those in algebra, realities other than those of sensible objects. Let us ardently pursue mathematics without trying to extend it beyond its domain." Different fields are different, Cauchy asserted: they rest on different forms of evidence, use different kinds of arguments, and generate different kinds of knowledge. Cauchy's mathematics was an austere and rigidly circumscribed subject for which he offered no justification outside of its own integrity.

According to Cauchy, "let us then admit that there are truths other than those in algebra, realities other than those of sensible objects. Let us ardently pursue mathematics without trying to extend it beyond its domain."

As time advanced during the nineteenth century, Cauchy's view tended to win out over Monge's view. Public assertions that scientific knowledge is irrelevant to humanistic education can be found as early as 1802. The final years of Gaspard Monge epitomized this change from a holistic to separatist view of mathematics. Monge taught geometry at the Ecole Polytechnique until the restoration in 1816 at which time the school was closed for several months. What is more, he was summarily removed from his position in the Institute and another appointed to take his place. He died miserably soon thereafter.

If Cauchy's separatist view of mathematics began to prevail in French mathematics in the early nineteenth century, then one must also conclude that d'Alembert's view of mixed mathematics must be waning during this period. If mathematics is to be done separately from other disciplines, how could it be mixed with these disciplines? On the other hand, it makes perfectly good sense to talk about how a body of mathematics could be *applied* to some discipline. Since the mathematics had been already established, one could simply "lift" the needed mathematics from its total embodiment and "insert" it where it could be applied in that discipline.

(b) William Whewell

As Cauchy's approach to mathematics was winning out over Monge's approach on the European Continent, the changes in mathematics in England were far more gradual. The English had a more "exemplary" view of mathematics which tended to grow stronger, both institutionally and philosophically, well into the middle of the 19th Century. Their view was based upon Newton's approach to calculus using fluxions and to the unified Enlightenment view of science. By 1800, Newton's approach to the calculus was still being taught at institutions like Cambridge. This approach was more rigorous than the Ancient Greeks' perspective of geometry but more difficult to learn and apply to problems than the calculus of Cauchy. There was an early 19th century awakening of English interest in Continental Mathematics led by a short-lived organization of Cambridge students known as the Analytic Society (1812-1813). This group led by Charles Babbage, George Peacock and John Herschel advocated the more analytical methods of calculus using Lebnitz notation. Although this group died out, many of its aspirations had effects on how mathematics was taught at Cambridge. By the 1820s, the Cambridge mathematics exam, the Tripos was radically modified to include some of these analytic techniques. However, the most powerful theme at Cambridge still remained the *connectedness and universality of knowledge*. The person that epitomized this approach was William Whewell.

In 1800 the mathematics curriculum at Cambridge included arithmetic, algebra, trigonometry, geometry, fluxions, mechanics, hydrostatics, optics and astronomy. The focus was from Newton's *Principia*. When Whewell came along there was pressure from the Analytic Society to change the curriculum towards Continental mathematics. Whewell forged a compromise. He wanted the students to be grounded in *physical realities*—in pulley machines, and forces rather than mathematical symbols. In his textbook *Elementary Treatise on Mechanics* (1819) he placed a considerable portion of mechanics prior to any discussion of differential calculus. Today, we argue that calculus should be taught before and at worst concurrently with mechanics. Whewell thought the best way to achieve rigorous results was through *geometric-physical reasoning* and to confirm the results thus achieved by "more direct" reasoning. It was his mission to prevent the establishment of the study of abstract analysis as a discipline independent of, and as prestigious, as mixed mathematics. He accomplished this mission with his many textbooks that were primarily used during the first half of the 19th century and his influence in the writing of the Tripos exams. This extended even as late as 1848 when many changes were being considered with regard to the Tripos exam. By 1854 (James Clerk Maxwell's year) the Tripos exam consisted of a ratio of three applied mathematics problems to every two pure mathematical problems; problems calling for synthetic-geometric solutions, including Newtonian ones, made up more than 40% of the examination. Obviously the emphasis was still on

Whewell thought the best way to achieve rigorous results was through *geometric-physical reasoning* and to confirm the results thus achieved by "more direct" reasoning.

mixed mathematics, i.e. geometry over algebra, intuitive rather than abstract rigor, on detail more than generization, extensiveness more than intensiveness, on problem solving rather than mathematical processes, and on Newton rather than Lagrange.

Even in the 2nd half of the 19th century mixed mathematics did not go away. There were battles between Arthur Cayley supporting Continental Mathematics and Henry Airy supporting mixed mathematics. Only after another generation of

mathematicians led by Bertrand Russell and G.H. Hardy do we clearly see a decline in mixed mathematics.

VI. THE DECLINE OF MIXED MATHEMATICS IN THE NINETEENTH CENTURY

(a) A Consequence of the French Revolution—Changing Views of Probability Theory

As mentioned earlier, the rise of military and engineering schools brought an increase in the study of mixed mathematics. Furthermore, the importance of mathematics as a method of searching for the truth was important to some Enlightenment philosophes. All of these trends collided with the tremendous upheavals of the French Revolution.

The French Revolution seemed to create a discontinuity and a rethinking of some of the trends of the 18th century. Philosophes of the 18th century stressed the importance of enlightening the individual. Society would ultimately prosper only if its citizens were enlightened individuals. This meant the individual had to master rational thinking which was based upon the reasoning of geometry (synthesis) and to a lesser extent algebra (analysis). Mathematically, this mean that psychological arguments could be mixed with mathematical technique in solving problems such as the St. Petersburg paradox and the Inoculation Problem.

The events of the French Revolution seemed to shatter this belief that "good sense" was monolithic and a constant for a selected few. Passions seemed to prevail over reason, and what was "good sense" and who practiced it was no longer so clear. The chaos and many political shifts from the Revolution in 1789 to the restoration of the Bourbons in 1814 shook the confidence out of the remaining philosophes, some of whom suggested that "good sense" is more intuitive than rational. Mixed mathematics involved the proper combination of theory with experience, but by the time of the restoration of the monarchy, these ideas seemed to some philosophers diametrically opposed.

While the philosophers of the Enlightenment thought the way to improve society was by concentrating on enlightening the individual (individual→society), the social scientists of the 19th century thought one can improve the lot of the individual by improving society (society→individual). This change of worldly views especially affected how probability theory was used. The philosophes used it by considering the rational individual while the social scientist used it in statistics to say something about the "average person with property A." For example, Quetelet used the frequentist interpretation of probability theory and the normal distribution to find the mean

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in the case of statistical regularities of data, and thus associated a number to some social trait of the average man. The average man could be assigned "a penchant for crime" equal to the number of criminal acts committed divided by the total population. In this way a set of discrete acts by distinct individuals was transformed into a continuous magnitude, "the penchant," which was an attribute of the average man. *It was the responsibility of the social scientist to apply the mathematical theory to the social chaos*; the normal distribution was *applied* to all kinds of social (and later biological) phenomena.

(b) Some Mathematical Discoveries of the 19th Century

If the rise of statistical methods affected how scholars viewed the role of mathematics in the "real world," then mathematical events occurring later in the 19th century had a further influence on this changing view of mathematics. Here I am referring to (1) the discovery of non-Euclidean geometry, (2) the discovery of non-commutative rings (i.e. Hamilton's quaternions), (3) the discovery of nowhere differentiable but everywhere continuous functions on the real line, (4) the changing relationship between geometry and mechanics.

The discovery of non-Euclidean geometry shifted the foundations of mathematics towards arithmetic

and away from geometry. No longer was there an example of absolute, infallible knowledge. Furthermore, the sensory perceptions could no longer be trusted. This was especially re-iterated with the discovery of non-intuitive mathematical entities like the quaternions and nowhere differentiable, everywhere continuous functions. The truth of such propositions must rely on more formalistic arguments and less on experience.

With regard to mechanics, recall d'Alembert used his rational mechanics as a model for the other mixed mathematical sciences. His rational mechanics was kind of an extension of geometry with three axioms of motion added. However, with the discovery of non-Euclidean geometry and the rise of experimental physics in the 19th century some physicists thought that mechanics was less a branch of geometry and more one of analysis (calculus).

VII. THE CHANGING MEANING OF APPLIED MATHEMATICS TODAY

In a recent article appearing in the July, 1993 issue of the *Bulletin of the American Mathematical Society*, Arthur Jaffee and Frank Quinn discuss "Theoretical Mathematics: Toward a Cultural Synthesis of Mathematics and Theoretical Physics."

According to Jaffe and Quinn, mathematics should be divided into "theoretical" and "rigorous" mathematics. They argue that applied mathematics is generated in two stages: First intuitive insights are developed, conjectures are made, and speculative outlines of justifications are suggested. This they call "theoretical" mathematics. Then the conjectures and speculations are corrected and are made reliable by proving them. This they call "rigorous" mathematics. Later in their article, they say "pure" was used in the past instead of "theoretical" but the term "pure" is "no longer common."

Jaffe and Quinn argue that mathematicians have even better experimental access to mathematical reality than the laboratory sciences have to physical reality. They say, "this is the *point of modeling*: a physical phenomena is approximated by a mathematical model; then the model is studied precisely because it is more accessible." Today, the same person is likely to be doing "theoretical mathematics" and "rigorous mathematics" in solving a problem. This is especially true in the areas of *string theory*, *conformal field theory* and *topological quantum field theory*.

Finally, they argue that the bifurcation of mathematics into theoretical and rigorous communities has partially begun but has been inhibited by consequences of *improper speculation*. They argue that "speculative mathematics" ought to be publishable but with the stipulation that it can be acknowledged as speculative.

VIII. CONCLUSION

I wish to argue that applied mathematics should be taught "mixed" allowing for "speculative mathematics." Solving mathematical problems has always involved factoring in "experience" even if that experience could be non-intuitive. To say that some mathematics was applied to some physical problem is a distortion and misleading. This view tends to trivialize the modeling process.

