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Super-Symmetric Three-Cycles in String Theory

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Associative 3-manifolds in $\mathbb{R}^7$

by

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Abstract

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We determine several families of so-called associative 3-dimensional manifolds in $\mathbb{R}^7$. Such manifolds are of interest because associative 3-cycles in G_2 holonomy manifolds such as $\mathbb{R}^6 \times S^1$, whose universal cover is $\mathbb{R}^7$, are candidates for representations of fundamental particles in String Theory.

We apply the classic results of Harvey and Lawson to find 3-manifolds which are graphs of functions $f : \operatorname{Im}\mathbb{H} \to \mathbb{H}$ and which are invariant under a particular 1-parameter subgroup of G_2, the automorphism group of the Cayley numbers, $\mathbb{O}$. Systems of PDEs are derived and solved, some special cases of a classic theorem of Harvey and Lawson are investigated, and theorems aiding in the classification of all such manifolds described here are proven. It is found that in most of the cases examined, the resulting manifold must be of the form of the graph of a holomorphic function crossed with $\mathbb{R}$. However, some examples of other types of graphs are also found.

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Acknowledgments

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Chapter 1

Introduction and Background

1.1 The Problem

The objective of this paper is to investigate a problem which is of some importance to theoretical physics. We would ultimately like to find supersymmetrc three cycles in the G_2 holonomy manifold $\mathbb{R}^6 \times S^1$. These cycles are characterized by being volume minimizing in their homology class.

The problem was first posed by Edward Witten in [3]. Witten suggests that a three cycle as mentioned above could represent a BPS saturated domain wall in string theory, and the identification of such cycles is important to the development and understanding of the physical theory. He does not, however, prove the existence of such a cycle which obeys the appropriate boundary conditions, and has no concrete examples of such cycles.

In this paper we make a first step towards this goal, by investigating supersymmetric 3-manifolds in $\mathbb{R}^7$, which is the universal cover of $\mathbb{R}^6 \times S^1$. We apply the classic results of Harvey and Lawson to find 3-manifolds which are graphs of functions $f : \operatorname{Im}\mathbb{H} \to \mathbb{H}$ and which are invariant under the 1-parameter subgroup of G_2 whose elements are of the form

$$h_t(a + b\mathbf{e}) = e^{it} a e^{-it} + e^{qt} b e^{-it}\mathbf{e} \tag{1.1}$$

for a fixed $q \in \operatorname{Im}\mathbb{H}$ and for all $t \in \mathbb{R}$. We find the special case systems of PDEs which result from Harvey and Lawson's classic PDE

$$Df = \sigma f \tag{1.2}$$

and solve them. When our function f is real-valued on the $\{i, j\}$ plane, we find that the general form for f is that of a complex power/root function in x_2 and x_3 coordinates, and is independent of the x_1 coordinate. We prove that graphs of f, where f is a complex function satisfying the Cauchy-Riemann equations, crossed with a line, are always associative. We then investigate the cases when f is not necessarily real valued on the $\{i, j\}$ plane. The PDEs for some special forms for f are investigated. In most of the cases examined we find that the resulting manifold must be the graph of a function satisfying Cauchy-Riemann equations crossed with $\mathbb{R}$. However, some examples of other types of graphs are also found. In particular, a graph of a function satisfying Cauchy-Riemann equations and a particular set of its rotations are found to form an associative manifold.

1.2 The Method of Calibrations

We will use the method of calibrations to identify our manifolds, a method pioneered by Harvey and Lawson in their ground breaking paper [1]. The basic idea behind calibrations is as follows. Suppose we are given a Riemannian manifold M. A *calibration* on M is a closed p-form ϕ such that

$$\phi(\mathbf{e}_1, \ldots, \mathbf{e}_p) \leq 1 \tag{1.3}$$

on all orthonormal p-tuples of tangent vectors at all points of M, i.e. on all tangent p-planes $e_1 \wedge e_2 \wedge \cdots \wedge e_p$ with $|e_1 \wedge e_2 \wedge \cdots \wedge e_p| = 1$. A tangent plane is called calibrated if ϕ achieves 1 on it. A p-dimensional submanifold of M is called calibrated if all of its oriented tangent planes are calibrated. The fundamental result is that any calibrated closed oriented p-dimensional cycle $N \subset M$ is of absolutely minimal volume in its homology class. This is easily demonstrated:

If N' is another p-cycle of M in the same homology class as N, then there is some oriented (p+1)-cycle C such that $\partial C = N' - N$. Now

$$\text{Vol}(N') \geq \int_{N'} \phi = \int_N \phi + \int_C d\phi = \text{Vol}(N) \tag{1.4}$$

where we have used the calibration property of ϕ, Stokes' theorem, and the fact that $d\phi = 0$, in that order. Thus N is absolutely minimal in its homology class. If Vol(N') = Vol(N) then N' must also be calibrated by ϕ.

The so-called *associative calibration* on $\mathbb{R}^7$ described below leads to a particularly elegant algebraic characterization of the manifolds we seek.

1.3 The Associative Calibration on $\mathbb{R}^7$

We will consider a calibration on the imaginary Cayley numbers (isomorphic to $\mathbb{R}^7$). The Cayley numbers and Quaternion numbers are discussed at length in Appendix A. Most of the results stated in this section can be found in [1].

Consider the 3-form

$$\phi(x, y, z) = \langle x, yz \rangle \tag{1.5}$$

where $x, y, z \in \text{Im}\,\mathbb{O}$. ϕ is alternating:

$$\phi(x, x, z) = \langle x, xz \rangle = \langle \bar{x}x, z \rangle = |x|^2 \langle 1, z \rangle = 0 \tag{1.6}$$

since $z \in \text{Im}\,\mathbb{O}$, and similarly $\phi(x, y, x) = 0$. Also,

$$\phi(x, y, y) = \langle x, y^2 \rangle = -\langle x, y\bar{y} \rangle = -|y|^2 \langle x, 1 \rangle = 0 \tag{1.7}$$

Since ϕ is alternating we can take x, y, z orthogonal. On any orthonormal triple u_1, u_2, u_3 we have

$$\phi(u_1, u_2, u_3) = \langle u_1, u_2 u_3 \rangle \leq |u_1||u_2 u_3| = |u_1||u_2||u_3| = 1 \tag{1.8}$$

by the Schwartz inequality and the norm preserving property of Cayley multiplication. It is easy to verify that ϕ is closed. Thus ϕ is a calibration on $\text{Im}\,\mathbb{O}$, and so ϕ

must achieve 1 on the tangent spaces on homologically minimizing 3-dimensional subvarieties of $\operatorname{Im}\mathbb{O}$.

Note that ϕ achieves its maximum of 1 only when $u_1 = u_2 u_3$. It follows easily from Theorem A.2.2 that this equality holds iff u_1, u_2, u_3 span a 3-plane isomorphic to the canonically oriented imaginary part of the quaternion algebra. Because these are exactly the associative subalgebras of $\mathbb{O}$ we call ϕ the *associative calibration* on $\operatorname{Im}\mathbb{O}$. Tangent planes and submanifolds calibrated by ϕ will be called *associative*.

Later in the paper when we search for associative manifolds invariant under groups of automorphisms of $\mathbb{O}$ we will need the following theorem:

Theorem 1.3.1. *For all $\gamma \in G_2$,*

$$\gamma^* \phi = \phi \tag{1.9}$$

Proof. Since γ is linear we only need to show that

$$\langle \gamma x, (\gamma y)(\gamma z) \rangle = \langle x, yz \rangle \tag{1.10}$$

But γ is an automorphism, so

$$(\gamma y)(\gamma z) = \gamma(yz) \tag{1.11}$$

and automorphisms of $\mathbb{O}$ are rotations by Theorem A.2.5, so the inner product is preserved, and the theorem is proven. □

1.4 The Partial Differential Equations of Associative Manifolds

One special type of associative submanifold we can look for is a graph of a function $f : \Omega \subset \operatorname{Im}\mathbb{H} \to \mathbb{H}$. That is, manifolds parameterized as $(x, f(x)) \in \operatorname{Im}\mathbb{H} \oplus \mathbb{H} = \operatorname{Im}\mathbb{O}$.

Definition 1.4.1. *If points in $\operatorname{Im}\mathbb{H}$ are denoted by $x = x_1 i + x_2 j + x_3 k$ then the* Dirac *operator D is defined on f as*

$$Df = -\frac{\partial f}{\partial x_1} i - \frac{\partial f}{\partial x_2} j - \frac{\partial f}{\partial x_3} k \tag{1.12}$$

The first order Monge-Ampere operator *on f is defined as*

$$\sigma f = \frac{\partial f}{\partial x_1} \times \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3} \tag{1.13}$$

where we employ the triple cross product of Cayley numbers, defined by (A.53).

Theorem 1.4.1. *Suppose $f : \Omega \subset \operatorname{Im} \mathbb{H} \to \mathbb{H}$ is C^1. The graph of f is an associative manifold iff f satisfies the differential equation*

$$Df = \sigma f \tag{1.14}$$

Proof. We need only prove the theorem when f is linear. In this case the graph of f is spanned by

$$x = i + f(i)e \tag{1.15}$$

$$y = j + f(j)e \tag{1.16}$$

$$z = k + f(k)e, \tag{1.17}$$

and one can verify using properties of Cayley multiplication that

$$\operatorname{Im} x \times y \times z =$$

$$= \operatorname{Im} \{ i\,(f(j) \times f(k)) + j\,(f(k) \times f(i)) + k\,(f(i) \times f(j)) \} + (\sigma(f) - D(f))\,e \tag{1.18}$$

By Theorem A.3.2 we see that the graph of f is associative iff $\operatorname{Im} x \times y \times z = 0$. Thus if the graph of f is associative then each component vanishes, and in particular, we have (1.14). Now assume (1.14) holds. By Theorem A.3.3 we must have that $[x, y, z]$ is orthogonal to x, y, and z. But $[x, y, z] \in \operatorname{Im} \mathbb{H}$ and this, together with (1.15), shows that $[x, y, z] = 0$. Thus, given the proper orientation, the graph of f is associative. □

Although we now have a 1^{st} order PDE that completely characterizes which f have associative manifolds as their graphs, the PDE is highly non-linear because of the σf term. No one has been able to solve it. Therefore we must search for solutions which exhibit certain symmetries in order to make the problem more tractable.

Chapter 2

Examples of Associative 3-Manifolds in $\mathbb{R}^7$

2.1 A Family of Associative Manifolds

In this section we will identify a set of associative manifolds in $\mathbb{R}^7 \cong \operatorname{Im}\mathbb{O}$ using a technique inspired by Harvey and Lawson [1].

Theorem 2.1.1. *For any $z_0 \in \mathbb{C}$, the graph*

$$M_{z_0} = \left\{ x + \left(\frac{z_0}{\sqrt{r_x}} e^{-\frac{1}{2}i\theta_x} \right) \mathbf{e} \;\middle|\; x = x_1 i + r_x \cos\theta_x j + r_x \sin\theta_x k \right\} \tag{2.1}$$

is an associative manifold in $\mathbb{R}^7$.

Note that (x_1, r_x, θ_x) are the cylindrical coordinates of x taking cylinder's axis to be the i axis. If we take r_x and θ_x to vary continuously we get a graph of a branch of the complex square root function crossed with a line. See Figures 2.1–2.1 for some plots of M.