"Mixing mathematics" involves breaking down a given problem into simpler parts until one arrives at "first principles." One is supposed to create a chain of truths starting from these first principles and logically arrive at a solution to the problem. It seems reasonable in today's changing mathematical world, that this chain could include speculative statements, conjectures, refutations, data accumulation, experimental strategies and even some incorrect conclusions as long as this chain is constructed in a logical progression. The student could be required in some sort of notebook to include any analytical or synthetic way of thinking involved in the solving of a problem. This may include the stripping down to "first principles" that have no obvious or apparent relevance to the original problem. It could include the use of "common experience" to make conjectures or

Why should acceptable mathematics always be in some finished form free from dead ends and speculations? Why should acceptable mathematics be free from the motivations that lead to the theorems?

refutations. It would be the opposite of the lean, economical and formalized kind of written mathematics emphasized today. Why should acceptable mathematics always be in some finished form free from dead ends and speculations? Why should acceptable mathematics be free from the motivations that lead to the theorems? Why does it have to appear from the finished product that mathematics has been applied to a problem when it was really mixed? Why do we continue to reinforce this distorted view of applied mathematics? Mathematics should be taught mixed and clearly advocated to students as being mixed.

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Footnotes

* This approach to modeling appears in Bittinger (1992), page 76. It would be an interesting study to examine to what extent the Bittinger diagram represents the presentation of modeling in elementary mathematics textbooks over the last fifty years. I am conjecturing that it would typlify other such diagrams.

** For a more thorough, scholarly explanation, including complete citations and footnotes, of the historical evolution of the term "mixed mathematics" see Brown (1991) and Brown (to appear).

Female Voices in Mathematics: A New Course

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SUMMARY:

For the last two January terms, I have offered a new college level course: Female Voices In Mathematics at St. John's University and the College of St. Benedict, Collegeville, MN. This paper is an attempt to share with others not only the content and syllabus of the course, but also the experience of teaching it so far. My hope is that this paper will not only help me get in touch with colleagues who are offering similar courses in their institutions so we can exchange ideas, but also prompt others to offer similar courses. I also wish that more courses of this nature would be offered everywhere so as to help raise awareness of the trials and tribulations of women in mathematics.

The thought of offering a course on women in mathematics occurred to me at the MAA Summer Conference at Boulder, CO, in 1989 when I attended Prof. Miriam Cooney's workshop on women and mathematics.[1] I felt that such a course was vital not only to underscore the contributions of the women mathematicians, but to make our students aware of these women's uphill struggles to study mathematics in spite of the social and cultural discriminations of their times. It is often said that women have contributed little to the field of mathematics, but when we read the stories of the few women mathematicians who did, we can not but help wonder how they ever contributed at all! In comparison, it would be difficult to find even a single male mathematician who faced as many difficulties as these women and still persisted in developing his passion for mathematics.

This course also seemed an ideal place to talk about the contemporary gender issues such as the supposed differences in mathematical ability, social and cultural stereotyping of sex-role attitudes, and subtle sexism in the classrooms—displayed sometimes by teachers and present in some text books. To make our students aware of the societal attitudes that supported discrimination against women's education in mathematics in Europe and America even up to the 19th century, I decided to bring in the stories of the struggles of women through the centuries to get education. This provided them with an excellent perspective to appreciate the discrimination faced by Christine Ladd Franklin when Johns Hopkins University in 1882 refused to grant her a Doctorate in mathematics—a degree which was eventually awarded to her when she was 74 years old.[2] Looking at the continuum of women's struggle to get education helped create in the students a sensitivity to how much women in the United States had achieved in the last one hundred years. It also made it easy for them to understand that even a century later in 1980, highly qualified women Ph.D's were still facing problems getting jobs and especially tenure at highly prestigious mathematics departments.[3] The nature of barricades restraining women's entry into a male-dominated field such as mathematics simply seemed to have shifted with the times, but they were still there! And yet in the 1990s, the struggle was not completely over—there still existed a glass ceiling preventing women from attaining prestigious positions in the Ivy League Institutions, probably one of the very last frontiers challenging women in mathematics! [4,5]

COURSE DESCRIPTION:

This is a college core curriculum course that is offered with a gender flag for two credits at Saint Johns University and the College Of Saint Benedict, Collegeville, MN, during the short three week January Term—2 hours of class every day for 3 weeks. (Total 30 hrs.)

The Course Catalogue listed the objective of the course as—

(i) To discuss topics such as is math anxiety due to social and cultural stereotypical images and is there really a

gender gap in the mathematical abilities of boys and girls.

(ii) To build appreciation of women's struggle to contribute to the field of mathematics and especially to help students debunk the stereotypical image that women can not do math.

The course has no prerequisites and the enrollment is limited to 30 students. In January '92 when the course was first offered, the class consisted of about 16 men and 12 women, whereas in January '93 there were 17 men and 8 women registered for the course. In retrospection I can say that to facilitate more meaningful discussions and dialogue on gender-based issues, it would be ideal to restrict the class to about 15 men and 15 women.

The course content is covered using informal lectures, videos, and group discussions on assigned readings. The three weeks are planned approximately as follows:

Part 1 (12 hrs.) - Math and Gender: Biology or Culture?; Math anxiety in women; Sexism in language, media, textbooks and classrooms.

Part 2 (8 hrs.) - History of European women's struggle (up to the 18th century) to educate themselves in mathematics: Societal attitudes towards women's studying of mathematics - an excellent background to talk about the life stories of Mary Somerville, Sonya Kovalevskaya, Sophie Germain and others.

Part 3 (8 hrs.) - American women's struggle to get education, and life stories of some American women mathematicians of the 20th century.

Student evaluations are based on participation in group discussions in the class on assigned materials, a paper on math and gender at the end of the first week, a research paper and presentation (by pairs of students) on the life story of a woman mathematician not covered in class, and a final examination consisting of short essay type questions.

COURSE:

On the first day I acquaint the class with the story of my career in mathematics and collect data from the students on (i) the number of math courses taken during school and college; (ii) attitude toward mathematics along with reasons for their attitude; (iii) people who helped shape this attitude—parents, teachers, peers, etc.; (iv) names of men and women mathematicians they had heard of; and (v) attitude towards women math majors that they knew (if any).

During both January '92 and '93, the class had a maximum of two or three math majors only. The rest of the students were from a variety of majors such as accounting, psychology, economics, management, biology, etc. On the average most of them had a fair attitude towards mathematics, had parents who mostly had encouraged them to study math, and of course, they had not heard of any women mathematicians in any of their courses (the math majors had heard of Emmy Noether only!). Most did not view women math majors differently, but the only two women math majors in the class in Jan. '93 did not think so. One wrote that whenever she informed other women students about her major, "...they would krinkle up their nose and say, 'Oh, I could never do that - I'm not good with numbers.'" The other wrote, "I was identified as the brain in my high school. My classmates thought I was so smart. This made me afraid to ask questions for they would then tend to emphasize the fact that I was the one asking questions. It was okay for any one else but me to ask questions." To my question if they had ever noticed any sexist attitudes on the part of their teachers, they all gave a clear negative answer on the first day. This perception however changed by the end of the first week during which we talked about the subtle gender inequities present in many math class rooms from the grade level upwards, not excluding the discrimination by teachers (both male and female) towards women students.[6,7] Some women students in their papers on math and gender at the end of the first week remembered instances such as:

"In high school I took an Algebra II class and the teacher gave everyone a hard time, but I seem to remember him encouraging the men to get more involved and ask more questions, whereas if you were a female you didn't want to ask questions because there was a good chance that you'd be ridiculed. The teacher's ego was incredible."

"My sister told me a story once of a friend of hers who had a most humiliating experience her first day of math class at a large university. She had

just walked in and sat down in the front row with four other female students. The professor walked in soon after and asked the five students to sit in the back because they obviously weren't going to learn anything in the class and (that) they would take the other male students' attentions."

"I had negative experience in math in (the) 9th and 11th grades and my freshman year in college. In 9th grade, my teacher wasn't willing to help me because I didn't ask my question while he was up at the board. I was a nuisance to him. In 10th grade, my teacher showed no interest in girls unless they played basketball and were mathematically gifted. The freshman year my professor seemed scolding to me and made me feel like a peon."

"During my freshman year of high school, the male (math) teacher was extremely intimidating, and gave out hours of detention like they were candy. He would then become upset with the class if we did not ask questions about what we did not understand, but when we did, he would ask us how to do the problem. We obviously did not know. He, then, would become frustrated because we did not know how to do it. It was a horrid class."

Next I start the class with discussion on the controversy about genetic ability in mathematics as provided by Benbow and Stanley's study and the resulting hoopla in the media.[8-13] Fennema and Pat Rogers' articles on sex-related differences in mathematical ability and achievement are discussed in small groups.[14,15] Other articles, books, and videos provide excellent resources for thought-provoking discussions.[16-21] I share Fausto-Sterlings view with the class that in spite of the growth in the literature devoted to the measurement of sex differences, on close examination the material appeared to be empirically bankrupt.[22] And yet, even in Aug.'92, the media still made juicy headlines such as: "Female brain smaller..."[23]

I use some excellent videos to emphasize how language and culture draw such subtle images of stereotypical gender roles in any society.[24-26] These videos provide not only a relevant alternative to lecturing, but also lead to a good exchange and sharing of related personal experiences. Group discussions are followed by each group sharing with the class some examples of how mostly subtle and at times not so subtle messages are given out to women as to what is expected of them.

Next we discuss sexist representations of gender roles and professions in elementary math text book illustrations and problem sets.[27-29] We look for examples of sexist bias in the form of illustrations, professions depicted as performed by women, and problem sets. I share with them some old mathematics text books, very similar to the text I had used in my elementary school.[30,31] Most of the illustrations show little boys doing some activity and little girls always watching—giving the impression that male figures were strong, silent, active and assertive. In most of the problem sets either males were depicted as farmers, doctors, shoemakers, or owners of shops etc., or characterized by distinguished activity like buying property, whereas women were categorized either by physical characteristics such as brown hair, blue eyes, etc., or shown to be doing sewing, shopping and other womanly chores. The class is given an assignment to leaf through elementary text books in the education department or the school libraries and come up with an evaluation using Kepner's format which is shown below. For each specific text at any grade level, they are asked to record not only the frequency of male/female representation in illustrations and problems, but also the type of activities and occupations listed.

Textbook:				
	Illustrations		Problems	
	Male	Female	Male	Female
1. Frequency				
2. Occupations				
3. Activities				
4. Exs. of sexually neutral language				
5. Exs. of sexually blatant language				

Students share, compare and discuss their findings first in small groups and then with the class. One male math/ed. major in Jan. '93 expressed that he was first surprised by even the mention of this topic and confessed that as a student he had never given any thought to the subtle sex stereotyping in math texts. I also share with the class the findings of Kuhnke and Kepner which have reported that the elementary math text books of today were very different from those of the 1940s. An all out effort had been made in the 1970s to eliminate sexist bias

It is often said that women have contributed little to the field of mathematics, but when we read the stories of the few women mathematicians who did, we can not but help wondering how they ever contributed at all!

and discriminatory sex-role stereotyping from the curriculum. The sex role stereotyping had continued to change in the text books between 1975 and 1980. Women were being shown in active roles and capable of achievement and men making mistakes and doing household tasks. Though gender neutral language was being used more often than in the past, very few models for aspiring female scientists and mathematicians had yet appeared.[30-31] The publishers seemed to choose a safe nonsexist route but not an openly antisexist route. That the situation in sciences had not been all too different was reported by a student who shared Nilsen's findings with the class.[32] I can think of a similar evaluation activity that students may engage in for the college level math and science text books to find out how women scientists and mathematicians had fared in them.

We end the first segment of the course by looking at the psychological determinants of educational and occupational choices, and discussing two 1980 appraisals of female mathematicians and their personalities.[33-35] I share recent data regarding participation of women in mathematics and sciences with the class and make them aware that the situation has greatly improved today, but even in 1993 the National Council of Teachers of Mathematics (NCTM) sometimes has to raise its voice against the general perceptions portrayed to little girls by Barbie dolls that mathematics is tough.[36-58]

At the end of the first part of the course, students are given an assignment to write a paper on mathematics and gender using any of the readings, videos, and/or discussions in the class. I encourage them to do their own search for other materials. I have listed below some of the topics they have written on:

(a) Sexism in language/ media / elementary math text books.
(b) Math anxiety in women.
(c) Teacher's role in a math class room.
(d) Women and the nature of mathematics.
(e) Are women less mathematically able?

Both in January '92 and '93 when I have taught this course, I have felt that the amount of literature/videos available on math and gender is quite exhaustive, and provides more material for thought and discussion than can be handled in 12 hours of class time. Students need time to ponder over these issues and one suggestion to keep track of their individual thoughts might be to ask them to keep a daily journal of reflections during this part of the course. Some other lively activities that can be developed to conclude this topic can be as follows:

(a) Class dialogue on some current stereotypes in our culture concerning women and mathematics (or science).
(b) Class debate on "Should professional women be like professional men?" (I have tried this with great success both times. Students became quite excited and enjoyed challenging their opponents. Thanks to Ruth Hubbard's paper for the topic.)[40]
(c) Class skits on teacher/student interactions in the mathematics classroom (Leder, Berson).[41,42]

The second part of the course moves on to sharing stories of women mathematicians. We start from the golden era (c.3000 - 1000 B.C.) when "a wave of feminine genius passed over the fragrant valleys and the vine-clad plains of ancient Egypt. Never in any other place or time..was there a more perfect flowering of female intelligence of the highest order".[43,44] The story is unfolded in a sequential fashion so as to give the students a perspective of women's struggles through the centuries. From the classical times of Aspasia, and Hypatia we see a gradual decline in the status of women through

the middle ages to the Renaissance to more modern times. Immanuel Kant's, "All abstract speculations, all knowledge which is dry, however useful it may be, must be abandoned to the laborious and solid mind of man.... For this reason women will never learn geometry", or De Lamennais' sharing of Kant's opinion concerning woman's inferiority, "I have never met a woman who was competent to follow a course of reasoning the half of a quarter of an hour—un demi quart d'heure, but, in the matter of reason, logic, the power to connect ideas, to enchain principles of knowledge and perceive their relationships, woman, even the most highly gifted, rarely attains to the height of a man of mediocre capacity." Mozan expressed attitudes that clearly reflected existing prejudices against women's studying of math or science for many centuries. Hypatia's cruel death, Mary Somerville's "saving Euclid for the night", Sophie Germain's "M. le Blanc", Hilbert's "Meine Herren,....the Senate is not a bathhouse" in support of Emmy Noether's employment at Gottingen, and Sonya Kovalevskaya's marriage of convenience, all provide us with some very touching stories that inform and inspire students and fill them with awe at the courage, determination, and strength of character of these women to persist in their studies in spite of the great odds against them.[46,47]

The struggle of the British ladies of the 18th century to acquaint themselves with the happenings in mathematics and science through the Ladies Diary, magazines, visits to the museums, and the public lectures is also very well documented.[48,49] Students are surprised to learn of a precursor of today's feminist ideology as reflected in a "Lady Of Quality" writing in 1721—"I think it is high time to

"Most Americans of the 1790s and early 1800s considered educated women a threat rather than an asset to society," wrote M. Rossiter.

look about us, and to vindicate Our sex; to let them know the value we ought in justice to set upon ourselves; to rouse up our courage, and fire our Breast with a worthy Indignation, and Resentment against such inhumane Treatment as we daily meet with that we may no longer give Pre-eminence to such vain, thoughtless, and ungovernable Animals, as Men of what Denomination soever". This historical scenario offers an excellent insight into the life sketches of Ada Lovelace, Mary Somerville, and Caroline Herschel.

Moving on to the period of colonization in U.S. history, students are shocked to learn about the prevailing antifeminist attitudes of the times. "Most Americans of the 1790s and early 1800s considered educated women a threat rather than an asset to society," wrote M. Rossiter.[50] Emma Willard's "Republican motherhood strategy" to make school education accessible to women under the pretext of making them "better mothers and wives", the establishment of women's colleges in 1830s, the struggle for co-education and eventually for graduate admission to prestigious research universities in the late 19th century surely surprised my students.[51] Most of them had never heard about these struggles so close to home and they expressed a feeling of being enlightened to learn that the education today that they take so much for granted had become available to women after such long struggles.

Armed with this knowledge of the history of women's efforts to get education, I let students work in pairs to prepare life sketches of women mathematicians (and scientists) not covered in class. Each pair submits a paper and gives a presentation on these autobiographies. Students have mentioned enjoying this assignment since it gave them a chance to really understand the character and idiosyncrasies of some of these mathematicians. These presentations also helped cover in class some recent biographies of Mary Rudin, Cathleen Morawetz, Julia Robinson and others.

Moving onto the 1970s, I introduce to students the Association For Women In Mathematics [AWM] and its role in encouraging participation of women in mathematics through different activities.[52-54] Lastly we look at some recent data from the 1980s and talk about the glass ceiling as experienced by women professors in getting jobs and tenure at some of the prestigious Ivy League institutions even today.[55-57] The Dec. '92 CNN video 'Beyond The Glass Ceiling' really opens their eyes to the prejudices women still face in today's corporate world and shows them how these women at the frontier are fighting to shatter this glass ceiling.[58]

On the last day of lectures I have invited one or two female colleagues to share with the class their experiences of first being students and then professors of mathematics. Students have enjoyed listening to these stories of struggle and survivals in the context of the current world scenario. One can also invite women math majors working in business and industry to talk to the class.

CONCLUSION:

This course has been taught twice so far and the organization of the course as described above provides a comprehensive story of female voices in mathematics. Discussing women mathematicians' struggles in the context of the over all struggle of women over the centuries to get education draws a very compelling picture. The earlier part of the course discussing the sexist biases prevalent in society, culture and language, classrooms, etc. helps to quickly involve the students into the course, since they are able to relate to what is discussed in class. Some of the comments I have received on the course evaluations have asked for more discussions, group projects and class activities. With the time constraint for a January term course I am limited in how much can be done in three weeks, but I do look forward to comments and suggestions from the readers that will help me expand this course into a full semester course.

I am delighted by the enthusiastic support this course has received so far:

"I have really enjoyed this course. It brings a lot of hidden facts out front."

"There is a need to see where women fit into math and how sexism plays a role in society, and there are no other courses that address all these topics."

"Before the course began I was not looking forward to it. I thought that it was going to be boring. But after it began, I really started to enjoy it. It amazes me that there is so much discrimination in the field of mathematics. I was never aware of this. I definitely think that this course should be offered again."

In closing, I take the following comment by a male student as a compliment to the success of this course in projecting the voices of women in mathematics:

"My reaction to this course is a sense of accomplishment. I came in apprehensive because of the subject matter, but I enjoyed it. Sure offer it again—the more knowledge that can be spread about this topic, the more educated us dumb men can get!"

NOTES

1. Miriam P. Cooney, MAA workshop on women and mathematics, Summer 1989, MAA Conference, Boulder, CO.

2. Margaret W. Rossiter, *Women Scientist in America: Struggles and Strategies to 1940*, (Baltimore, MD.: Johns Hopkins Univ. Press), 1982.

3. See "More Women Are Earning Doctorates.....by Top Universities", *Chronicle of Higher Education*, December 6, 1989, A 13.

4. See "Does Harrison Case Reveal Sexism.....?", *Science* 28, June 1991, 252 1781-83.

5. Jenny Harrison, "Escher Staircase", *Notices of the AMS*, September 1991, 729-734. (This entire issue is devoted to women and Mathematics.)

6. Myra Sadker and David Sadker, "Sexism in the Classroom: From Grade School to Graduate School", *AWM Newsletter* 20, 6, Nov.-Dec. 1990, 11-14.

7. Elizabeth Fennema, "Teachers and Sexbias in Mathematics", *The Mathematics Teacher*, 73, March 1980, 169-73; See also "Teacher's Beliefs and Gender Differences in Mathematics" in *Mathematics And Gender*, ed. Elizabeth Fennema and Gilah Leder (New York, Teacher's College Press, 1990).

8. C.P. Benbow and J.C. Stanley, "Sex Differences in Mathematical Ability: Fact or Artifact", *Science* 210, 1981, 1262-64.