Proof. It can be checked by direct computation that the tangent 3-plane at every point of M_{z_0} is associative. Parameterize the graph by the coordinates of x:

$$S(x_1, x_2, x_3) = x_1 i + x_2 j + x_3 k + \left(\frac{z_0}{\sqrt{r_x}} e^{-\frac{1}{2}i\theta_x} \right) \mathbf{e} \tag{2.2}$$

The change of coordinate derivatives for cylindrical coordinates are

$$\begin{aligned} \frac{\partial r_x}{\partial x_2} &= \cos\theta_x & \qquad \frac{\partial r_x}{\partial x_3} &= \sin\theta_x \\ \frac{\partial \theta_x}{\partial x_2} &= -\frac{\sin\theta_x}{r_x} & \qquad \frac{\partial \theta_x}{\partial x_3} &= \frac{\cos\theta_x}{r_x} \end{aligned} \tag{2.3}$$

To take the derivatives with respect to the rectangular coordinates we differentiate using the product rule. Our calculations are simplified by Lemma A.2.2 for the $\mathbf{e}$ and $i\mathbf{e}$ components, since it allows us to commute and associate products such as $(ab)(c\mathbf{e}) = (abc)\mathbf{e} = (bac)\mathbf{e}$, etc, for $a, b, c \in \mathbb{C}$.

$$\frac{\partial S}{\partial x_1} = i \tag{2.4}$$

$$\frac{\partial S}{\partial x_2} = j - \frac{z_0}{2r_x^{3/2}}\left(\cos\theta_x - i\sin\theta_x\right)e^{-\frac{1}{2}i\theta_x}\mathbf{e} \tag{2.5}$$

$$\frac{\partial S}{\partial x_3} = k - \frac{z_0}{2r_x^{3/2}}\left(\sin\theta_x + i\cos\theta_x\right)e^{-\frac{1}{2}i\theta_x}\mathbf{e} \tag{2.6}$$

From this it is very clear that

$$\frac{\partial S}{\partial x_1}\frac{\partial S}{\partial x_2} = \frac{\partial S}{\partial x_3} \tag{2.7}$$

at every $x \in \operatorname{Im}\mathbb{H}$, which shows that the tangent planes are associative, hence M_{z_0} is associative for each $z_0 \in \mathbb{C}$. $\square$

Although this proof is valid, it offers no insight into how we obtained such a manifold. We will now demonstrate how to obtain the manifold by solving the differential equation $Df = \sigma f$, using symmetry constraints to simplify our task. In particular, we will impose the restriction that our graph be invariant under a 1-parameter subgroup of the exceptional Lie group G_2. Recall that G_2 is the automorphism group of $\mathbb{O}$, that is,

$$G_2 = \{\, g \in GL_8(\mathbb{R}) \mid g(xy) = g(x)g(y),\, \forall x, y \in \mathbb{O}\,\} \tag{2.8}$$

For more details on the structure of G_2, see Appendix A, page 37.

Lemma 2.1.1. *The following is an automorphism of $\mathbb{O}$ for every $t \in [0, 2\pi)$:*

$$h_t(a + b\mathbf{e}) = e^{it}ae^{-it} + be^{-it}\mathbf{e} \tag{2.9}$$

where $a, b \in \mathbb{H}$. The set $\{h_t \mid t \in \mathbb{R}\}$ forms a 1-parameter subgroup of G_2.

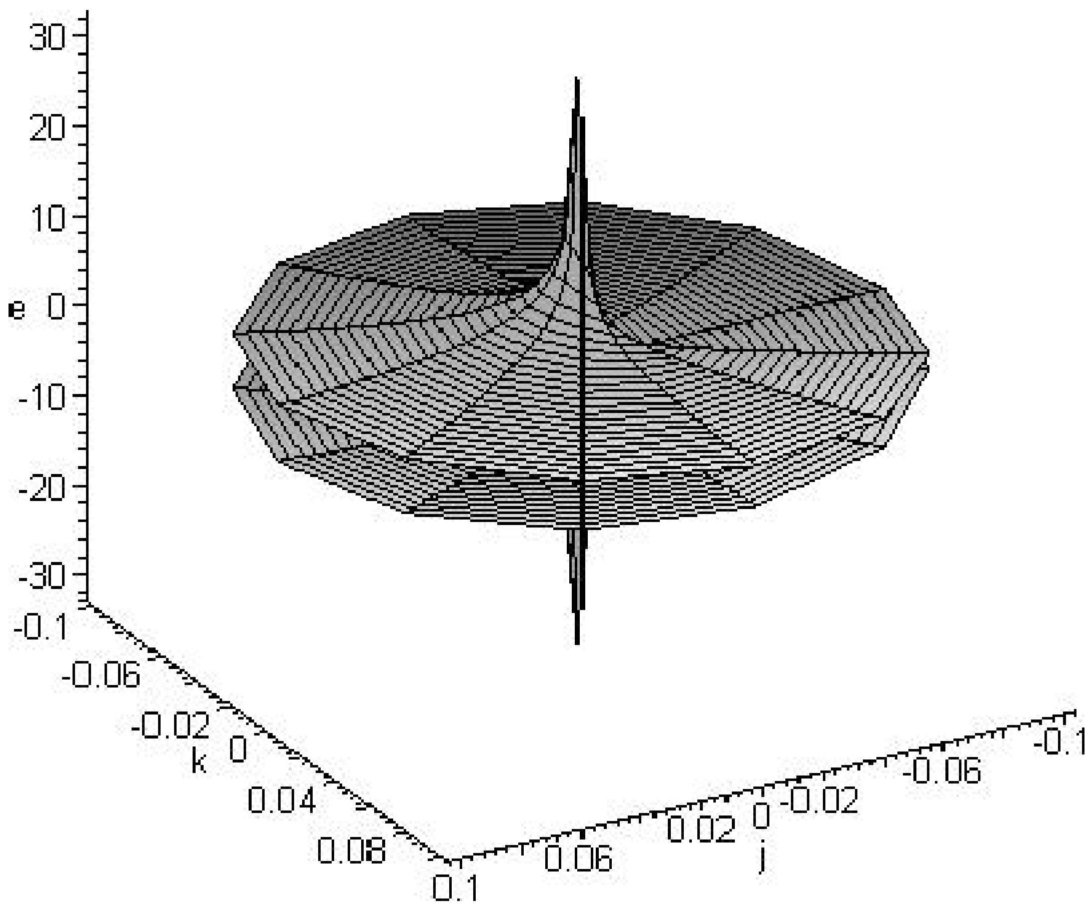

Figure 2.1: A slice of the associative manifold of Thm 2.1.1 in $\{j, k, e\}$ space

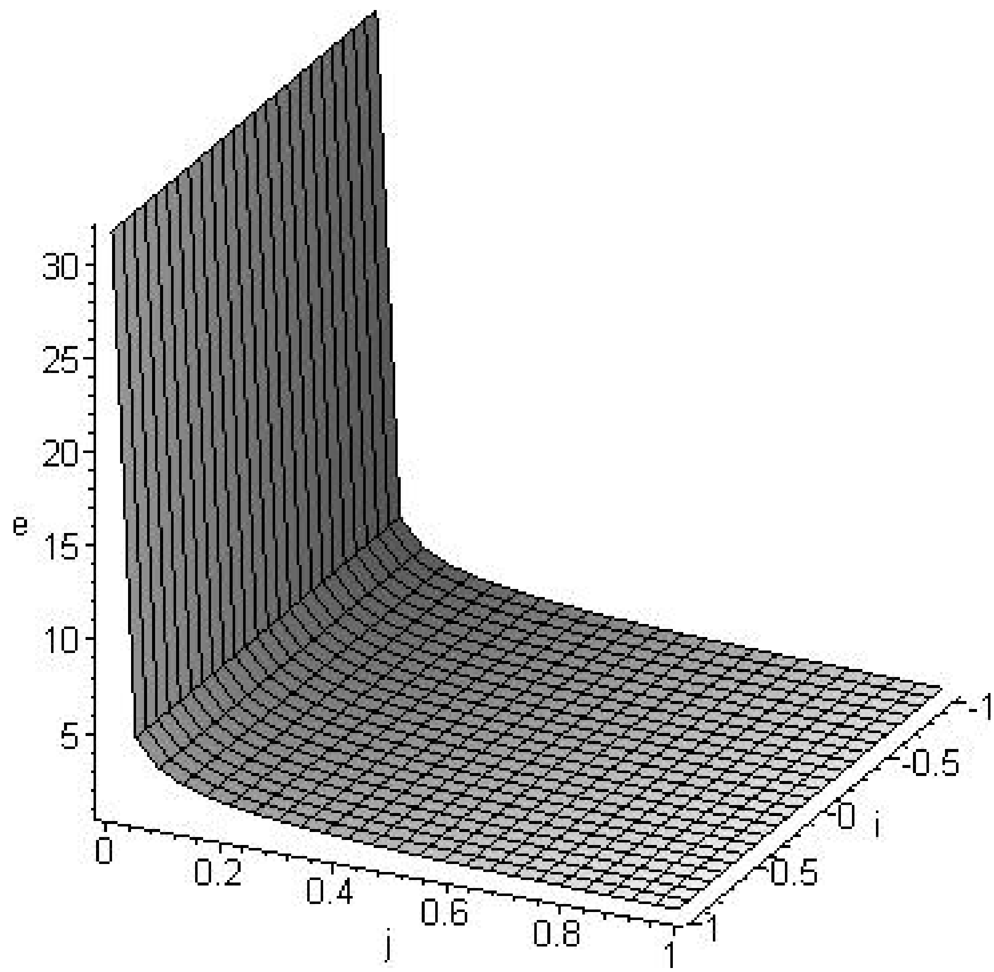

Figure 2.2: A slice of the associative manifold of Thm 2.1.1 in $\{i, j, e\}$ space

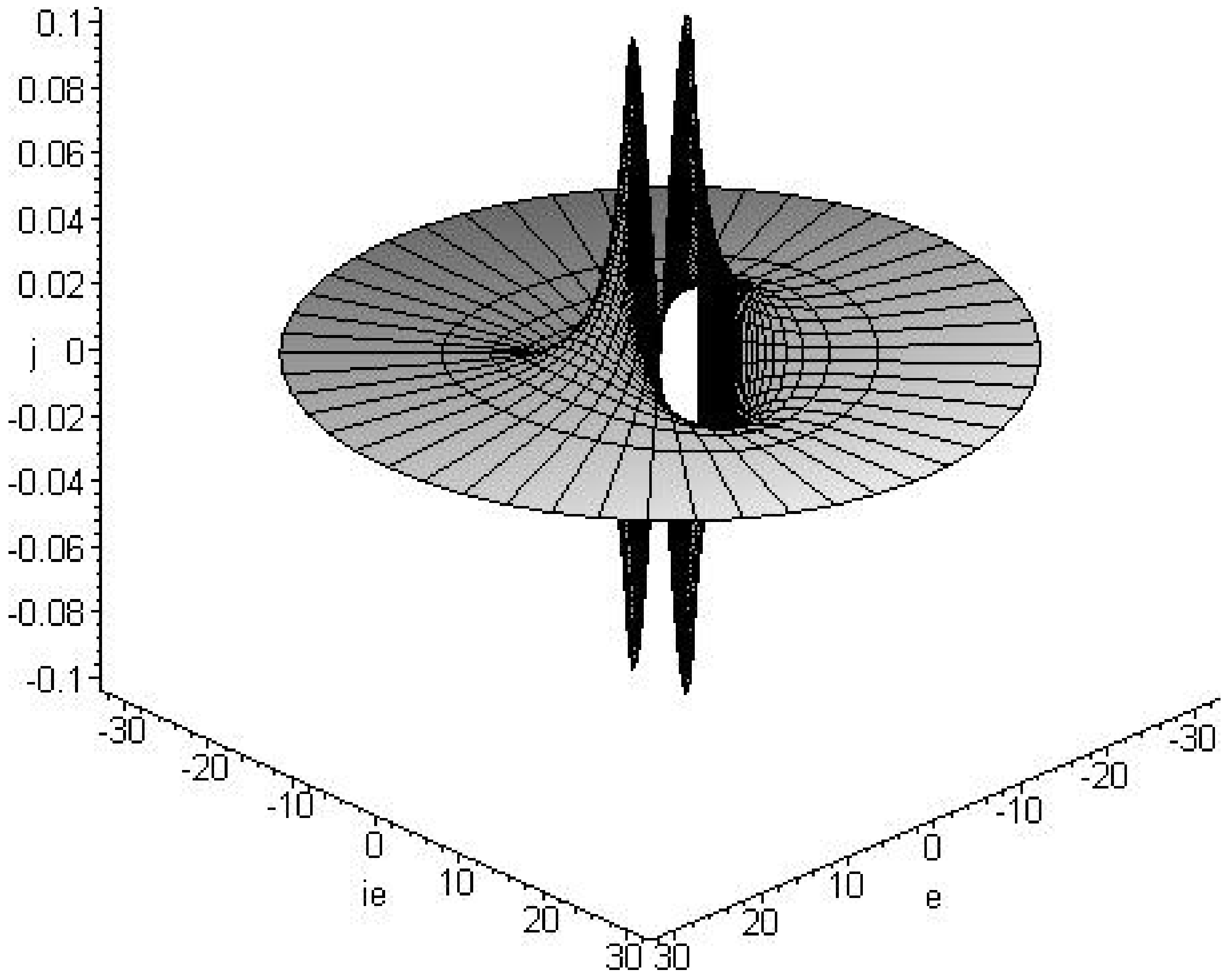

Figure 2.3: A slice of the associative manifold of Thm 2.1.1 in $\{j, e, ie\}$ space

Proof. From the definition of the Cayley number product, and using $\overline{xy} = \bar{y}\bar{x}$,

$$\begin{aligned} h(a,b)h(c,d) &= (e^{it}ae^{-it}, be^{-it})(e^{it}ce^{-it}, de^{-it}) \\ &= (e^{it}ae^{-it}e^{it}ce^{-it} - \overline{de^{-it}}be^{-it}, de^{-it}e^{it}ae^{-it} + be^{-it}\overline{e^{it}ce^{-it}}) \\ &= (e^{it}ace^{-it} - e^{it}\bar{d}be^{-it}, dae^{-it} + b\bar{c}e^{-it}) \\ &= (e^{it}(ac - \bar{d}b)e^{-it}, (da + b\bar{c})e^{-it}) = h\left((a,b)(c,d)\right) \end{aligned} \tag{2.10}$$

□

We will sometimes refer to this subgroup as the circle group or the circle action on $\mathrm{Im}\,\mathbb{O}$.