9. Alice T. Schafer and M. W. Gray, "Sex and Mathematics", *Science* 211, 1981, 231.

10. See "Mathematical Ability: Is Sex a Factor?", *Science* 212, April 10, 1981, 114-21.

11. See "The Gender Factor in Mathematics", *Time*, December 15, 1980, 57.

12. See "Math and Sex: Are Girls Born With Less Ability?", *Science* 210, December 12, 1980, 1236. Also "Do Males Have a Math Gene?", *Nature*, December 1980.

13. See "Are Boys Better at Math?", *The New York Times*, December 7, 1980, 102.

14. Elizabeth Fennema, "Sex-Related Differences in Mathematics Achievement: Where and Why", in *Women and The Mathematical Mystique*, ed.Lynn H. Fox, Linda Brody, and Diane Tobin (Baltimore, MD: Johns Hopkins Univ., 1980),76-90; also Margaret R. Meyer and Mary Koehler, "Internal Influences on Gender Mathematics", in Mathematics and Gender, as in note 7 above.

15. Pat Rogers, "Gender Differences in Mathematical Ability: Perceptions vs. Performance", *AWM Newsletter*, 19, 4, July-August 1989, 6-10; See also Elyse Sutherland and Joseph Veroff, "Achievement Motivation and Sex Roles", in *Women Gender and Social Psychology* (Hillsdale, N.J.: Erlbaum, 1985).

16. See "Sizing Up the Sexes", *Time*, January 20, 1992, 42-51.

17. See "Men vs. Women", *U.S. News and World Report*, August 8, 1988, 50-56.

18. Londa Schiebinger, *The Mind Has No Sex* (Cambridge, .Mass.: Harvard Univ. Press, 1989).

19. See *Proc. Of the National Conf. On Women In Math And Sciences*, ed. Sandra and Phillip Keith, St. Cloud State University, MN., November 1989; Also Dorothy Buerk, "The Voices of Women Making Meaning in Mathematics", *Journal of Education*, 167, 3, 1985, 59-70.

20. Ann Fausto-Sterling, *Myths of Gender: Biological Theories About Women and Men* (New York: Basic Books, 1985); Also Susan F. Chipman, Lorelei R. Brush, and Donna M. Wilson, in *Women and Mathematics: Balancing the Equation* (Hillsdale, N.J.: Erlbaum, 1985).

21 See video, *Sexual Brain* Community Television Of Southern California, Princeton, N.J., 1987. Also Bill Moyer's, *Science and Gender With Evelyn Fox Keller*, PBS, Alexandria, VA., 1990.

22. Ann Fausto-Sterling, "Women and Science", *Women's Studies International Quarterly*, 4,1, 1980, 41-50; See also Ann E. Kramer, *Science, Sex, and Society* (New York: WEEA Publ., 1979).

23. See "Female Brain Smaller, Rushton Report Says", *The Toronto Star*, August 16, 1992.

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* I am extremely thankful to Professor Miriam Cooney, Saint Mary's College, Notre Dame, Indiana for her constant guidance and encouragement.

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MATH
It's Not Just a Four Letter Word

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Ahhh, arithmetic—those were the days. Addition, subtraction, multiplication tables. Then things got complicated. Along with those familiar numbers suddenly they added x, y, π, and all kinds of other symbols that certainly looked more at home in the alphabet than combined with numbers.

Like most people, I made it through algebra and geometry. To this day I do not know how, but I did. However, a life long fear of higher math has plagued me ever since those long ago high school days.

The decision to further my education came many years later. Every semester the "math requirement" for my associate degree loomed overhead. Like death and taxes it was inevitable. I continued to put off signing up for a math course, knowing full well my days were numbered.

If only math were something more than *just* numbers. Well, Virginia, there is more, much, much more.

In the Fall of 1991, I approached one of our math professors and discussed the possibility of taking an independent study course relating to math. With his approval and guidance, it was decided that I would study independently and write papers pertaining to mathematics and culture. One book in particular, given to me by my professor, opened new doors of understanding and a desire to learn more. My enthusiasm for this book is unqualified.

Marcia Ascher's book *Ethnomathematics*[1] has a wealth of information that should appeal to teachers, math majors, and those who consider themselves math-phobic. It certainly appealed to me; it opened new paths to understanding mathematics and its origins. Instead of square roots and multiplication tables, I learned about the Quipus invented by the Incas of Peru. Consisting of a variety of colored and knotted cords attached to a base rope, the Quipu was used for calculating and recording numbers. Perhaps after reading and learning about this interesting "calculator" of long ago, children could make a Quipu and use it in some project. What a great way to learn Math!

A section of the book explores number words with many examples of how cultures make different use of the counting facility; some generate lots of number words and a very few generate none. Nahuatl is a language of central Mexico. Nahuatl and Mayan numerals have a cyclic pattern based on twenty. One of my favorite examples is the numeral classifier, wich is a term that is included when number words are spoken with nouns. Their purpose is to convey information about the nouns that is qualitative rather than quantitative but a necessary part of a quantitative statement. While some languages have as few as two classifications others have as many as two hundered.

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The language of the Maori, the indigenous people of New Zealand, illustrates the numeral classifier. When a statement is made about a number of human beings, it must contain the classifier for humans. For example, "five humans women" or "five humans Americans." A clear distinction between classes is possible, human versus everything else.

The use of numeral classifiers by the people of the Gilbert Islands shows us how a language uses its

classifiers to reflect the specifics of their environment. There are 18 classifiers from (a) animates (except for fish longer or larger than people), spirits, and ghosts to (r) general (often replacing other classifiers of inanimates and used for serial counting). The letter, (p), means all modes of transportation. So, if we were to tell someone that we had six boats it would be as if we were to say "sixp boats." The Gilbertese classifiers become affixed to the number words. It was fun to use the list of classifiers given and try to make up sentences using this method of communicating quantities of objects, people, anything.

The Kusaiean language has just two classifications which also combine into number words. It is much simpler than Gilbertese and yet reflects the specifics of their environment.

While the drawing of continuous figures in the sand superficially appears to be a children's game, it is actually part of a widespread storytelling tradition among the Tshokwe. The figures called sona are drawn exclusively by men. It is primarily older men who are knowledgeable and proficient in the drawing skill. To draw the sona, an array of dots is first constructed. The continuous figure is drawn surrounding the dots without touching them. Aspects of the Tshokwe culture need to be presented to help comprehend the figures and their associated stories and names. Learning about the Tshokwe culture and their sand tracings is a wonderful walk through another world. This is a recognizable form of what we now call "graph theory".

Ascher tells the story of Euler in Königsberg and the beginnings of modern graph theory. According to the story, seven bridges spanned a forked river that separated the town into four land masses. The townspeople were interested in knowing if, on their Sunday walks, they could start from home, cross each bridge once and only once and end at home. Euler showed that for the particular situation, such a route was impossible. We read on to find out why and are given many different graphs to illustrate what became known as Eulerian paths and also graphs where no Eulerian paths exist.

We are even given a look at kinship of people and how this can be related to mathematics. The Warlpiri, who live in a desert area in Australia's Northern Territory, have a particularly complex kin system. It consists of eight sections and each person is in one of them. Preferred marriages are to take place with persons from other specified sections, and their children are in another section, which depends on the section of the mother. We are given sections 1 - 8 and also a diagram of the marriage rules. One can trace through several generations using this diagram. What a great way to introduce math to children or any age group for that matter.

And, of course, we can all identify with games of chance and those involving strategy. Games of chance such as bingo and roulette are games in which players may bet but they make no choices that affect the outcome of the game. Winning or losing is out of their control. However, in games such as checkers or chess, each player does make choices. In games of chance, there is involvement with concepts of probability.

A popular game of chance among Native Americans involves the use of a dish and some small flat disks. The objects are shaped and decorated or colored so that each has two faces that are different from each other. Usually there are six

From Quipus to sand tracing, it becomes evident that Math is more than just numbers. It's a fascinating world of information that stretches far back into the past, a past that holds a wealth of knowledge for even the most skeptical person.

or eight disks. One of the two players places the disks in the dish and, by striking or shaking the dish, causes the disks to jump and resettle. The resulting assortment determines the number of points won and whether or not the player goes again or must pass the dish to his opponent. In some cultures, such as the Cayuga, the dish was a wooden bowl and the disks were six flattened peach stones which were smooth and had been blackened by burning on one side. Along with the description of this game, we are also given a look at some of the probabilistic implications. This game reminded me of many childhood games

which were similar in several ways. While playing these games as a child, it would have been of interest to learn in school that the games my friends and I had been playing were like those enjoyed around the world in many other cultures. Examples of these games and possibly giving the opportunity to play the games of other peoples by making the game pieces as they would, could be a helpful tool in building a foundation for math.

From Quipus to sand tracing, it becomes evident that Math is more than just numbers. It's a fascinating world of information that stretches far back into the past, a past that holds a wealth of knowledge for even the most skeptical person.

Didn't you ever wonder why people, even very young children, are able to sing a song and remember each and every word? Perhaps that is because we view singing a song as pleasurable—it makes us feel good. Maybe a similar approach to teaching math and sciences is the answer. When the fear is removed and one finds learning pleasant and no longer intimidating, one can begin to explore new worlds that once were thought to be unreachable.

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Poetry by Helen Lewy

Presented at the Mathematics Poetry Reading, Jan 14, 1994
Joint Mathematics Meetings, Cincinnati, OH

THE MATHEMATICIAN COMES HOME

"Hello, my dear,—and how are you?"
"I'm fine. I baked a cake!
And how was your day, Husband Mine?"
" Oh, just the 'standard take'."

"What kind of goodie did you bake?"
"Upside down" she quoth
" Oh, no! Not that!!" he cried in pain,
"It' s happened to us both!

For some eccentric Fate of ours
Is playing Cosmic Clown:
The math I did today,—it, too,—
Came out quite upside down!!"

A programmer living in Gates
Was subject to odd sorts of states;
In a rage, a while back,
He hacked up his Mac,—
Now his future is up to the Fates!

A researcher in Algebra (Linear)
Didn't dare to talk math during dinnear!

Mathematics in Literature and Poetry

by
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This article is a written version of my presentation, "Encourage a Variety of Interests," at the AMS-MAA Joint Meetings in Cincinnati, Ohio on January 15, 1994.

I find that students are pleased to learn that mathematics can be found in literature and poetry as well as in the sciences and finance and other traditional "applications." Acquainting students with readings that involve mathematics helps to achieve two goals:

1. To broaden their views of "WHAT IS MATHEMATICS?".
2. To interest more students in mathematics.

In my courses, I have asked students to read poems or brief literary passages as out-of-class assignments. Although we occasionally have discussion of a passage in class, more often I ask them to follow their reading with:

1. Writing their reactions to their reading (perhaps in a journal they are keeping for the course);

2. Talking to several other people about their reading and comparing the reactions of others with their own; usually I ask them to put this in writing since some would avoid this assignment (because they feel awkward carrying it out) unless I require a tangible product—something I will collect and read.

I especially like to use item 2; students do not often have the opportunity to fit ideas from their mathematics courses into general conversation and are, in a way, deprived when their friends in other majors can informally exchange ideas from their courses. Being able to talk about mathematical ideas increases student enjoyment of mathematics and also increases their appreciation (and that of their listeners) of its relevance.

Here are some readings that I have enjoyed using; a list of references at the end gives full bibliographic information for these and suggests others.

This passage, from *Six Degrees of Separation* (a play, made into a movie) by John Guare, introduces the graph-theory conjecture that in the acquaintanceship graph for people alive in the world today, between any two vertices may be found a path of seven or fewer edges:

> I read somewhere that everybody on this planet is separated by only six other people. Six degrees of separation. Between us and everybody else on this planet. The president of the United States. A gondolier in Venice. Fill in the names. I find that A] tremendously comforting that we're so close and B] Chinese water torture that we're so close. Because you have to find the right six people to make the connection. It's not just big names. It's *anyone*. A native in a rain forest. A Tierra del Fuegan. An Eskimo. I am bound to everyone on this planet by a trail of six people. It's a profound thought. . . .

The next passage is from the beginning of a short story by Jorge Luis Borges, *The Library of Babel*. Many of Borges stories involve mathematical ideas—symmetry, recursion, infinity, and so on.

> The universe (which others call the Library) is composed of an indefinite and perhaps infinite number of hexagonal galleries, with vast air shafts between, surrounded by very low ceilings. From any one of the hexagons one can see, interminably, the upper and lower floors. The distribution of the galleries is invariable. Twenty shelves, five long per side, cover all the sides except two; their height, which is the distance from floor to ceiling, scarcely exceeds that of a normal bookcase. One of the free sides leads to a

narrow hallway which opens onto another gallery, identical to the first and to all the rest. . . .

The Library is a sphere whose exact center is any one of its hexagons and whose circumference is inaccessible.

There are five shelves for each of the hexagon's walls; each shelf contains thirty-five books of uniform format; each book is of four hundred and ten pages; each page, of forty lines, each line of some eighty letters which are black in color. . . .

The next passage comes from *Gone with the Wind* (Chapter 36) by Margaret Mitchell; Scarlett O'Hara and Frank Kennedy have recently married; the following paragraph is about Frank:

It had begun to dawn on him that this same sweet pretty little head was a "good head for figures." In fact, a much better one than his own and the knowledge was disquieting. He was thunderstruck to discover that she could swiftly add a long column of figures in her head when he needed a pencil and paper for more than three figures. And fractions presented no difficulties to her at all. He felt there was something unbecoming about a woman understanding fractions and business matters and he believed that, should a woman be so unfortunate as to have such unladylike comprehension, she should pretend not to. Now he disliked talking business with her as much as he had enjoyed it before they were married. Then he had thought it all beyond her mental grasp and it had been pleasant to explain things to her. Now he saw that she understood entirely too well and he felt the usual masculine indignation at the duplicity of women. Added to it was the usual masculine disillusionment in discovering that a woman has a brain.

The relationship between formal logic and logical reasoning in the "real" world is considered in this passage from *Don Quixote* (Chapter 51) by Cervantes; I like to have my students in Discrete Mathematics read it and think and talk about it. A foreigner presents a problem to Sancho Panza:

"My lord," he began, "there was a large river that separated two districts of one and the same seignorial domain—and let your Grace pay attention, for the matter is an important one and somewhat difficult of solution. To continue then: Over this river there was a bridge, and at one end of it stood a gallows with what resembled a court of justice, where four judges commonly sat to see to the enforcement of a law decreed by the lord of the river, of the bridge, and of the seignory. That was the following: 'Anyone who crosses this river shall first take oath as to whither he is bound and why. If he swears to the truth, he shall be permitted to pass; but if he tells a falsehood, he shall die without hope of pardon on the gallows that has been set up there.' Once this law and the rigorous conditions it laid down had been promulgated, there were many who told the truth and whom the judges permitted to pass freely enough. And then it happened that one day, when they came to administer the oath to a certain man, he swore and affirmed that his destination was to die upon the gallows which they had erected and that he had no other purpose in view.

"The judges held a consultation. 'If,' they said, 'we let this man pass, without hindrance, then he has perjured himself and according to the law should be put to death; but he swore that he came to die upon that scaffold, and if we hang him that will have been the truth, and in accordance with the same law he should go free.' And now, my Lord Governor, we should like to have your Grace's opinion as to what the judges should do with the man . . .

Here is Sancho's first response:

"Well, then," said Sancho, "my opinion is this: that part or the man that swore to the truth should be permitted to pass and that part of him that lied should be hanged, and thus the letter of the law will be carried out."

The questioner reminds Sancho that this would result in the man's death and Sancho tries again:

"See here, my good sir," said Sancho, "either I am a blockhead or this man you speak of deserves to die as much as he deserves to live

and cross the bridge; for if the truth saves him, the lie equally condemns him. And this being the case, as indeed it is, it is my opinion that you should go back and tell those gentlemen who sent you to me that, since there is as much reason for acquitting as for condemning him, they ought to let him to free, as it is always more praiseworthy to do good than to do harm. . . .

Usually in mathematics classes we don't talk about our feelings toward our work. Here are poems that describe feelings about geometry and algebra.

GEOMETRY

by Rita Dove, selected in 1993 to be Poet Laureate of the United States, the youngest person and the first Afro-American to be selected for that honor.

I prove a theorem and the house expands:
the windows jerk free to hover near the ceiling,
the ceiling floats away with a sigh.

As the walls clear themselves of everything
but transparency, the scent of carnations
leaves with them. I am out in the open

and above the windows have hinged into butterflies,
sunlight glinting where they've intersected.
They are going to some point true and unproven.

ALGEBRA

by Linda Pastan, Poet Laureate of Maryland, a poet who often uses mathematical images in her poetry; this poem appears in *Against Infinity*

I used to solve equations easily.
If train A left Sioux Falls
at nine o'clock, traveling
at a fixed rate,
I knew when it would meet train B.
Now I wonder if the trains will crash;
or else I picture naked limbs
through Pullman windows, each
a small vignette of longing.
And I knew X, or thought I did,
shuttled it back and forth
like a poor goat
across the equal sign.
X was the unknown on a motor bike,
those autumn days when leaves flew past
the color of pencil shavings.
Obedient as a genie, it gave me answers
to what I thought were questions.

Unsolved equations later, and winter now,
I know X better than I did.
His is the scarecrow's bitter mouth
sewn shut in cross-stitch;
the footprint of a weasel on snow.
X is the unknown assailant.
X marks the spot
towards which we speed like trains,
at a fixed rate.

This next poem, by Howard Nemerov, is a favorite of mine. It capture the wonder that we feel when we see mathematics fit nature. Nemerov (1920-92) is a former Poet Laureate. It appears in his collection, *The Western Approaches*.

FIGURES OF THOUGHT

To lay the logarithmic spiral on
Sea-shell and leaf alike, and see it fit,
To watch the same idea work itself out
In the fighter pilot's steepening, tightening turn
Onto his target, setting up the kill,
And in the flight of certain wall-eyed bugs
Who cannot see to fly straight into death
But have to cast their sidelong glance at it
And come but cranking to the candle's flame—

How secret that is, and how privileged
One feels to find the same necessity
Ciphered in forms diverse and otherwise
Without kinship—that is the beautiful
In Nature as in art, not obvious,
Not inaccessible, but just between.

It may diminish some our dry delight
To wonder if everything we are and do
Lies subject to some little law like that,
Hidden in nature, but not deeply so.

The Czechoslovakian poet, Miroslav Holub, has written a number of poems that use mathematical

images. This one is a favorite of my students; it appears in *Against Infinity* (translated by editor Jet Wimp, a mathematician and poet).

ZITO THE MAGICIAN

to amuse the king Zito changes water into
wine   frogs into footmen   beetles
into bailiffs   he makes a Prime Minister
out of a rat   he bows: daisies
grow from his fingertips
a talking bird perches on his shoulder

so there

think up something else   demands the king
think up a black star   Zito thinks up a black
star
think up dry water   Zito thinks up dry water
think up a lake in a wicker basket   Zito does

so there

up comes a student: think up an angle alpha
whose sine is bigger than one

Zito pales;   I'm sorry
the sine of any angle is between minus one
and plus one   he stutters
nothing can be done
about it

he leaves the royal chambers   shuffling
through the throng of
courtiers back to his home
in a nutshell

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What Changes Should Be Made for the Second Edition of the NCTM *Standards?*

Zalman Usiskin
University of Chicago

UCSMP Director Zalman Usiskin presented this talk at the Eighth Annual UCSMP Secondary Conference, held November 7-8, 1992. This talk has been edited slightly for publication.

Those of you who are unfamiliar with the *Curriculum and Evaluation Standards* or the *Professional Teaching Standards,* and even those of you who are very familiar with these documents, may have thought, when you saw the title of this talk, who cares about the *Standards?* If you have a great deal of freedom at your school to teach what you want to, you might think that the *Standards* are merely rhetoric with little power. But in fact over half of the states in the country have changed their testing programs or curriculum recommendations in light of the *Standards.* Textbook publishers boast that their books agree with the *Standards,* and standardized test publishers are changing the tests because of the *Standards.* And within the past two years the National Science Foundation has funded 13 multi-year curriculum projects—including our own elementary materials component—to help implement what is often called the "vision" of the *Standards.*

Perhaps as significant, dissent from the *Standards* has been meager, primarily because in its journals, books, and conferences NCTM has followed a cheerleading policy that discourages any criticism of the *Standards.* If one is not for the *Standards,* one must be against good mathematics, against good teaching, against good evaluation. Thus you should care about the *Standards* because they affect the materials that will be available for you to teach, the tests your students take, what you hear at conferences you attend, and what you see in the journals you read.