The basic idea is as follows: we may use the action defined above to rotate any $x \in \mathbb{H}$ into the plane spanned by i and j (this follows from the geometric interpretation of quaternion conjugation, as in Theorem A.4.1). We will seek functions $f : \mathrm{Im}\,\mathbb{H} \to \mathbb{H}$ whose graph is invariant under the circle action defined above. In this case the function's value on the $\{i, j\}$ plane is sufficient to determine all of f. Applying the requirement that $Df = \sigma f$ will yield a much simplified partial differential equation whose solution gives us f on the $\{i, j\}$ plane. Because we are using a subgroup of G_2 to generate the full 3-manifold from this, and by Theorem 1.3.1, $\gamma^*\phi = \phi$ for all $\gamma \in G_2$, we are assured that the associative calibration is preserved and hence we need only worry about solving our PDE at points in the $\{i, j\}$ plane.

Lemma 2.1.2. *A graph of $f : \mathrm{Im}\,\mathbb{H} \to \mathbb{H}$ is invariant under the circle action iff*

$$f(x) = f(x_1 i + r_x j)e^{-\frac{1}{2}i\theta_x} \tag{2.11}$$

Here x_1, r_x, and θ_x are the standard cylindrical coordinates for x taking the axis about i.

Proof. First assume the graph of f is invariant under the circle action. Then given any $x \in \mathrm{Im}\ \mathbb{H}$ we can take $t = -\frac{1}{2}\theta_x$ and apply the automorphism. Invariance implies that there is a $y \in \mathrm{Im}\,\mathbb{H}$ such that

$$e^{-\frac{1}{2}\theta_x}xe^{\frac{1}{2}\theta_x} + f(x)e^{\frac{1}{2}\theta_x}\mathbf{e} = y + f(y)\mathbf{e} \tag{2.12}$$

Equating the quaternion parts of the equation and applying the geometric meaning of quaternion conjugation gives us that $y = x_1 i + r_x j$. Equating the remaining part of the equation gives

$$f(x) = f(x_1 i + r_x j) e^{-\frac{1}{2} i\theta_x} \tag{2.13}$$

Now suppose (2.11) holds. Apply any circle action. We need to find a $y \in \operatorname{Im}\mathbb{H}$ such that

$$e^{it} x e^{-it} + f(x_1 i + r_x j) e^{-\frac{1}{2} i\theta_x} e^{-it} \mathbf{e} = y + f(y_1 i + r_y j) e^{-\frac{1}{2} i\theta_y} \tag{2.14}$$

It is clear we need to take $y = e^{it} x e^{-it}$. Thus y is just x rotated about the i axis by some angle $2t$ radians. Hence $y_1 = x_1$ and $r_y = r_x$. So we need only show that $e^{-\frac{1}{2} i\theta_x} e^{-it} = e^{-\frac{1}{2} i\theta_y}$. That is, we need $\theta_x + 2t = \theta_y$. But this is precisely how θ changes from the rotation which takes x to y, so indeed this relation holds. $\square$

We can now prove a slightly better version of Theorem 2.1.1:

Theorem 2.1.2. *A manifold invariant under the circle action defined above and obtained as the graph of an $f : \operatorname{Im}\mathbb{H} \to \mathbb{C}$ is associative iff it is of the form (2.1).*

Proof. We will consider graphs of the form

$$M = \left\{ x + g(x_1 i + r_x j) e^{-\frac{1}{2} i\theta_x} \mathbf{e} \mid x = x_1 i + r_x \cos\theta_x j + r_x \sin\theta_x k \right\} \tag{2.15}$$

for some function $g(x_1, r_x) : \{i, j\}$ plane $\to \mathbb{C}$ to be determined, whose choice makes M into an associative manifold. Note that we are assuming g complex-valued here. We require $Df = \sigma f$ to hold for M. It is not difficult to calculate the partials of f. Making use of (2.3), we find

$$\frac{\partial f}{\partial x_1} = \frac{\partial g}{\partial x_1} e^{-\frac{1}{2} i\theta_x} \tag{2.16}$$

$$\frac{\partial f}{\partial x_2} = \left(\frac{\partial g}{\partial r_x} \cos\theta_x + \frac{1}{2} g \frac{\sin\theta_x}{r_x} i \right) e^{-\frac{1}{2} i\theta_x} \tag{2.17}$$

$$\frac{\partial f}{\partial x_3} = \left(\frac{\partial g}{\partial r_x} \sin\theta_x - \frac{1}{2} g \frac{\cos\theta_x}{r_x} i \right) e^{-\frac{1}{2} i\theta_x} \tag{2.18}$$

Now, we calculate from this

$$\begin{aligned} Df = -\Bigg[\frac{\partial g}{\partial x_1} e^{-i\theta_x} i + \left(\frac{\partial g}{\partial r_x} \cos\theta_x + \frac{1}{2} g \frac{\cos\theta_x}{r_x} \right) j + \\ + \left(\frac{\partial g}{\partial x_r} \sin\theta_x + \frac{1}{2} g \frac{\sin\theta_x}{r_x} \right) k \Bigg] e^{\frac{1}{2} i\theta_x} \end{aligned} \tag{2.19}$$

Since each of the partials of f lie in the complex plane, their triple cross product vanishes, and so

$$\sigma f = 0 \tag{2.20}$$

So we get that M is associative iff $Df = 0$. We could, at this point, restrict our attention to the $\{i, j\}$ plane, i.e. set $\theta = 0$ and solve. But we can see that regardless of the value of θ we get the two equations

$$\frac{\partial g}{\partial x_1} = 0 \tag{2.21}$$

$$\frac{\partial g}{\partial r_x} + \frac{1}{2r_x} g = 0 \tag{2.22}$$

The solution for g is readily obtained by solving the first order differential equation for its second parameter. The general solution is

$$g(x_1, r_x) = \frac{z_0}{\sqrt{r_x}} \tag{2.23}$$

for any $z_0 \in \mathbb{C}$. Substituting this back into (2.15) yields (2.1). □

Remark 2.1.1. *We note that applying any automorphism of $\mathbb{O}$ to this manifold gives another associative manifold, although the resultant manifold may not be a graph of some $f : \mathrm{Im}\,\mathbb{H} \to \mathbb{C}$.*

2.2 Examples of Associative Manifolds Obtained by a More General Action in G_2

We can extend the method above further by considering more general actions of automorphisms. If we wish to use a subset of G_2 to impose symmetries on our graph in order to reduce the number of dimensions on which we solve the differential equations, we must use a subgroup of G_2. Because we are dealing with graphs of functions from $\operatorname{Im}\mathbb{H}$ to $\mathbb{H}$, for simplicity we would like to consider automorphisms that fix $\operatorname{Im}\mathbb{H}$ and $\mathbb{H}\mathbf{e}$ as sets. Finally, our experience in proving Lemma 2.1.2 suggests that the set of automorphisms should at least be closed under composition. Hence, we consider 1-parameter subgroups of G_2.

Lemma 2.2.1. *If G is a 1-parameter subgroup of G_2 that fixes $\operatorname{Im}\mathbb{H}$ and $\mathbb{H}\mathbf{e}$. Then the elements of G are of the form*

$$h_t(a+b\mathbf{e}) = e^{pt}ae^{-pt} + e^{qt}be^{-pt}\mathbf{e} \tag{2.24}$$

for fixed $p, q \in \operatorname{Im}\mathbb{H}$ and $\forall t \in \mathbb{R}$.

Proof. It is proven in the appendix (Lemma A.4.1) that all automorphisms of $\mathbb{O}$ fixing $\operatorname{Im}\mathbb{H}$ are of the form

$$h(a+b\mathbf{e}) = e^{p}ae^{-p} + e^{q}be^{-p}\mathbf{e} \tag{2.25}$$

for $p, q \in \operatorname{Im}\mathbb{H}$. By a countability argument, we can find some $h_1 \in G$ with $\frac{|p|}{\pi}$ or $\frac{|q|}{\pi}$ irrational. Thus, by the closure property of G, we get that $h_n \in G$ for all $n \in \mathbb{N}$. By our choice of h_1 and the periodicity of exponentials, the h_n's form a dense subset of the set of all h_t for $t \in \mathbb{R}$. By continuity of h_t as a function of t, we must have (2.24). $\square$

Although our most general 1-parameter subgroup involves arbitrary imaginary quaternions p and q, we restrict our attention to the $p = i$ case. Analogous results of everything that follows hold for the general case.

Consider the more generalized circle action, for any $q \in \operatorname{Im}\mathbb{H}$:

$$h_t(a+b\mathbf{e}) = e^{it}ae^{-it} + e^{qt}be^{-it}\mathbf{e} \tag{2.26}$$

A note on the meaning of e^{qt}: q is an arbitrary imaginary quaternion, so $q = |q|\hat{q}$ where $\hat{q}$ is a unit imaginary quaternion. The algebra generated by 1 and $\hat{q}$ is isomorphic to $\mathbb{C}$, thus

$$e^{qt} = \cos(|q|t) + \hat{q}\sin(|q|t) \tag{2.27}$$

Of course, since quaternion multiplication does not commute in general, we *cannot* decompose e^{qt} into a product, that is,

$$e^{qt} \neq e^{q_1 it} e^{q_2 jt} e^{q_3 kt} \tag{2.28}$$

for $q = q_1 i + q_2 j + q_3 k$, unless all but one of the q_i is zero.

Lemma 2.2.2. *The action above is an automorphism for any $q \in \operatorname{Im}\mathbb{H}$ and any $t \in \mathbb{R}$. A graph $f : \operatorname{Im}\mathbb{H} \to \mathbb{H}$ is invariant under this automorphism group iff*

$$f(x) = e^{\frac{1}{2}q\theta_x} g(x_1, r_x) e^{-\frac{1}{2}i\theta_x} \tag{2.29}$$

Here, again, we make use of cylindrical coordinates using the i axis as the axis of cylindrical symmetry, and $g(x_1, r_x)$ is a function from the $\{i, j\}$ plane to $\mathbb{H}$. We forgo a proof of the lemma since it is so similar to the proof of Lemma 2.1.2.