A second thing that you might wonder is, When will the second edition appear? Well, I should tell you that there is no official date for their appearance, because there is no official plan for a second edition. Indeed, one of the reasons for this talk is to encourage discussion of a second edition in the hope that there will be one.

I began thinking about the second edition of the *Standards* when we began thinking about the second edition of the UCSMP secondary books. From 1983 through 1990, we worked on a complete secondary curriculum, and finally, in May of 1991, the last bits of copy for the final pieces of the teacher's edition for the last of the six

If you have a great deal of freedom at your school to teach what you want to, you might think that the *Standards* are merely rhetoric with little power. But in fact over half of the states in the country have changed their testing programs or curriculum recommendations in light of the *Standards.*

books went to the publisher. It was a Tuesday; I remember feeling so good and so relieved to have it behind me. On Friday, I had lunch with the president of ScottForesman, and—wouldn't you know it—the purpose of the lunch was to discuss how we felt about doing a second edition! Please, I said—we just finished the first one!

You may be thinking the same thing about the NCTM *Standards.* Didn't they just come out? The *Curriculum and Evaluation Standards* appeared in the spring of 1989. The draft of the *Professional Teaching Standards* appeared in 1991. But still it is not too early to think about a second edition, because it takes a couple of years to get a committee together, a couple of years to write, a year to get comments from everyone and get it in final form. So even were the committees to be named now, it might not be until 1997 that the second edition appeared. I myself think the second

edition should appear in 1999 but be announced as soon as possible.

Reasons for a Second Edition

You may also be thinking, we haven't yet implemented much of what is in the first edition of the *Standards* in our district. So why do a second edition? There are a few fundamental reasons. First, the *Standards will* die if there is not a second edition. They will die just as every other report in mathematics education has died. Here is a very brief history of such documents. In 1918 a committee from the Mathematical Association of America and mathematics teacher organizations from New Jersey, Chicago, and a few other areas got together to plan a national report that was ultimately titled *The Reorganization of Mathematics in Secondary Education*. It took five years to do the report, which came out in 1923 and so is known as the '23 Report, and in the interim NCTM was formed by some of these same teacher organizations. The '23 Report was very influential in moving mathematics education away from the view that mathematics should be taught to develop general mental faculties, and towards the view that the practical should be given strong consideration. In this it has much in common with the current *Standards*. The influence of the '23 Report lasted until the effects of the depression caused less attention to be paid to mathematics in schools.

The first major report in which NCTM played a role was a joint report with the MAA in 1940. This report responded to the problems of the depression by promoting a two-track system for high schools, one which would be more academic, with algebra and geometry and so on, and the other a general mathematics track in which the subjects were integrated.

After the war, in 1946, NCTM promoted a set of short reports from a committee called the Post-War Plans Committee. These reports dealt with a set of functional competencies that all students needed to have. If you look at these reports, which were published in *The Mathematics Teacher*, you might be surprised to see some of the same things that reappeared in the 70s as minimal competencies.

Generally, there are national reports only when people see problems. And so it is interesting to note that there was no big NCTM report during the time of new math, indicating that on the whole, NCTM was happy with the new math. The major report of that era came from the College Entrance Examination Board. NCTM largely abdicated its role as a policy leader during the 60s and 70s, and did not attempt to reassume this role until An Agenda for Action, a brief document more political than substantive, appeared in 1980. It called for a curriculum organized around problem solving but also said we still need to determine what problem solving is. Whenever a recommendation is put forward for something that has never been tried, it must be understood to be either political or philosophical. An *Agenda for Action* was not so much a document *for* something as one *against* the back-to-basics movement, that is, against the concentration of energy on the teaching of paper-and-pencil skills at the expense of everything else.

As soon as the *Standards* appeared, *An Agenda for Action* was forgotten. There is virtually no history in the *Standards*, no memory of what had been recommended before and failed, what had never previously been recommended, and there is no indication of what if anything is truly new in the *Standards*. Just three weeks ago at the Illinois Council of Teachers of Mathematics meeting, a speaker who ought to know better—a former president of the Illinois council—announced that the history of mathematics education begins in 1989 with the *Standards*. This is a very dangerous view. If there is no second edition of the *Standards*, then like all other reports, the *Standards* will be forgotten. They will be viewed as a short-lived fad, and the credibility of NCTM as a player in the arena of mathematics education policy will be diminished if not destroyed. Our major professional voice will have lost power. In the second edition of the *Standards* there need to be some historical perspectives.

Whenever a recommendation is put forward for something that has never been tried, it must be understood to be either political or philosophical.

As everyone knows, there are people who do not agree with many of the general goals presented in

the *Standards*, people who are waiting for the *Standards* to go away. This is not as farfetched a strategy as you might think. A school district that adopted textbooks in 1990 could easily have said at

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the time that the ideas in the *Standards* were then too new, giving the always-phony excuse, "They are wonderful goals, but we are just not ready at this time." The next adoption for such a district will not likely occur until 1995 or 1996, and the one after that somewhere around the turn of the century. So if things go as they have in the past, if only the district can get past one more major adoption, they can be rather certain that the movement will go away.

We felt we would have the same problem with regard to the UCSMP secondary curriculum: If we did not do a second edition, we would be perceived to have failed. And so I told the president of the publishing company in that Friday lunch that there had to be a second edition of UCSMP, because if there were no second edition people would think that we had failed, despite our influence on the *Standards*, and despite our being used in a huge number of classrooms in the country. We estimate that in the past three years something like 15-20 percent of all purchases of new mathematics textbooks from prealgebra to precalculus have been UCSMP texts, and our primary books are in increasing demand. We have also been quite influential on other textbooks—which was our main goal—but memories in mathematics education are amazingly short.

There is a third reason to have a second edition of the *Standards*. It is because times have changed, even in just a few years. When the *Standards* were written, the first graphing calculator, the Casio fx-7000, had just appeared. There was only one geometry drawing program, the Geometric Supposer. There was no complete six-year secondary mathematics program like UCSMP's. There was no national goal to be first in mathematics in the world. There was no major movement to change the nature of assessment, not just in mathematics but in all of education. And the middle school movement was not as strong as it is today.

The leaders of NCTM themselves recognize that the times have changed. After I settled on the title for this talk, I heard that there is to be a third volume in the "Standards" series, a volume devoted to Assessment Standards.

A fourth reason to have a second edition is that the *Standards* have been interpreted in various ways by teachers, curriculum developers, teacher trainers, and administrators. The Addenda project of NCTM is an indication that the authors of the *Standards* feel that they have not always been interpreted correctly. And so a second edition can make clarifications.

A fifth reason to have a second edition is that it may just be possible that the *Standards* have some errors, some things that were unwise, and a second look may give the opportunity to correct some of these things. After all, many recommendations in the *Standards* were never tested on a large scale. We have never had a twelve-year curriculum like that in the *Standards*. Would a student actually going through this curriculum meet the goals we desire? To put this in UCSMP perspective, only earlier this year did the first group of students graduate who studied from UCSMP texts in all grades 7 through 12.

A sixth reason to have a second edition is to acknowledge that there are many districts that have already made significant moves that implement or go a long way towards implementing the recommendations of the current *Standards*. Thousands and thousands of school districts have adopted our books or others that cover the wide range of content in the *Standards*, and the wider range of mathematical processes, and some follow if not exceed the technology recommendations in the *Curriculum Standards*. Perhaps thousands of teachers have changed the way they teach their classes along the lines of the *Professional Teaching Standards*. And some of the largest test creators in the country have been revising their tests to fit the *Evaluation Standards*. We should not speak as if

no one has changed, and we should begin to look beyond these changes. Many of the most forward-looking districts are already asking what they should do now. What more needs to be done? Where should they be going? Or perhaps the first-edition *Standards* are so visionary that if you have adopted them, you don't have to examine what you are doing any more—you are sitting pretty for the next umpteen years. If so, it would be nice to know that, too. It is as important to know what to keep as it is to know what to change.

A child entering first grade in the September after the Standards first appeared in the spring of 1989 will not finish high school until the year 2000. We should keep ideas going long enough to see their impact.

And still another reason to have a second edition is the time required for educational reform to have impact. A child entering first grade in the September after the Standards first appeared in the spring of 1989 will not finish high school until the year 2000. We should keep ideas going long enough to see their impact.

Content of the Second Edition

Many of these reasons for a second edition suggest that the second edition should be of the same general form as the first, with the same number of volumes of about the same length. But what should be in those volumes? I will try to indicate what I think should be kept and what should be changed.

Aspects of Mathematics at All Levels

Let me begin with the common threads of the *Curriculum and Evaluation Standards*. The first thing that should be kept are the three aspects of learning mathematics: problem solving—which includes the ability of mathematics to handle a variety of pure and applied problems, communication—which includes all of the aspects of language, and reasoning—which includes both induction and deduction. These are three quite different aspects of mathematics, and to identify and make them prominent was a brilliant idea.

But a fourth aspect needs to be added: mathematics as procedures. The use and study of algorithms is an important part of mathematics that is not addressed by the first three aspects, and since it is the thing to which most teachers give the most time, it needs to be addressed. Regardless of whether you rely on calculators, computers, paper-and-pencil, or your memory in obtaining an answer to a mathematical question, even in dealing with rich problems in real-world settings, there is still almost always some aspect that is mechanical. Doing the calculations in the Pythagorean Theorem, or finding the root of an equation, or finding percents, or rewriting fractions as decimals is an important part of mathematics.

The current fourth aspect of the *Standards*, mathematical connections, is a theme that has permeated all of my curriculum work. But it is not parallel to the other aspects of mathematics, and it may even be out of place as a standard.

The Grade Levels

The division of standards into K-4, 5-8, and 9-12 was not the result of some discovery that there are huge differences between fourth graders and fifth graders, or between eighth graders and ninth graders. It was simply because thirteen grades are too much to deal with, and three parts seems manageable, and many high schools begin at grade 9. Still, I believe these rather arbitrary divisions should be kept in order to maintain continuity from first to second edition.