Now let us again assume g is real valued. We again solve $Df = \sigma f$ at points where $\theta_x = 0$. Taking partials we get

$$\frac{\partial f}{\partial x_1} = e^{\frac{1}{2}q\theta_x} \frac{\partial g}{\partial x_1} e^{-\frac{1}{2}i\theta_x} \tag{2.30}$$

$$\frac{\partial f}{\partial x_2} = e^{\frac{1}{2}q\theta_x} \left[-\frac{1}{2}\frac{\sin\theta_x}{r_x} g(x_1, r_x)\hat{q} + \frac{\partial g}{\partial r_x}\cos\theta_x + \frac{1}{2}\frac{\sin\theta_x}{r_x} g(x_1, r_x) i \right] e^{-\frac{1}{2}i\theta_x} \tag{2.31}$$

$$\frac{\partial f}{\partial x_3} = e^{\frac{1}{2}q\theta_x} \left[\frac{1}{2}\frac{\cos\theta_x}{r_x} g(x_1, r_x)\hat{q} + \frac{\partial g}{\partial r_x}\sin\theta_x - \frac{1}{2}\frac{\cos\theta_x}{r_x} g(x_1, r_x) i \right] e^{-\frac{1}{2}i\theta_x} \tag{2.32}$$

At $\theta_x = 0$ we get the much simpler equations:

$$\frac{\partial f}{\partial x_1}(x_1, r_x, 0) = \frac{\partial g}{\partial x_1} \tag{2.33}$$

$$\frac{\partial f}{\partial x_2}(x_1, r_x, 0) = \frac{\partial g}{\partial r_x} \tag{2.34}$$

$$\frac{\partial f}{\partial x_3}(x_1, r_x, 0) = \frac{g(x_1, r_x)}{2r_x}\,(\hat{q} - i) \tag{2.35}$$

We only need to solve for $Df = \sigma f$ for these simpler partials. Since two of these are real valued, again we get $\sigma f = 0$ so we need to solve $Df = 0$. Now, if $q = q_1 i + q_2 j + q_3 k$,

$$Df(x_1, r, 0) = \left[\frac{1}{2} q_3 \frac{g(x_1, r_x)}{r_x}\right] - \left[\frac{\partial g}{\partial x_1} + \frac{1}{2} q_2 \frac{g(x_1, r_x)}{r_x}\right] i - \left[\frac{\partial g}{\partial r_x} + \frac{1}{2}(1 - q_1)\frac{g(x_1, r_x)}{r_x}\right] j \tag{2.36}$$

We get three PDEs to solve for $g(x_1, r_x)$:

$$\frac{1}{2} q_3 \frac{g(x_1, r_x)}{r_x} = 0 \tag{2.37}$$

$$\frac{\partial g}{\partial x_1} + \frac{1}{2} q_2 \frac{g(x_1, r_x)}{r_x} = 0 \tag{2.38}$$

$$\frac{\partial g}{\partial r_x} + \frac{1}{2}(1 - q_1)\frac{g(x_1, r_x)}{r_x} = 0 \tag{2.39}$$

If we want non-trivial solutions the first equation shows that q_3 must be zero. For any value of q_1 we have the general solution to the third equation

$$g(x_1, r_x) = K(x_1) r_x^{\frac{1}{2}(q_1 - 1)} \tag{2.40}$$

where $K(x_1)$ is any function of just x_1. We assume for non-trivial solutions that $K(x_1)$ is not identically zero. Plugging this into the second equation, we get

$$K'(x_1) r_x^{\frac{1}{2}(q_1 - 1)} = -\frac{1}{2} q_2 K(x_1) \frac{r_x^{\frac{1}{2}(q_1 - 1)}}{r_x} \tag{2.41}$$

or, rearranging and canceling terms,

$$\frac{K'(x_1)}{K(x_1)} = -\frac{q_2}{2r_x} \tag{2.42}$$

which can only happen if $q_2 = 0$ and $K(x_1) = K \in \mathbb{R}$ is a constant function.

Thus, the general form for $g(x_1, r_x)$ is

$$g(x_1, r_x) = K r_x^{\frac{1}{2}(q_1 - 1)} \tag{2.43}$$

Plugging this back into our original form for f we get that the following is an associative manifold for any $K, q_1 \in \mathbb{R}$:

$$\left\{ x + \left(K r_x^{\frac{1}{2}(q_1-1)} e^{\frac{1}{2} i q_1 \theta_x} e^{-\frac{1}{2} i \theta_x} \right) \mathbf{e} \;\middle|\; x = x_1 i + r_x \cos\theta_x j + r_x \sin\theta_x k \right\} \tag{2.44}$$

To simplify, we write $C = \frac{1}{2}(q_1 - 1)$. We have just proven the following:

Theorem 2.2.1.

$$M_{K,C} = \left\{ x + K \left(r_x e^{i\theta_x} \right)^C \mathbf{e} \;\middle|\; x = x_1 i + r_x \cos\theta_x j + r_x \sin\theta_x k \right\} \tag{2.45}$$

is associative for all real K and C, where we take a holomorphic branch of the complex power/root function.

Remark 2.2.1. *Taking $C = -\frac{1}{2}$ yields as a special case our original family of manifolds. We note that again, the image of the $M_{K,C}$'s under any automorphism of $\mathbb{O}$ gives other associative manifolds which are rotations of these.*

2.3 The Relation to the Cauchy-Riemann Equations

The results obtained here are special cases of the following more general theorem:

Theorem 2.3.1. *Suppose $f(x) = f_0(x) + i f_1(x)$ for real-valued f_0, f_1. Then the graph of f in $\operatorname{Im}\mathbb{O}$ is associative iff $\frac{\partial f}{\partial x_1} = 0$ and f satisfies the Cauchy-Riemann equations in x_2, x_3:*

$$\frac{\partial f_0}{\partial x_2} = \frac{\partial f_1}{\partial x_3} \tag{2.46}$$

$$\frac{\partial f_0}{\partial x_3} = -\frac{\partial f_1}{\partial x_2} \tag{2.47}$$

Since the graphs examined so far have been graphs of holomorphic functions from the $\{x_2,x_3\}$ plane into $\mathbb{C}$, crossed with the x_1 axis, this theorem shows that they are associative.

Proof. Since the image of f is entirely in the complex plane, it is easy to see that $\sigma f = 0$. Thus f is associative iff $Df = 0$. Now,

$$-Df = \frac{\partial f}{\partial x_1}i + \frac{\partial f}{\partial x_2}j + \frac{\partial f}{\partial x_3}k = 0 \tag{2.48}$$

Since the partials are all complex-valued, the first term in this sum is entirely complex valued, while the second two terms are entirely within the $\{j,k\}$ plane. Thus they vanish separately:

$$\frac{\partial f}{\partial x_1} = 0 \tag{2.49}$$

$$\frac{\partial f}{\partial x_2}j + \frac{\partial f}{\partial x_3}k = \left(\frac{\partial f_0}{\partial x_2} - \frac{\partial f_1}{\partial x_3}\right)j + \left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_0}{\partial x_3}\right)k = 0 \tag{2.50}$$

Requiring components to vanish separately in the second equation gives the Cauchy-Riemann equations. □

Another result of the same flavor is also possible:

Theorem 2.3.2. *Suppose $f : \operatorname{Im}\mathbb{H} \to \mathbb{H}$ with $f = f_0+f_1i+f_2j+f_3k$ where $f_i : \operatorname{Im}\mathbb{H} \to \mathbb{R}$ for each $i = 0,1,2,3$, and $\frac{\partial f}{\partial x_1} = 0$. Then the graph of f is associative iff the projections of f, $f_0 + f_1i$ and $f_3k + f_2j$, satisfy the following Cauchy-Riemann equations:*

$$\frac{\partial f_0}{\partial x_2} = \frac{\partial f_1}{\partial x_3} \tag{2.51}$$

$$\frac{\partial f_0}{\partial x_3} = -\frac{\partial f_1}{\partial x_2} \tag{2.52}$$

and

$$\frac{\partial f_3}{\partial x_2} = \frac{\partial f_2}{\partial x_3} \tag{2.53}$$

$$\frac{\partial f_3}{\partial x_3} = -\frac{\partial f_2}{\partial x_2} \tag{2.54}$$

Note that the in the second set of Cauchy-Riemann equations, f_3 plays the role of the real variable and f_2 plays the role of the imaginary variable.

Proof. The graph of f is associative iff $Df = \sigma f$. Since $\frac{\partial f}{\partial x_1} = 0$, $\sigma f = 0$ and we get that the graph of f is associative iff

$$\frac{\partial f}{\partial x_2} j + \frac{\partial f}{\partial x_3} k = 0 \tag{2.55}$$

Writing out the components of f and multiplying through using the rules of quaternion multiplication, we get

$$\left(-\frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3}\right) + \left(-\frac{\partial f_3}{\partial x_2} + \frac{\partial f_2}{\partial x_3}\right) i + \left(\frac{\partial f_0}{\partial x_2} - \frac{\partial f_1}{\partial x_3}\right) j + \left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_0}{\partial x_3}\right) k \ = \ 0 \tag{2.56}$$

Requiring components to vanish separately gives the desired result. $\square$

Theorem 2.3.2 allows us to construct a number of new associative manifolds. For example,

$$\Big\{\, x_1 i + x_2 j + x_3 k + \\ (x_2^2 - x_3^2)\mathbf{e} + 2x_2x_3 i\mathbf{e} + e^{x_2}\sin x_3 j\mathbf{e} + e^{x_2}\cos x_3 k\mathbf{e} \;\Big|\; x_1, x_2, x_3 \in \mathbb{R} \,\Big\} \tag{2.57}$$

is associative since it is a graph of complex square and exponential functions, which are holomorphic, and hence satisfy the C-R equations. For another example, we can note that

$$\left\{ x + K_1\left(r_x e^{i\theta_x}\right)^{C_1}\mathbf{e} + kK_2\left(r_x e^{i\theta_x}\right)^{C_2}\mathbf{e} \;\middle|\; x = x_1 i + r_x\cos\theta_x j + r_x\sin\theta_x k \right\} \tag{2.58}$$

is associative.

On the other hand, the theorem shows that if a function is independent of x_1 then it must be of a very specific form. In order to obtain more interesting examples of associative manifolds with this technique we must examine the cases where g is not necessarily real-valued. We will investigate this in the next chapter.

Chapter 3

Further Investigation of the PDEs of Associative Manifolds

3.1 A Result on the Symmetries of the Problem

In the previous chapter we considered only real-valued functions $g : \mathbb{R}^2 \to \mathbb{R}$ which give rise to functions f defined by (2.29). This greatly simplified the PDEs because σf vanished at $\theta_x = 0$, eliminating the non-linear part of the PDE system. We will now consider the more general case of $g : \operatorname{Im}\mathbb{H} \to \mathbb{H}$ not necessarily real-valued. In order to more systematically study the solutions to the PDEs for arbitrary g and q, we first study the invariance properties of the solutions:

Theorem 3.1.1. *Let $q \in \operatorname{Im}\mathbb{H}$ be fixed. Suppose*

$$M = \{\, x + e^{\frac{1}{2}q\theta_x} g(x_1 i + r_x j) e^{-\frac{1}{2}i\theta_x}\mathbf{e} \mid x = x_1 i + r_x \cos\theta_x j + r_x \sin\theta_x k \,\} \tag{3.1}$$

is associative for some $g : \operatorname{Im}\mathbb{H} \to \mathbb{H}$. Then we also get an associative graph if we replace g with $e^{q\phi} g$ for some $\phi \in \mathbb{R}$.

Proof. We've shown that

$$h(a + b\mathbf{e}) = a + e^{q\phi} b\mathbf{e} \tag{3.2}$$

is a linear automorphism of $\mathbb{O}$. Thus, it takes associative manifolds to associative manifolds in $\operatorname{Im}\mathbb{H}$. Applying h to M gives

$$h(M) = x + e^{q\phi} e^{\frac{1}{2}q\theta_x} g(x_1 i + r_x j) e^{-\frac{1}{2}i\theta_x}\mathbf{e} = \tag{3.3}$$

$$= x + e^{\frac{1}{2}q\theta_x} \left[e^{q\phi} g(x_1 i + r_x j) \right] e^{-\frac{1}{2}i\theta_x}\mathbf{e} \tag{3.4}$$

where we use the commutativity of quaternions in the $\{1, q\}$ plane. $\square$

This result is a useful tool in characterizing symmetric solutions to $Df = \sigma f$. For example, it tells us that the graphs obtained from $g = g_1 i + g_2 j$ and $q = ai$ satisfy the same differential equations as (3.8) below with g_0 replaced by g_1 and g_3 replaced by $-g_2$, by taking $\phi = \frac{\pi}{2}$.

3.2 Some Solutions for Quaternion-Valued g

We will now examine the equations derived from our method for various special cases of g and q. Because for these cases σf no longer vanishes even at $\theta_x = 0$, the PDE system is especially complicated, and so we employ the commercial mathematics package Maple to aid in our calculations of the PDEs. See Appendix B for the code used to generate the equations.

In all of the following we assume $g = g_0 + g_1 i + g_2 j + g_3 k$ where $g_i : \mathrm{Span}\{i, j\} \to \mathbb{R}$. First, we show that the more general quaternion-valued g does indeed give us new associative manifolds.

Theorem 3.2.1. *Let $g = g_0 + g_3 k$ satisfy the Cauchy-Riemann equations:*

$$\frac{\partial g_0}{\partial x_1} = \frac{\partial g_3}{\partial r} \tag{3.5}$$

$$\frac{\partial g_0}{\partial r} = -\frac{\partial g_3}{\partial x_1} \tag{3.6}$$

Then the graph

$$M = \{\, x + e^{\frac{1}{2} i\theta_x} g(x_1 i + r_x j) e^{-\frac{1}{2} i\theta_x} \mathbf{e} \mid x = x_1 i + r_x \cos\theta_x j + r_x \sin\theta_x k \,\} \tag{3.7}$$

is associative.

Note that this is a manifold obtained by suitable *rotations* of a the graph of a function satisfying the C-R equations, not from a translation, and so it does not fall under the conditions of Theorem 2.3.2.