However, two years ago this month I gave a talk in Toronto, and I learned that education leaders in the province of Ontario had convened a committee to determine what the standards meant to them. This committee concluded that the K-4 standards were almost entirely devoted to what their schools covered in K-2, and that the 5-8 standards focused primarily on their grades 7-8. This left a big gap in grades 3-6.

It is not surprising that there would be a gap in what is recommended for grades 3-6, because these are the years in which teachers in the past have spent almost all their time on paper-and-pencil computation.

Because the *Standards* do not adequately discuss grades 3-6, students finishing grade 6 in a curriculum aspiring only to the *Standards* do not go as far as they could. And here we have another broad weakness of the *Standards:* although the governmental and business support for change in mathematics education is based in great part on the low performance of U.S. students in international comparisons, the *Standards* simply have not taken some of the best ideas from what is done in other countries. Indeed, the curricula in the countries of the Orient and the former Soviet Union—which had quite a good mathematics curriculum at these levels—have been ignored.

You may wonder why the work in these countries would be ignored. One reason is that the curricula of these countries do not follow the philosophy of the writers of the *Standards*. They do not believe that children always have to construct knowledge for themselves. They do not believe that symbolic mathematics needs to be delayed. They don't believe in Piaget. They don't use calculators.

We may disagree with the philosophies that underlie mathematics in those countries, but we should not ignore them, because as the researchers in the Second International Mathematics Study concluded, we in the United States have had an underachieving curriculum. In particular, we expect less at the elementary school level than almost all other industrialized countries. As a result, students in seventh and eighth grade in almost every industrialized country study what about 75 percent of our students reach only in ninth and tenth grade. Only those students in the U.S. who take a rich course in seventh grade and algebra in eighth grade, our *Transition Mathematics* and *Algebra* or their equivalent, come close to having a curriculum like almost all students in many other countries. For this reason, I believe the standards for grades 5-8 are really more appropriate for grades 5-7.

Curriculum Content in Grades 9-12

The scope of the content of the *Standards* at these levels is wonderful and needs no major changes: every good curriculum should have algebra, geometry, functions, statistics, discrete mathematics, geometry from a synthetic viewpoint, geometry from an algebraic viewpoint, and conceptual foundations of calculus. As you know, the UCSMP secondary curriculum does, and we are proud of it.

However, it is a weakness of the first edition that what they assign to grades 9-12 cannot be done in four years. We know this because of our experience actually writing such a curriculum. Put most succinctly, the current 9-12 standards need to begin in grade 8. We have had great success in schools that have adopted the entire UCSMP curriculum with algebra offered for the vast majority of students in grade 8.

The Common Core

Although the *Standards* are pessimistic about how early some mathematics can be covered, they are optimistic about who can learn mathematics once it is taught. Individual differences are not considered until grades 9-12, and then they are dealt with by a core curriculum. This is an impossible dream. Children come to school in some communities years ahead of children in other communities, and to assume that they are all the same is a failure to acknowledge their reality. It is not tracking to give children the same opportunities at different ages any more than it is tracking to put both 18-year-old and 22-year-old marine recruits through the same boot camp.

We may disagree with the philosophies that underlie mathematics in those countries, but we should not ignore them, because as the researchers in the Second International Mathematics Study concluded, we in the United States have had an underachieving curriculum.

A second-edition *Standards* should be more mature and less doctrinaire than the first edition. It should distinguish the ideal we must strive for from the attainable. It should ask that all students be given the same curriculum through algebra and geometry, but not necessarily at the same time. And I must tell you that we have learned in UCSMP that even beginning at different years is not enough. For a

variety of reasons—jobs outside of school, lack of attention to homework, learning style, attitude towards school— some students do not learn at the same pace as others.

Schools that have taught all their seventh-grade students *Transition Mathematics* have one by one come to the conclusion that they must slow down for some students. Some do *Transition Mathematics* and *Algebra* in three years with slower students; that seems a reasonable solution. But if it takes them four years, those students are simply not ready for*TM* and the school should wait.

Role of Exploration

The *Standards* promote what has been called "active learning", and they pay particular attention to the role of exploration in learning mathematics. Recommended are all sorts of tools to do this exploration, with particular attention to concrete materials. One of the things to happen since the *Standards* appeared is the increasing appearance of more powerful technology to engage in exploration: spreadsheets, geometry drawing programs, and algebra programs that combine graphics with symbol manipulation; graphics calculators that enable graphs to be drawn at will.

With all this ability to generate examples and confirm patterns with examples, I worry about the future of deduction, that aspect of mathematical reasoning that is unique. Induction may generate patterns, but it does not tell us that the patterns hold. Former President Bush knows this better than anyone. His economic advisors kept telling him last year and early this year: Don't worry, recessions last only so long; by the time the election comes around, the economy will have started to pick up, and you will be reelected.

Reasoning using deduction needs to be in the curriculum of all students, from grade 1 up. It is the way we decide whether something is true in mathematics, and to avoid it is akin to teaching science with no experiments. We need to look again at the roles of assumptions, logic, definitions, theorems, and proof in an exploratory environment. It is not enough to say that students will want to confirm the patterns they find: our research indicates that many *PDM* students consider confirmation by example as powerful as confirmation by proof.

Evaluation

I understand that the upcoming assessment standards will review the current evaluation standards. Thus my suggestions may not be for the second edition, but for the first edition of them.

It goes without saying that we should keep the goal of multiple roles and methods of evaluation. I'm not certain that portfolios are the answer, but certainly we should fight the notion that standardized tests or multiple-choice tests suffice. However, there are two changes I would like to see in the evaluation standards.

First, we need some attention to the problems of grading students. I believe strongly that all assessment should be a learning experience, and I believe in the importance of evaluation for diagnosis and remediation, but the fact is that after

One of the things to happen since the *Standards* appeared is the increasing appearance of more powerful technology to engage in exploration

the early elementary grades, there is a bottom line: teachers need to obtain a relatively impartial way of assigning grades to students. The newer assessment rhetoric needs to be fitted into the reality of a very important aspect of the job of many teachers; the requirement that they come up with defensible grades periodically during the year and a final grade at the end.

My second bit of advice is more of a warning: Let us drop this overstated rhetoric about all the old tests being bad. Those tests were used because they are quite effective in fitting a particular mathematical model of performance—a single number that has some value to predict future performance. Until it can be shown that the alternate assessment techniques do a better job at prediction, let us not knock what is there. The mathematics education community has forgotten that it is poor performance on the old tests that rallied the public behind our desire to change. We cannot very well pick up the banner but then say

the tests are no measure of performance. We cannot have it both ways.

Let me be more specific. I believe as strongly as anyone in this room that long division is obsolete. But by the time a child is through with fourth or fifth grade, that child better know how to get the answer in a division situation. By seventh grade or so, that student needs to be able to divide very large or very small numbers. We had better be able

If it is so easy to demonstrate that using calculators helps, then let's demonstrate it and advertise the huge improvements! Let's let the public know how much better today's students are because they have better technology.

to show that, with technology allowed, today's students can outperform their counterparts of years ago. I know it seems obvious that this can be done, but I have not seen many studies of this, and some skills, like finding the unknown number in a proportion or rewriting a decimal as a fraction or solving a simple equation, are not necessarily helped by calculators. If it is so easy to demonstrate that using calculators helps, then let's demonstrate it and advertise the huge improvements! Let's let the public know how much better today's students are because they have better technology.

We might use the following rhetoric. When today's algorithms in arithmetic and algebra were invented, mostly about 400-500 years ago, they used new hardware—paper and more recently pencils, and new software—algorithms like partial product multiplication and long division and the quadratic formula. They were the best at the time, but now we have better technology.

Teaching with Technology

Nothing has changed in the past few years more than technology. For this reason, the *Professional Teaching Standards* need to give direct attention to the use of technology in teaching, and by technology I mean here specifically calculators and computers. Guidance is needed regarding the incorporation of spreadsheets, geometry drawing tools, statistics packages, function graphers, and calculators. It may be time that we begin to recommend that every student have a computer at home, and that we begin to work with social agencies in low-income communities to achieve this goal.

Role of the Teacher

We should not expect as much from the teaching standards as we do from the curriculum standards. Although there is a long history of rather detailed suggestions for curriculum, there does not exist such a long history of recommendations for the teaching process. The *Professional Teaching Standards* venture into generally untrod ground.

The verbs on page 1 of the teaching standards point out the desired role of the teacher: selecting mathematical tasks, providing opportunities, orchestrating classroom discourse, using and helping students use technology, seeking and helping students seek connections, guiding the work of the class. It is a wonderful vision.

But there is something missing. At times direct instruction is needed. To give directions, to set the stage for a new unit, to summarize, to tell a story, to emphasize what is important and what is not, to bring cohesion to the class. All these times and others are suitable for the traditional instruction. There is a reason why direct instruction is so pervasive and so difficult to change; it is because there are ways in which it can be very effective. The importance of the teacher as facilitator should continue to receive emphasis, but the writers of the teaching standards need to take the best from traditional practice as well.

Students

I believe NCTM has placed too much of a burden on teachers. We teachers can change our curriculum, our ways of teaching, and the way we evaluate, but we also need students to change. These changes do not always come automatically even with the greatest teachers. There need to be reasonable expectations about how much students need to work, about the tools we should expect them to have, and about the attitudes they should bring to school. There need to be statements about the roles of homework, and of parents, and of guidance counselors, and of administrators, and of

school boards, and of the other important players in a child's education. But the key has to be the students.

Last year I spoke about this point in great detail, and if you do not have our UCSMP Newsletter No. 10, please write us to ask for a copy.

Finally, I would like to say a few remarks regarding the way we should look at what we do. There are those who wish us to expect our treatments to cure everyone. But why should we expect practices to succeed in education any more than we always expect medical practices to succeed? We should point out to the public that we recommend something not because it is a sure-fire cure, but because on the whole it is the best treatment we know.