Proof. The differential equations obtained from setting $Df = \sigma f$ at $\theta_x = 0$ when g maps entirely into the $\{1, k\}$ plane and when $q = ai$ are

$$\frac{\partial g_3}{\partial r} - \frac{\partial g_0}{\partial x_1} + \left(\frac{1-a}{2}\right)\frac{g_3}{r}\left(1 + \frac{\partial g_0}{\partial x_1}\frac{\partial g_3}{\partial r} - \frac{\partial g_3}{\partial x_1}\frac{\partial g_0}{\partial r}\right) = 0 \tag{3.8}$$

$$\frac{\partial g_0}{\partial r} + \frac{\partial g_3}{\partial x_1} + \left(\frac{1-a}{2}\right)\frac{g_0}{r}\left(1 + \frac{\partial g_0}{\partial x_1}\frac{\partial g_3}{\partial r} - \frac{\partial g_3}{\partial x_1}\frac{\partial g_0}{\partial r}\right) = 0 \tag{3.9}$$

Taking $a = 1$ yields the Cauchy-Riemann equations required, and M is simply the graph corresponding to this particular g. □

Thus, consideration of arbitrary g gives us more interesting results. Indeed, the graph of any g satisfying (3.8), suitably rotated, is associative.

Although we get some new results from these considerations, many choices for the form of g give only trivial solutions to the PDEs, or solutions covered by Theorems 2.3.1 or 2.3.2. Directly from Theorem 2.3.1 we get that a complex-valued g with a complex q can only be the graph of a function satisfying the Cauchy-Riemann equations crossed with the real line, but we can show similar results for other forms of g as well.

Lemma 3.2.1. *$g = g_0 + g_2 j$ and $q = ai$ only admits solutions of the form covered by Theorem 2.3.2.*

Proof. The equations obtained by setting $Df = \sigma f$ at $\theta_x = 0$ are

$$\frac{\partial g_2}{\partial r} + \left(\frac{1-a}{2}\right)\frac{g_2}{r} = 0 \tag{3.10}$$

$$\frac{\partial g_0}{\partial r} + \left(\frac{1-a}{2}\right)\frac{g_0}{r} = 0 \tag{3.11}$$

$$\frac{\partial g_0}{\partial x_1} + \left(\frac{1-a}{2}\right)\frac{g_2}{r}\left[\frac{\partial g_2}{\partial x_1}\frac{\partial g_0}{\partial r} - \frac{\partial g_0}{\partial x_1}\frac{\partial g_2}{\partial r}\right] = 0 \tag{3.12}$$

$$\frac{\partial g_2}{\partial x_1} + \left(\frac{1-a}{2}\right)\frac{g_0}{r}\left[\frac{\partial g_0}{\partial x_1}\frac{\partial g_2}{\partial r} - \frac{\partial g_2}{\partial x_1}\frac{\partial g_0}{\partial r}\right] = 0 \tag{3.13}$$

From the first two equations, we must have

$$g_0(x_1, r) = K_0(x_1) r^{\frac{a-1}{2}} \tag{3.14}$$

$$g_2(x_1, r) = K_2(x_1) r^{\frac{a-1}{2}} \tag{3.15}$$

for some differentiable $K_0, K_2 : \mathbb{R} \to \mathbb{R}$ dependent only upon x_1. It is readily verified that constant K_0 and K_2 give solutions. We will now show that there are no non-constant K_0 or K_2 satisfying the other two equations. Plugging our form for g_0 and g_2 into the second two equations, rearranging terms, and canceling common factors of r (we assume $r \neq 0$ here)gives us

$$K_0' = -\left(\frac{1-a}{2}\right)^2 K_2 \left(K_0' K_2 - K_0 K_2'\right) r^{a-3} \tag{3.16}$$

$$K_2' = -\left(\frac{1-a}{2}\right)^2 K_0 \left(K_2' K_0 - K_2 K_0'\right) r^{a-3} \tag{3.17}$$

If $a \neq 3$ then we must have $K_0' = K_2' = 0$, otherwise there would be an r dependence in K_0 and K_2. We now examine the case $a = 3$. The remaining equations to be satisfied reduce to

$$K_0' + K_2^2 K_0' - K_2 K_2' K_0 = 0 \tag{3.18}$$

$$K_2' + K_0^2 K_2' - K_0 K_0' K_2 = 0 \tag{3.19}$$

Rearranging terms, we can solve the first of these equations for K_0 in terms of K_2:

$$K_0 = c_1 \exp\left(\int \frac{K_2' K_2}{1 + K_2^2}\, dx\right) \tag{3.20}$$

where $c_1 \in \mathbb{R}$. Substituting this into the second equation, canceling K_2' (we seek solutions in which K_2' is not identically zero), and rearranging terms,

$$\frac{1 + K_2^2}{c_1^2 K_2^2} = \exp\left(2 \int \frac{K_2' K_2}{1 + K_2^2}\, dx\right) \tag{3.21}$$

Taking logs and derivatives gives, after some manipulation and cancellation,

$$K_2^2 = -1 \tag{3.22}$$

which is a contradiction. Hence, all solutions to the original set of equations satisfy $\frac{\partial g}{\partial x_1} = 0$. □

Lemma 3.2.2. *$g = g_0 + g_2 j$ and $q = bj$ only admits solutions of the form covered by Theorem 2.3.2.*

Proof. The equations are

$$\frac{\partial g_2}{\partial r} + \frac{1}{2}\frac{g_2}{r} = 0 \tag{3.23}$$

$$\frac{\partial g_0}{\partial r} + \frac{1}{2}\frac{g_0}{r} = 0 \tag{3.24}$$

$$\frac{\partial g_0}{\partial x_1} + \frac{1}{2}\frac{g_2}{r}\left[\frac{\partial g_2}{\partial x_1}\frac{\partial g_0}{\partial r} - \frac{\partial g_0}{\partial x_1}\frac{\partial g_2}{\partial r}\right] + \frac{1}{2}b\frac{g_0}{r} = 0 \tag{3.25}$$

$$\frac{\partial g_2}{\partial x_1} + \frac{1}{2}\frac{g_0}{r}\left[\frac{\partial g_0}{\partial x_1}\frac{\partial g_2}{\partial r} - \frac{\partial g_2}{\partial x_1}\frac{\partial g_0}{\partial r}\right] + \frac{1}{2}b\frac{g_2}{r} = 0 \tag{3.26}$$

By a similar technique as the previous lemma, we can prove that there are no solutions to these equations with an x_1 dependence. Solving the first two equations gives

$$g_2 = K_2(x_1)r^{-\frac{1}{2}} \tag{3.27}$$

$$g_0 = K_0(x_1)r^{-\frac{1}{2}} \tag{3.28}$$

Plugging these into the last two equations gives, after some manipulation and the cancellation of common factors of r,

$$(K_0')r^3 + \left(\frac{bK_0}{2r}\right)r^2 - \left[\frac{1}{4}K_2\left(K_2'K_0 - K_2K_0'\right)\right] = 0 \tag{3.29}$$

$$(K_2')r^3 + \left(\frac{bK_2}{2r}\right)r^2 - \left[\frac{1}{4}K_0\left(K_0'K_2 - K_0K_2'\right)\right] = 0 \tag{3.30}$$

The linear independence of the functions $\{r^3, r^2, 1\}$ over coefficients depending only on x_1 then implies that each coefficient vanishes, and this implies that both K_0 and K_2 vanish. □

Lemma 3.2.3. *$g = g_0 + g_2 j$ and $q = ck$ only admits solutions of the form covered by Theorem 2.3.2.*

Proof. The equations are

$$\frac{\partial g_2}{\partial r} + \frac{1}{2}\frac{g_2 + cg_0}{r} = 0 \tag{3.31}$$

$$\frac{\partial g_0}{\partial r} + \frac{1}{2}\frac{g_0 - cg_2}{r} = 0 \tag{3.32}$$

$$\frac{\partial g_0}{\partial x_1} + \frac{1}{2}\frac{g_2 + cg_0}{r}\left[\frac{\partial g_2}{\partial x_1}\frac{\partial g_0}{\partial r} - \frac{\partial g_2}{\partial r}\frac{\partial g_0}{\partial x_1}\right] = 0 \tag{3.33}$$

$$\frac{\partial g_2}{\partial x_1} - \frac{1}{2}\frac{g_0 - cg_2}{r}\left[\frac{\partial g_2}{\partial x_1}\frac{\partial g_0}{\partial r} - \frac{\partial g_2}{\partial r}\frac{\partial g_0}{\partial x_1}\right] = 0 \tag{3.34}$$

First note that if $g_0 = cg_2$ then the equations force $\frac{\partial g_2}{\partial x_1} = 0$ and hence $\frac{\partial g_0}{\partial x_1} = 0$ as well, giving us the hypotheses of Theorem 2.3.2 (actually, we can see that this forces g to be constant). We therefore consider the case where $g_0 - cg_2$ is not identically zero. We can assume that $\frac{\partial g_2}{\partial x_1}\frac{\partial g_0}{\partial r} - \frac{\partial g_2}{\partial r}\frac{\partial g_0}{\partial x_1}$ is also not identically zero; otherwise, we would certainly have $\frac{\partial g}{\partial x_1} = 0$. Now, taking ratios of the equations gives

$$\frac{\partial g_2}{\partial r}\Big/\frac{\partial g_0}{\partial r} = \frac{g_2 + cg_0}{g_0 - cg_2} \tag{3.35}$$

$$\frac{\partial g_0}{\partial x_1}\Big/\frac{\partial g_2}{\partial x_1} = -\frac{g_2 + cg_0}{g_0 - cg_2} \tag{3.36}$$

from which we get

$$\frac{\partial g_0}{\partial r}\frac{\partial g_0}{\partial x_1} + \frac{\partial g_2}{\partial r}\frac{\partial g_2}{\partial x_1} = 0 \tag{3.37}$$

which implies that the partials are linearly dependent, and so either

$$\frac{\partial g_0}{\partial x_1} = -\psi\frac{\partial g_2}{\partial r} \tag{3.38}$$

$$\frac{\partial g_2}{\partial x_1} = \psi\frac{\partial g_0}{\partial r} \tag{3.39}$$

or

$$\frac{\partial g_0}{\partial r} = -\psi\frac{\partial g_2}{\partial x_1} \tag{3.40}$$

$$\frac{\partial g_2}{\partial r} = \psi\frac{\partial g_0}{\partial x_1} \tag{3.41}$$

for some real-valued function ψ which we assume is not identically zero. Since we assumed $g_0 - cg_2$ is not identically zero, by our second original equation we have that $\frac{\partial g_0}{\partial r}$ is not identically zero, thus we can always use (3.38). Substituting (3.38) into our original third equation and canceling terms gives

$$\frac{\partial g_2}{\partial r} = \frac{1}{2}\frac{g_2 + cg_0}{r}\left[\left(\frac{\partial g_0}{\partial r}\right)^2 + \left(\frac{\partial g_2}{\partial r}\right)^2\right] \tag{3.42}$$

Now, using our first original equation, we get

$$\frac{g_2 + cg_0}{2r}\left[\left(\frac{\partial g_0}{\partial r}\right)^2 + \left(\frac{\partial g_2}{\partial r}\right)^2\right] = -\frac{g_2 + cg_0}{2r} \tag{3.43}$$

which implies that $g_2 = -cg_0$. But this also implies g is constant by the form of the original equations. □

Lemma 3.2.4. *$g = g_0 + g_1 i$ and $q = bj$ only admits solutions of the form covered by Theorem 2.3.2.*

Proof. The equations are

$$\frac{\partial g_1}{\partial x_1} + \frac{b}{2}\frac{g_1}{r} = 0 \tag{3.44}$$

$$\frac{\partial g_0}{\partial x_1} + \frac{b}{2}\frac{g_0}{r} = 0 \tag{3.45}$$

$$\frac{\partial g_1}{\partial r} + \frac{1}{2}\frac{g_1}{r} + \frac{b}{2}\frac{g_0}{r}\left(\frac{\partial g_0}{\partial r}\frac{\partial g_1}{\partial x_1} - \frac{\partial g_0}{\partial x_1}\frac{\partial g_1}{\partial r}\right) = 0 \tag{3.46}$$

$$\frac{\partial g_0}{\partial r} + \frac{1}{2}\frac{g_0}{r} + \frac{b}{2}\frac{g_1}{r}\left(\frac{\partial g_1}{\partial r}\frac{\partial g_0}{\partial x_1} - \frac{\partial g_1}{\partial x_1}\frac{\partial g_0}{\partial r}\right) = 0 \tag{3.47}$$

Solving the first two gives

$$g_1 = K_1(r)e^{-\frac{bx_1}{2r}} \tag{3.48}$$

$$g_0 = K_0(r)e^{-\frac{bx_1}{2r}} \tag{3.49}$$

for differentiable K_0, K_1 functions only of r. Plugging these into the last two equations and rearranging terms gives

$$\left(\frac{b}{2r^2}\right)x_1 + \left[\left(\frac{b}{2r}\right)^2 K_1\left(K_0'K_1 - K_1'K_0\right)\right]e^{-\frac{bx_1}{r}} + \\ \left[\left(\frac{b}{2r}\right)^3\frac{K_1}{r}\left(K_1 - K_0\right)\right]x_1 e^{-\frac{bx_1}{r}} + \frac{K_0 + K_0'}{2r} = 0 \tag{3.50}$$

By the linear independence of the functions $\{1, x_1, e^{-\frac{bx_1}{r}}, x_1 e^{-\frac{bx_1}{2r}}\}$ we must have each coefficient function vanish. Thus $b = 0$, and it is evident from the form of the original equations that this implies $\frac{\partial g}{\partial x_1} = 0$. $\square$

Chapter 4

Conclusion and Future Work

We have found several families of associative manifolds in $\mathbb{R}^7$ which are mathematically interesting. By requiring our manifolds to be graphs of functions $f : \operatorname{Im}\mathbb{H} \to \mathbb{H}$ that are also invariant under 1-parameter subgroups of G_2 we greatly simplified the PDEs involved and found tractable special cases. We found that when our function f is real-valued on the $\{i, j\}$ plane, the general form for f is that of a complex power/root function in x_2 and x_3 coordinates, and is independent of the x_1 coordinate, a special case of Theorem 2.3.1. Together with Theorem 2.3.2 these allow us to prove that a large class of manifolds are indeed associative. When f is not necessarily real-valued on the $\{i, j\}$ plane, we've shown that a number of the PDEs derived only admit solutions already covered by these two theorems. We did, however, find a new example of an associative manifold not covered by Theorems 2.3.1 or 2.3.2. This manifold also involves the graph of a function satisfying the Cauchy-Riemann equations, but while the previous manifolds were formed as a set of translations of this graph, this new manifold is formed by a particular set of rotations of the graph in seven-dimensional space.

Our results are all mathematically interesting in their own right, but we would also like to determine which of these may prove useful to theoretical physicists in the future. Therefore one important area of future work is the application of the results here to finding homologically volume-minimizing 3-cycles in $\mathbb{R}^6 \times S^1$, and identifying which, if any, of these resulting cycles are useful in String Theory.

Another potentially useful area of research is the complete characterization of which forms of $g : \mathbb{R}^2 \to \mathbb{H}$ give rise to "interesting" associative manifolds as discussed in chapter 3. Ideally, we would like to know all possible manifolds achieved by solving $Df = \sigma f$ for f invariant under our circle actions.

Appendix A

Cayley Numbers and Quaternion Numbers

A.1 Definitions and Overview

In this appendix we will define and derive basic properties of the Cayley numbers and Quaternion numbers. Most of the results here can be found in either [1] and [2]; a few are original variations on the presentations offered there. The presentation in [1] begins with a general normed algebra rather than Cayley numbers, and goes on to prove that Cayley numbers and its subalgebras are the only normed algebras over $\mathbb{R}$.

The Cayley numbers, denoted $\mathbb{O}$, comprise an eight dimensional algebra over $\mathbb{R}$, meaning it is a vector space over $\mathbb{R}$ isomorphic to $\mathbb{R}^8$, furnished with a vector multiplication rule (from $\mathbb{O} \times \mathbb{O} \to \mathbb{O}$) with unit, which associates with scalar multiplication. That is, if $x, y \in \mathbb{O}$ and $k \in \mathbb{R}$ then

$$k(xy) = (kx)y \tag{A.1}$$

Moreover, we shall prove that $\mathbb{O}$ is *normed*, meaning that if $x, y \in \mathbb{O}$ then

$$|xy| = |x||y| \tag{A.2}$$

where $|\cdot|$ is the standard Euclidean norm on $\mathbb{R}^8$.

The canonical basis for $\mathbb{O}$ as a vector space is denoted $\{1, i, j, k, e, ie, je, ke\}$. Any $x \in \mathbb{O}$ can be written as

$$x = x_1 + x_2 i + x_3 j + x_4 k + x_5 e + x_6 ie + x_7 je + x_8 ke \tag{A.3}$$

where the $x_i \in \mathbb{R}$. We also have an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{O}$ which is identical to the standard Euclidean inner product on $\mathbb{R}^8$:

$$\langle x, y \rangle = \Sigma_{i=1}^{8} x_i y_i \tag{A.4}$$

The subalgebra spanned by $\{1, i, j, k\}$ is called $\mathbb{H}$, the Quaternion numbers. The subalgebra spanned by $\{1, i\}$ is the complex numbers $\mathbb{C}$. It can be proven that $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the only normed algebras over $\mathbb{R}$.

The multiplication rule is intimately tied to the geometry of Euclidean space. In addition to preserving norm, left and right multiplication by Cayley numbers has an additional geometric interpretation as a rotation of $\mathbb{R}^8$, as will be demonstrated below.

Definition A.1.1. *The* real part *of a Cayley number x as in (A.3) is denoted* $\operatorname{Re} x = x_1$. *The* imaginary part *of x is* $\operatorname{Im} x = x_2 i + x_3 j + x_4 k + x_5 e + x_6 ie + x_7 je + x_8 ke$. *The* conjugate *of x is $\bar{x} = \operatorname{Re} x - \operatorname{Im} x$.*

Note that according to our definition, the imaginary part of $a + bi$ is bi, not b, as is commonly defined in Complex Analysis. We will now define multiplication on $\mathbb{H}$ and $\mathbb{O}$. Associate to each $x = x_1 + x_2 i + x_3 j + x_4 k$ a pair of complex numbers $a = x_1 + x_2 i$ and $b = x_3 + x_4 i$. Then if $x, y \in \mathbb{H}$ have representations in $\mathbb{C} \oplus \mathbb{C}$ given by (a, b) and (c, d) respectively, then

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}) \tag{A.5}$$

We say that $\mathbb{H}$ was obtained by applying the *Cayley-Dickson process* to $\mathbb{C}$. It is not hard to verify that this definition yields a 4 dimensional algebra over $\mathbb{R}$. Note that if we break $\mathbb{C}$ into $\mathbb{R} \oplus \mathbb{R}$ a similar definition gives the multiplication rule for $\mathbb{C}$; that is, we obtain C by applying the Cayley-Dickson process to $\mathbb{R}$. Also note that this definition of multiplication is equivalent to defining

$$ij = -ji = k \tag{A.6}$$

$$jk = -kj = i \tag{A.7}$$

$$ki = -ik = j \tag{A.8}$$

$$i^2 = j^2 = k^2 = -1 \tag{A.9}$$

and requiring multiplication to distribute over addition. From this is it clear that quaternion multiplication is *not* commutative in general. It can be directly verified, however, that it is associative. We will show it is normed when we show $\mathbb{O}$ is.

To define multiplication on Cayley numbers we break $x \in \mathbb{O}$ into $a = x_1 + x_2 i + x_3 j + x_4 k$ and $b = x_5 + x_6 i + x_7 j + x_8 k$ and again apply the Cayley-Dickson process, this time to $\mathbb{H}$:

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}) \tag{A.10}$$

Again, this can be seen to yield a valid 8 dimensional algebra over $\mathbb{R}$. Note that this rule satisfies

$$(i)(e) = ie \tag{A.11}$$

$$(j)(e) = je \tag{A.12}$$

$$(k)(e) = ke \tag{A.13}$$

where we denote multiplication on the left and the basis vectors on the right. Note also that since quaternion multiplication is not commutative, the order of the product defined here is important. We can verify also that Cayley multiplication is *not associative* in general, since by our rule, $i(je) = -ke$, while $(ij)e = ke$. However, there is a weak associativity rule that Cayley numbers satisfy, which we will demonstrate below.

A.2 Some Algebraic Properties

Lemma A.2.1. *For $x, y \in \mathbb{O}$, $\langle x, y\rangle = \mathrm{Re}\, \bar{x}y$ and $\overline{xy} = \bar{y}\bar{x}$. Also, $|x|^2 = \bar{x}x$.*

Proof. We will actually prove the result for any algebra over $\mathbb{R}$ obtained from repeated application of Cayley-Dickson. We induct on n, the number of times Cayley-Dickson was applied. Thus for $\mathbb{R}$, $n = 0$, for $\mathbb{C}$, $n = 1$, for $\mathbb{H}$, $n = 2$, and for $\mathbb{O}$, $n = 3$. The lemma holds for higher n as well, although these are no longer normed algebras.

The lemma clearly holds for the case $n = 0$. Now, suppose it holds for k. We'll show it for $k + 1$. By the definition of multiplication in Cayley-Dickson and using the fact that $\overline{(a, b)} = (\bar{a}, -b)$,

$$\mathrm{Re}\, \overline{(a, b)}(c, d) = \mathrm{Re}\,(\bar{a}, -b)(c, d) = \tag{A.14}$$

$$= \mathrm{Re}\,(\bar{a}c + \bar{d}b, d\bar{a} - b\bar{c}) = \tag{A.15}$$

$$= \mathrm{Re}\,(\bar{a}c) + \mathrm{Re}\,(\bar{d}b) = \tag{A.16}$$

$$= \langle a, c\rangle + \langle b, d\rangle = \langle (a, b), (c, d)\rangle \tag{A.17}$$

which proves the first assertion. Similarly, applying the induction hypothesis and the linearity of conjugation,

$$\overline{(a, b)(c, d)} = \overline{(ac - \bar{d}b, da + b\bar{c})} = \tag{A.18}$$

$$= (\overline{ac - \bar{d}b}, -da - b\bar{c}) = \tag{A.19}$$

$$= (\bar{c}\bar{a} - \bar{b}d, -b\bar{c}) - da = \tag{A.20}$$

$$= (\bar{c}, -d)(\bar{a}, -b) = \overline{(c, d)}\;\overline{(a, b)} \tag{A.21}$$

which proves the second assertion. The final assertion follows from the previous two:

$$|x| = \mathrm{Re}\, \bar{x}x = \frac{1}{2}\left(\bar{x}x + \overline{\bar{x}x}\right) = \frac{1}{2}\left(\bar{x}x + \bar{x}x\right) = \bar{x}x \tag{A.22}$$

□

Definition A.2.1. *The* associator *is defined as*

$$[x, y, z] = (xy)z - x(yz) \tag{A.23}$$

Lemma A.2.2. *The associator on $\mathbb{O}$ is alternating.*

Proof. Directly from the Cayley-Dickson definition of multiplication we may verify that if

$$x = a + \alpha e \tag{A.24}$$

$$y = b + \beta e \tag{A.25}$$

$$z = c + \gamma e \tag{A.26}$$

for $a, b, c, \alpha, \beta, \gamma \in \mathbb{H}$, then

$$[x, \bar{x}, y] = [a, \bar{\beta}, \alpha] + [\alpha, \bar{b}, a]e \tag{A.27}$$

Since $\mathbb{H}$ is associative, both parts vanish. Since the associator is trilinear and clearly vanishes when one of its arguments is real, this shows that

$$[x, x, y] = 0 \tag{A.28}$$

Similarly we can show the other required equations

$$[x, y, y] = 0 \tag{A.29}$$

$$[x, y, x] = 0 \tag{A.30}$$

□

Note that our proof also shows that $[x, \bar{x}, y] = 0$ and similar identities. We can now show that $\mathbb{O}$ is normed.

Theorem A.2.1. *$\mathbb{O}$ is normed.*

Proof. From Lemmas A.2.1 and A.2.2,

$$|xy|^2 = (xy)\overline{(xy)} = xy\bar{y}\bar{x} = x|y|^2\bar{x} = x\bar{x}|y|^2 = |x|^2|y|^2 \tag{A.31}$$

□

We can say more about the structure of $\mathbb{O}$, its subalgebras, and its automorphisms, but we must first prove a few lemmas.