It is a sign of maturity to say that there are things we do not know. The *Standards* should not recommend practices that have never been tried on a large scale, as if these practices are certain to succeed. In the second edition, there should be places where options are given—even on important issues. This is easily done in a second edition whose very existence proclaims that it is natural to think of revising the *Standards*. And if the second edition identifies when the third edition will come out, it will not have to think so far into the future. Then when the third edition appears, the work begun by the authors of the *Standards* may be said to have been institutionalized, and we will have a mechanism for an ongoing statement of policy in curriculum, evaluation, and instruction, instead of an isolated document. This would truly be a revolutionary achievement in mathematics education in our country.

NCTM Standards, Second Edition: A Review and Commentary on Zalman Usiskin's Address

Harald Ness
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Did you have the privilege of attending Zalman Usiskin's address, "What Changes Should be Made for the Second Edition of the NCTM Standards?" given at the Eighth Annual UCSMP Secondary Conference? Well, neither did I. In fact, when I received my copy of the UCSMP Newsletter where a slightly edited version of the talk was printed, I was so caught up with department and teaching responsibilities that I didn't have time to read it. However, I did have the time to digest it in the leisure of this past summer, when the load was lighter and the light was longer.

Professor Usiskin's work in mathematics education is well known, and his ideas are always well thought out. The ideas presented in this address are important, timely, and deserving of as wide dissemination as possible. I would like to pass them along to readers of the HMN Journal along with a few comments of my own.

The main premise of Professor Usiskin's talk is that although there are not current plans for a second edition of the "Standards", there should be.

1. First of all, he states, if there is not a second edition, the "Standards" will die just as all the other reports on mathematics education have died. He cites sundry such reports beginning with the report of a committee formed in 1918. I can relate to that. I have been cleaning out a twenty-five year accumulation in my office in preparation for my pending retirement. I was astounded at the number of reports from special committees, task forces, and what have you that had published reports (recommendations) on mathematics education reform over the years, usually supported with funds obtained from foundations. These had been placed in various piles with the intention that someday I would read them (actually I did read a few of them). The point is, as Professor Usiskin said, they have all died.

2. Secondly, there are people who do not agree with many of the general goals presented in the "Standards". He says there are school districts hoping that by the time they have to do their next textbook adoption, the movement will go away. Also, I think that those who disagree should be heard and their concerns debated.

3. Times have changed. Professor Usiskin cites the significant advances made in technology, changes in textbooks, and changes in attitudes and views on assessment even in the short time since the "Standards" came out. Curriculum should be dynamic, not static. A continuous evolution is much better than a series of discrete revolutions.

4. The "Standards" have been interpreted in different ways by teachers, curriculum developers, teacher trainers, and administrators. A second edition could provide clarification.

5. Professor Usiskin dares to say that there may be some errors in the "Standards"; and also, that with time and experience, some of the recommendations may have been shown to be unwise. Most importantly, some things in the "Standards" had not been tested, and now, with experience and results, we may gain insight into the wisdom of some of them.

6. A second edition could reflect the changes that have been implemented as a result of the

"Standards" and, in some cases, have gone beyond. Some may want to know what to do next; i.e., keep up the momentum of change.

Professor Usiskin then addresses what he thinks should be kept and what should be changed. Problem solving, the language of mathematics, and reasoning are the aspects he indicates should be kept. He argues for a fourth aspect, mathematics as procedures (algorithms). He feels that the fourth aspect of the "Standards", mathematical connections, should permeate all curricular work, but it is not parallel to the other aspects. Regardless of how you view it, it is important. Those of you who know me know that I think we should also stress the place of mathematics in our culture and the significant force mathematics has been in the development of our culture.

He feels the "Standards" do not adequately discuss grades 3-6 because, he states, these are years in which teachers (in the U.S.) have traditionally spent most of the time on paper/pencil computations. That is certainly food for thought, but a much more powerful suggestion he makes is that the "Standards" have not taken into consideration the best ideas of what is done in the other countries. Professor Usiskin contends that one reason the authors of the "Standards" ignored what was going on in other countries is that the authors' philosophy differs form the philosophy of those constructing curriculum in other countries. Among these differences, he states that curriculum

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designers in other countries do not believe that children always have to construct knowledge for themselves, do not believe that symbolic mathematics needs to be delayed, they don't believe in Piaget, and they don't use calculators. I must confess that I disagree with the constructionists in this country and agree with the foreigners on the first two items. In fact, I am greatly concerned that the constructionist movement has gone so far as to have a deleterious effect. As for the use of calculators (and computers), I think they should and must be used. However, they should be used wisely; unfortunately, from what I have seen, they are often used as a substitute for thinking, and this is not good. Regardless of whether we agree with the philosophies of the other countries, Professor Usiskin states, we cannot ignore their programs because the Second International Mathematics Assessment Study concluded that we have an underachieving curriculum. This, of course, assumes the International Assessment is assessing the things we think are important. I think there are other indications of great underachievement, also. Professor Usiskin agrees with the scope of the "Standards" 9-12 program, but based upon his experiences with the UCSMP program, it cannot be done in four years. I disagree that this cannot be done in four years because I have done it. However, it was done in a college prep math program where tracking existed in the high school and at a time of a different societal climate. Professor Usiskin also states that the "Standards" fail to acknowledge, and compensate, for individual differences. I heartily agree, and think the egalitarian ethic has been carried too far. I still believe with the statement made long ago by John Kemeny that it is much more in keeping with the democratic philosophy to have students in a program consistent with their abilities, and I agree with Zalman Usiskin that students do not, and indeed cannot, learn at the same pace.

I strongly agree with Professor Usiskin's concern about our overemphasis on the use of technology for exploration. As stated before, too often the calculator or computer is used as a substitute for thinking (deductive, if you will). It leads to fuzzy thinking and often false conclusions. We need to retain (or put back in, in some cases) the deductive process. He cites an anecdote about former President Bush, and it might be one reason he is former.

There are a couple of cogent statements Professor Usiskin makes about assessment:

1. We cannot say poor performance on former tests indicate a need for curricular reform and then say we must change our testing techniques because they are not valid measures of performances.

2. We better be able to show that the use of technology allows students to outperform their counterparts of the past. It should be easy to demonstrate this; and if we cannot, we might want to re-think the technology question or how we make use of technology.

He mentions questioning certain things such as multiple choice questions. I would like to interject some opinions on this. I have never believed that multiple choice questions tested students' knowledge of mathematics. They might test the students' skills at making choices from four or five options; realistically, there are more often an infinite number of options. As Peter Hilton has stated, the only place multiple choice questions are valid is for finite group theorists. Also, we should take cognizance of the research project in Britain (sorry, I temporarily lost the reference) that indicated that multiple choice tests (at least in mathematics) were biased in favor of males. The multiple choice test showed markedly higher scores for males while the "traditional" test (I assume they meant that students provided the answers) showed no difference between males and females. I think this is important, and we should strive for gender neutral assessment.

We cannot expect cure alls in education any more than in medical practice. We should communicate to the public that our recommendations are not anticipated to be "sure fire-cures", but they are the best treatment we know now.

On teaching, Professor Usiskin agrees with the importance of the NCTM stated role of the teacher, but says something is missing. He says there is still a time and place for the traditional direct instruction; e.g., to give directions, set the stage for new things, to summarize, to tell a story, to emphasize what is important, and to bring cohesion to the class.

As in many discussions on improving education, the "Standards" fail to address the most important aspect in the teaching/learning process, the students. As Professor Usiskin points out, we can change the curriculum, the ways of teaching, and the means of assessment; but the desired outcome will not be achieved without changes taking place in the students. We need to discuss reasonable expectations about how much students need to work, about tools we expect them to have, and their attitudes. Governor Lester Maddox said that if they wanted a more successful prison system, they should get a better class of prisoners. Why do we have such difficulty facing the fact that students are an important factor in the success of our educational system? The failure to consider the student factor, however, goes back a long time. I recall that shortly after the "new math" endeavor of the late fifties, one of the big guns (who shall remain nameless) in that movement wrote an article in *The Mathematics Teacher* about teacher effectiveness. He attempted to measure teaching effectiveness by student performance on tests and stated that they could not figure out why one teacher was very "effective" one year and the very next year was not. I wrote a letter to the editors of NCTM stating the obvious; that teacher had a different set of students. They refused to print it. I guess they didn't want to embarrass the author. It was also obvious that they were not measuring teacher effectiveness; they were measuring student effectiveness.

Professor Usiskin also addresses how we look at what we do. We cannot expect cure alls in education any more than in medical practice. We should communicate to the public that our recommendations are not anticipated to be "sure fire-cures", but they are the best treatment we know now. I would like to add that we must realize that when we change things because the changes will benefit some students, these changes often make things worse for other students.

Professor Usiskin states in closing that the second edition should include options; and by the time the third edition appears, the process will be "institutionalized", and we will have an ongoing study of a dynamic curriculum for mathematics education.

Epsilon and Delta: A Little Love Story

Bonnie Shulman
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She reads her Calculus text:
Given an epsilon
do si do
find a delta
if you can
Approach
oh so close
Delta on the domain
pursues epsilon on the range
along lazy eight lane
ad infinitum

She begins to doze:
Delta lasoos epsilon
they get married and go live
on the one-over-ex-square ranch
with an area they can paint
but never walk around
not enough fence in the universe
to contain it
but enough paint to cover it
so strange

She dreams:
Their herd of discrete cattle
roam the infinite range
bounded by zero below
with domain greater than one
heading off into the horizon
they live happily
ad infinitum

Humacrostic

Stephen I. Brown
SUNY at Buffalo

How does one conceive of math
Under paradigm emerging?
More as scouting out new path,
And less as product burgeoning?
Now we must towards history look
In seeking worlds fallible and nifty.
Sages centuries past mistook
The negatives for "numeri ficti."
Imaginaries had pallor somber,
Centering on their loss of order,

Masquerading as if they were "number,"
Assuming not what such things ought to.
Then philosophy will scare some schisms,
Helping pan positivism's vultures.
Eliminating prior "isms,"
Math is seen 'gainst evolving cultures.
And how we learn and how we teach logic,
That's partly philosophy's new find.
It's not purely province pedagogic—
Centered meaning of "math mind,"
Surpassing not what heart's defined.

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