Lemma A.2.3. *For $x, y, z \in \mathbb{O}$,*

$$\langle x, wy\rangle = \langle \bar{w}x, y\rangle \tag{A.32}$$

$$\langle x, yw\rangle = \langle x\bar{w}, y\rangle \tag{A.33}$$

Proof. First note that

$$\langle xw, yw\rangle = \langle x, y\rangle|w|^2 \tag{A.34}$$

$$\langle wx, wy\rangle = \langle x, y\rangle|w|^2 \tag{A.35}$$

The first equation follows from

$$|(x+y)w|^2 = |x+y|^2|w|^2 \tag{A.36}$$

since it is easily shown by expanding the inner product and using Theorem A.2.1 that

$$|(x+y)w|^2 = |x|^2|w|^2 + |y|^2|w|^2 + 2\langle xw, yw\rangle|w|^2 \tag{A.37}$$

and

$$|x+y|^2 = |x|^2 + |y|^2 + 2\langle x, y\rangle \tag{A.38}$$

Similarly we can prove the other assertion, using left multiplication by w.

Now, to prove our original lemma, note that we can assume $x, y, z \in \mathrm{Im}\ \mathbb{O}$ since the inner product is trilinear and clearly the equalities hold when one of the arguments is real (by Lemma A.2.1). Assuming imaginary w, and using the claim just established and linearity of the inner product, we have

$$\langle x, y\rangle(1+|w|^2) = \langle x(1+w), y(1+w)\rangle = \tag{A.39}$$

$$= \langle x+xw, y+yw\rangle = \langle x, y\rangle\left(1+|w|^2\right) + \langle x, yw\rangle + \langle xw, y\rangle \tag{A.40}$$

from which, using $\bar{w} = -w$, we deduce the required relation. A similar procedure establishes the result for left multiplication. □

Lemma A.2.4. *If $x, y, w \in \mathbb{O}$ and $\langle x, y\rangle = 0$ then*

$$x(\bar{y}w) = -y(\bar{x}w) \tag{A.41}$$

Proof. Note that in general,

$$2\langle x, y\rangle = \bar{x}y + x\bar{y} \tag{A.42}$$

from which we get

$$2\langle x, y\rangle w - x(\bar{y}w) - y(\bar{x}w) = [x, \bar{y}, w] + [y, \bar{x}, w] \tag{A.43}$$

Now, the right hand side of this equation vanishes since

$$0 = [x+y, \overline{x+y}, w] = [x, \bar{x}, w] + [x, \bar{y}, w] + [y, \bar{x}, w] + [y, \bar{y}, w] \tag{A.44}$$

and the first and last terms vanish again by Lemma A.2.2. Thus, setting $\langle x, y\rangle = 0$ gives the result. □

Theorem A.2.2. *Let A be a subalgebra of $\mathbb{O}$, let $\epsilon \in A^{\perp}$ with $|\epsilon| = 1$. Then $A\epsilon \perp A$ and*

$$(a+b\epsilon)(c+d\epsilon) = (ac - \bar{d}b) + (da + b\bar{c})\epsilon \tag{A.45}$$

Proof. Since A is a subalgebra it contains 1 and $\bar{a}$ for each $a \in A$. Now, if $a, b \in A$, by Lemma A.2.3,

$$\langle a, b\epsilon\rangle = \langle \bar{b}a, \epsilon\rangle = 0 \tag{A.46}$$

since $\bar{b}a \in A$, thus $A\epsilon \perp A$. Now ϵ must be imaginary since $1 \in A$, and hence by Lemma A.2.1, $\epsilon^2 = -1$. Now,

$$(a + b\epsilon)(c + d\epsilon) = ac + (b\epsilon)(d\epsilon) + a(d\epsilon) + (b\epsilon)c \tag{A.47}$$

and, using Lemmas A.2.1 and A.2.4,

$$(b\epsilon)(d\epsilon) = -\bar{d}((\overline{b\epsilon})\epsilon) = \bar{d}((\epsilon\bar{b})\epsilon) = -\bar{d}((\epsilon\bar{\epsilon})b) = -\bar{d}b \tag{A.48}$$

$$a(d\epsilon) = a(\epsilon\bar{d}) = \epsilon(\bar{a}d) = (\bar{a}\bar{d})\epsilon = (da)\epsilon \tag{A.49}$$

$$(b\epsilon)c = (b\bar{c})\epsilon \tag{A.50}$$

and so the result is proven. □

Theorem A.2.3. *If A is a subalgebra of $\mathbb{O}$ it is isomorphic to either $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

Proof. Clearly $\mathbb{R} \subset A$. If $\mathbb{R} = A$ we're done. Otherwise take some $\epsilon_1 \in \operatorname{Im} A$. By the previous theorem, $\mathbb{R}+\mathbb{R}\epsilon_1 \cong \mathbb{C}$. If $\mathbb{R}+\mathbb{R}\epsilon_1 = A$ we're done. If not, take $\epsilon_2 \perp \mathbb{R}+\mathbb{R}\epsilon_1$, etc. We repeat this process until we get all of A. It must stop by the time we get $\mathbb{O}$ since A is a subalgebra, hence its dimension is no more than 8. □

Theorem A.2.4. *Given the orthonormal triple $e_1, e_2, e_3 \in \operatorname{Im} \mathbb{O}$ satisfying $e_3 \perp e_1e_2$, there is a unique automorphism g of $\mathbb{O}$ such that $g(i) = e_1$, $g(j) = e_2$, $g(k) = e_3$.*

Proof. The uniqueness of the automorphism is clear, since the algebra homomorphism property determines g on all of $\mathbb{O}$ once we know $g(i)$, $g(j)$ and $g(k)$. To show existence, note that by Theorem A.2.2, $C \cong \mathbb{R} + \mathbb{R}e_1 = A_1$, $\mathbb{H} \cong A_1 + A_1e_2 = A_2$, and, $\mathbb{O} \cong A_2+A_2e_3 = A_3$. The automorphism sending $\mathbb{O}$ to A_3 satisfies the required properties. □

We denote the group of automorphisms of $\mathbb{O}$ by G_2.

Theorem A.2.5. *G_2 is a subgroup of O(7).*

Here we interpret O(7) as acting on $\operatorname{Im}\mathbb{O}$.

Proof. An automorphism of g must be a non-degenerate linear transformation that fixes $\mathbb{R} \subset \mathbb{O}$. Now, it is easily checked that if $x \notin \mathbb{R}$ then $x \in \operatorname{Im}\mathbb{O}$ iff $x^2 \in \mathbb{R}$, and so g must send imaginary Cayley numbers to imaginary Cayley numbers. These observations show that $g(\bar{x}) = \overline{g(x)}$. Thus

$$|g(x)|^2 = g(x)\overline{g(x)} = g(x)g(\bar{x}) = g(x\bar{x}) = g(|x|^2) = |x|^2 \tag{A.51}$$

which shows that g is an isometry of $\mathbb{R}^8$. Since g fixes $\mathbb{R}$ we can conclude the result. $\square$

A.3 The Cross Products of Cayley Numbers

In this section we define the cross product of two and three Cayley numbers. The cross product of three Cayley numbers is used to define the Monge-Ampere operator in Chapter 1. In the interest of saving space, and because the cross products only play a peripheral role in the paper, we offer only the results directly relevant to our paper, and we offer them here without proof. For further information, please consult [1].

Definition A.3.1. *The* cross product *of $x, y \in \mathbb{O}$ is defined as*

$$x \times y = -\frac{1}{2}\left(\bar{x}y - \bar{y}x\right) \tag{A.52}$$

The triple cross product *of $x, y, z \in \mathbb{O}$ is defined as*

$$x \times y \times z = \frac{1}{2}\left(x(\bar{y}x) - z(\bar{y}x)\right) \tag{A.53}$$

The following results are easily proven using Cayley number identities already established, and justify the use of the term "cross product":

Theorem A.3.1. *$x \times y$ and $x \times y \times z$ are alternating. Furthermore,*

$$|x \times y| = |x \wedge y| \tag{A.54}$$

and

$$|x \times y \times z| = |x \wedge y \wedge z| \tag{A.55}$$

The alternating property of the cross products is often useful in proving theorems since we can assume the arguments are pairwise orthogonal. Further calculations using properties of Cayley numbers yield:

Theorem A.3.2.

$$\operatorname{Im} x \times y = \frac{1}{2}[x, y] \tag{A.56}$$

$$\operatorname{Im} x \times y \times z = \frac{1}{2}[x, y, z] \tag{A.57}$$

We note one final related theorem we will need later.

Theorem A.3.3. $[x, y]$ *is orthogonal to* x *and* y*.* $[x, y, z]$ *is orthogonal to* x*,* y*,* z*, and each commutator* $[x, y], [y, z], [z, x]$*.*

A.4 The Geometric Interpretation of Quaternion Multiplication

We have already seen that the algebra of Cayley numbers is geometric in nature; it is norm-preserving and its multiplication rules may be used to formulate geometric notions such as cross products. Here we discuss further the tie between the Cayley algebra and geometry. We focus on the subalgebra of Quaternion numbers because these are elegant and most directly relevant to the rest of the paper. Many of the results described here may be extended to Cayley numbers. See [2] for details.

We begin our discussion with a geometric definition of quaternion multiplication. If $p, q \in \mathbb{H}$, it can be checked directly that we can define

$$pq = (\operatorname{Re} p)(\operatorname{Re} q) - (\operatorname{Im} p) \cdot (\operatorname{Im} q) + (\operatorname{Re} p)\operatorname{Im} q + (\operatorname{Re} q)\operatorname{Im} p + (\operatorname{Im} p) \times (\operatorname{Im} q) \tag{A.58}$$

where $\cdot$ and $\times$ denote the standard three dimensional vector dot product and cross product, respectively, in $\mathbb{R}^3 \cong \operatorname{Im} \mathbb{H}$.

Quaternions may be expressed in a polar form, much like complex numbers. If $p = p_r + p_i i + p_j j + p_k k$, we may pull out an overall magnitude of $|p|$, define a unit vector in $\{i, j, k\}$ space by

$$u_p = \frac{p_i i + p_j j + p_k k}{\sqrt{p_i^2 + p_j^2 + p_k^2}} \tag{A.59}$$

and an angle θ_p by

$$\theta_p = \cos^{-1}\left(\frac{p_r}{|p|}\right) \tag{A.60}$$

so that we may write p as

$$p = |p|(\cos\theta_p + u_p \sin\theta_p) \tag{A.61}$$

The polar form allows us to describe an elegant geometric interpretation for the multiplication of quaternions, much like in the complex case. Given a particular unit length quaternion

$$p_0 = \cos\theta + u_{p_0} \sin\theta, \tag{A.62}$$

multiplying an arbitrary quaternion q on the left by p_0 performs a 4-D rotation on q (we are treating $\mathbb{H}$ as $\mathbb{R}^4$ here, which, of course, it is isomorphic to as a vector space). Like all 4-D rotations, this quaternion induced rotation consists of simultaneous rotations of two completely orthogonal planes of basis vectors. The plane spanned by 1 and u_{p_0} is rotated counterclockwise by the angle θ, and the plane orthogonal to this (the plane in $\{i, j, k\}$ space that is perpendicular to u_{p_0}) is also rotated counterclockwise by θ. The orientation of the planes (and hence which direction is clockwise and which is counterclockwise) is as follows: in the first case, counterclockwise brings 1 towards u_p. In the second, the "right hand rule" pointing one's thumb in the direction of u_{p_0} determines the counterclockwise direction. If we multiply on the right side of q instead, we get counterclockwise rotation in the 1, u_p plane, but *clockwise* rotations in the $u_{p_0}^{\perp}$ plane.

Theorem A.4.1. *Let $p_0 \in \mathbb{H}$ be given. Then, for any $q \in \mathbb{H}$, left and right multiplication of q by p_0 is as described above.*

Proof. We need a convenient basis in which to express arbitrary q. Take any purely imaginary, unit quaternion v such that $v \perp u_{p_0}$. Note that, by our definition of quaternion multiplication,

$$u_{p_0} v = u_{p_0} \times v \tag{A.63}$$

where we have quaternion multiplication on the left side of the equation, and on the right side we treat the purely imaginary quaternions as vectors in $\mathbb{R}^3$. This shows that an orthonormal basis for $\mathbb{H}$ is $\{1, u_{p_0}, v, (u_{p_0} v)\}$. Now then, suppose

$$q = a + b u_{p_0} + c v + d(u_{p_0} v). \tag{A.64}$$

Then a straightforward calculation (using the fact that quaternion multiplication distributes over addition) gives

$$\begin{aligned} p_0 q = (a \cos\theta - b \sin\theta) + (a \sin\theta + b \cos\theta) u_{p_0} + \\ + (c \cos\theta - d \sin\theta) v + (c \sin\theta + d \cos\theta) u_{p_0} v \end{aligned} \tag{A.65}$$

We used the identity

$$u_{p_0}^2 = -1 \tag{A.66}$$

which is true of any purely imaginary unit quaternion (or Cayley number), as discussed earlier. The resultant vector is exactly what we would get from multiplying q by an appropriate rotation matrix in SO(4). The proof for right multiplication is identical, except we use the basis vector $v u_{p_0}$ in place of $u_{p_0} v$, which gives the opposite orientation to the rotations in the $u_{p_0}^{\perp}$ plane. □

Corollary A.4.1. *Any 3-D rotation of vectors in $\{i, j, k\}$ space may be achieved by conjugation by a unit quaternion. Up to sign, the quaternion we choose to conjugate by is uniquely determined.*

Proof. From the previous result, if $p_0 = \cos\theta + u_{p_0} sin\theta$ then for any $q \in \mathbb{H}$, the conjugation $p_0 q p_0^{-1}$ results in the rotation of the imaginary part of q by an angle 2θ

about the axis u_{p_0}. The real part is held fixed. It is clear that our choice is unique up to sign, since if

$$p_0 q p_0^{-1} = p_1 q p_1^{-1} \tag{A.67}$$

for all $q \in \mathbb{H}$ then

$$(p_1^{-1} p_0) q = q (p_1^{-1} p_0) \tag{A.68}$$

for all $q \in \mathbb{H}$, and so $p_1^{-1} p_0 \in \mathbb{R}$, so they differ by at most a multiplicative constant. If they are both unit quaternions, they differ by at most sign. □

Corollary A.4.2. *Aut($\mathbb{H}$) = SO(3)*

Here we regard SO(3) as acting on $\operatorname{Im} \mathbb{H}$.

Proof. Our earlier theorems on G_2 shows that Aut($\mathbb{H}$) is a subgroup of SO(3). Furthermore, it follows from the previous theorem that any rotation of $\operatorname{Im} \mathbb{H}$ is an automorphism of $\mathbb{H}$, since it may be expressed as a map

$$q \mapsto p q p^{-1} \tag{A.69}$$

for appropriately chosen $p \in \mathbb{H}$. We need only show that orientation reversing members of O(3) are not automorphisms. Suppose one were. Through composition with a rotation we could get an automorphism g such that $g(i) = -i$, $g(j) = j$, and $g(k) = k$. But this cannot be an automorphism, since

$$-1 = g(-1) = g(ijk) = g(i)g(j)g(k) = -ijk = 1, \tag{A.70}$$

a contradiction. □

Our theorem may be extended to another fundamental result regarding quaternion multiplications.

Theorem A.4.2. *Any* four *dimensional rotation may be achieved by a combination of left and right quaternion multiplications by unit length quaternions. The representing left and right multiplying quaternions are unique up to sign.*

When we say "unique up to sign" here, we mean that we could multiply both the left and the right multiplied quaternion by -1 and get the same result.

Proof. We will freely intermix interpretations as quaternions and as members of $\mathbb{R}^4$. Given any $A \in$ SO(4), let $a = A(1)$, the quaternion to which A sends the unit vector along the real axis. Since rotations preserve length, we know $|a| = 1$. The mapping

$$B(q) = a^{-1}A(q) \tag{A.71}$$

clearly is a composition of rotations, and hence a rotation itself. Also, it fixes the real axis, so $B \in SO(3)$ acting on the space of purely imaginary quaternions. But then from our previous theorem,

$$B(q) = pqp^{-1} \tag{A.72}$$

for some unit quaternion p. Therefore,

$$A(q) = (ap)qp^{-1} \tag{A.73}$$

and our construction shows that the quaternions are unique up to sign, since the representations of B is unique up to sign. □

Finally, we note a lemma we will use in Chapter 2:

Lemma A.4.1. *For each pair of non-zero quaternions q_1, q_2, the map $g : \mathbb{O} \to \mathbb{O}$ given by*

$$g(a, b) = (q_1 a q_1^{-1}, q_2 b q_1^{-1}) \tag{A.74}$$

is an automorphism. Moreover, these are the only *automorphisms of $\mathbb{O}$ which fix $\mathbb{H}$ set-wise.*

Proof. It can be directly verified by the definitions that each g defined above is indeed an automorphism. Now suppose $h \in G_2$ fixes $\mathbb{H}$. Since $h \in$ O(7), h fixes $\mathbb{H}^\perp$ too, and so we may regard h as a pair (h_1, h_2) where $h_1 \in$ SO(3) and $h_2 \in$ O(4). Now, from our discussion thus far, it is clear that we can find q_1 such that the resulting automorphism g satisfies $g|_{\mathbb{H}} = h|_{\mathbb{H}}$ and taking $q_2 = h_2(1)q_1$ insures that $g(e) = h(e)$. Then $g^{-1} \circ h$ is an automorphism that fixes i, j, and e. But this forces it to be the identity automorphism. Thus $g = h$. $\square$

Appendix B

Maple Code

To aid in the calculations of the more complicated PDEs in Chapter 3 we wrote a short Maple program. The code relies on a package that implements Cayley Number multiplication written by W. D. Joyner available at `http://web.usna.navy.mil/w̃dj/cayley.mpl`. We have suppressed most of the longer outputs in the interest of saving space.

```
> restart;read `cayley.mpl`;
> octonian_to_list := proc(w::list)
> RETURN([Re(op(1,op(1,w))), Im(op(1,op(1,w))),
> Re(op(2,op(1,w))), Im(op(2,op(1,w))),
> Re(op(1,op(2,w))), Im(op(1,op(2,w))),
> Re(op(2,op(2,w))), Im(op(2,op(2,w)))]);
> end:
> get_eqns := proc(v::list)
> RETURN(op(1,v)=0,op(2,v)=0,op(3,v)=0,op(4,v)=0,
> op(5,v)=0,op(6,v)=0,op(7,v)=0,op(8,v)=0);
> end:
> assume(a,real);assume(b,real);assume(c,real);
> additionally(a^2+b^2+c^2>0);
> nq:=sqrt(a^2+b^2+c^2):
> q:=[[a*I,b+I*c],[0,0]]:
> assume(t,real);expq:=expand([[cos(nq*t/2),0],[0,0]]+sin(nq*t/2)*q
> expi := [[cos(t/2)-I*sin(t/2),0],[0,0]]:
```

```
> g0 := 'g0':  g1 := 'g1':  g2 := 'g2':  g3 := 'g3':
> assume(g0,real);assume(g1,real);assume(g2,real);assume(g3,real);
> assume(x1,real);assume(r,real);
> g := [[g0 + I*g1, g2 + I*g3],[0,0]]:
> assume(x1,real);assume(r,real);f(x1,r,t) :=
evalc(multiply_octonian(g(x1,r),multiply_octonian(expq,expi))):
> _dfdx1:=diff(f(x1,r,t),x1):

> drdx2 := cos(t):  dtdx2 := -sin(t)/r:
> _dfdx2:=expand(diff(f(x1,r,t),r)*drdx2 +
diff(f(x1,r,t),t)*dtdx2):
> drdx3 := sin(t):  dtdx3 := cos(t)/r:
> _dfdx3:=expand(diff(f(x1,r,t),r)*drdx3 +
diff(f(x1,r,t),t)*dtdx3 ):
> dfdx1:=expand(eval(_dfdx1,t=0));
```

$$dfdx1 := [[(\frac{\partial}{\partial x1^{\sim}}\,\mathrm{g0}^{\sim}(x1^{\sim},\, r^{\sim})) + I\,(\frac{\partial}{\partial x1^{\sim}}\,\mathrm{g1}^{\sim}(x1^{\sim},\, r^{\sim})),$$
$$(\frac{\partial}{\partial x1^{\sim}}\,\mathrm{g2}^{\sim}(x1^{\sim},\, r^{\sim})) + I\,(\frac{\partial}{\partial x1^{\sim}}\,\mathrm{g3}^{\sim}(x1^{\sim},\, r^{\sim}))],\, [0,\, 0]]$$

```
> dfdx2:=expand(eval(_dfdx2,t=0));
```

$$dfdx2 := [[(\frac{\partial}{\partial r^{\sim}}\,\mathrm{g0}^{\sim}(x1^{\sim},\, r^{\sim})) + I\,(\frac{\partial}{\partial r^{\sim}}\,\mathrm{g1}^{\sim}(x1^{\sim},\, r^{\sim})),$$
$$(\frac{\partial}{\partial r^{\sim}}\,\mathrm{g2}^{\sim}(x1^{\sim},\, r^{\sim})) + I\,(\frac{\partial}{\partial r^{\sim}}\,\mathrm{g3}^{\sim}(x1^{\sim},\, r^{\sim}))],\, [0,\, 0]]$$

```
> dfdx3:=expand(eval(_dfdx3,t=0));
```

$$dfdx3 := [[-\frac{1}{2}\frac{\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})\,a\tilde{}}{r\tilde{}} - \frac{1}{2}\frac{\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{})\,b\tilde{}}{r\tilde{}} + \frac{1}{2}\frac{I\,\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{})\,a\tilde{}}{r\tilde{}}
- \frac{1}{2}\frac{\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})\,c\tilde{}}{r\tilde{}} - \frac{1}{2}\frac{I\,\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} - \frac{1}{2}\frac{I\,\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})\,b\tilde{}}{r\tilde{}}
+ \frac{1}{2}\frac{I\,\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{})\,c\tilde{}}{r\tilde{}} + \frac{1}{2}\frac{\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}}, \frac{1}{2}\frac{a\tilde{}\,\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} + \frac{1}{2}\frac{b\tilde{}\,\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}}
- \frac{1}{2}\frac{c\tilde{}\,\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} - \frac{1}{2}\frac{I\,a\tilde{}\,\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} + \frac{1}{2}\frac{I\,\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}}
+ \frac{1}{2}\frac{I\,c\tilde{}\,\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} - \frac{1}{2}\frac{\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} + \frac{1}{2}\frac{I\,b\tilde{}\,\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}}], [0,\ 0]]$$

```
> Df := -multiply_octonian(dfdx1,[[I,0],[0,0]])-
multiply_octonian( dfdx2,[[0,1],[0,0]])-
multiply_octonian(dfdx3,[[0,I],[0,0]]);
```

$$Df := [[-I\,(\frac{\partial}{\partial x1\tilde{}}\,\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{})) + (\frac{\partial}{\partial x1\tilde{}}\,\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})) + (\frac{\partial}{\partial r\tilde{}}\,\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{}))
+ I\,(\frac{\partial}{\partial r\tilde{}}\,\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})) - \frac{1}{2}\frac{a\tilde{}\,\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} + \frac{1}{2}\frac{\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}}
+ \frac{1}{2}\frac{c\tilde{}\,\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} + \frac{1}{2}\frac{b\tilde{}\,\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} -
I\,(\frac{1}{2}\frac{a\tilde{}\,\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} + \frac{1}{2}\frac{b\tilde{}\,\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} - \frac{1}{2}\frac{c\tilde{}\,\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}} - \frac{1}{2}\frac{\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}}),
I\,(\frac{\partial}{\partial x1\tilde{}}\,\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{})) - (\frac{\partial}{\partial x1\tilde{}}\,\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})) - (\frac{\partial}{\partial r\tilde{}}\,\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{}))
- I\,(\frac{\partial}{\partial r\tilde{}}\,\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})) + \frac{1}{2}\frac{\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{})\,a\tilde{}}{r\tilde{}} - \frac{1}{2}\frac{\mathrm{g0}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}}
- \frac{1}{2}\frac{\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})\,b\tilde{}}{r\tilde{}} + \frac{1}{2}\frac{\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{})\,c\tilde{}}{r\tilde{}} - I
(-\frac{1}{2}\frac{\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})\,a\tilde{}}{r\tilde{}} - \frac{1}{2}\frac{\mathrm{g2}\tilde{}(x1\tilde{},\ r\tilde{})\,b\tilde{}}{r\tilde{}} - \frac{1}{2}\frac{\mathrm{g3}\tilde{}(x1\tilde{},\ r\tilde{})\,c\tilde{}}{r\tilde{}} + \frac{1}{2}\frac{\mathrm{g1}\tilde{}(x1\tilde{},\ r\tilde{})}{r\tilde{}})]
, [0,\ 0]]$$

```
> Sigmaf := expand(multiply_octonian(dfdx1,multiply_octonian(
conjugate_octonian(df dx2),dfdx3)) - multiply_octonian(
dfdx3,multiply_octonian(conjugate_octonian(dfdx2),df
dx1)))/2:
```

```
> theScore := evalc(octonian_to_list(expand(Df - Sig-
maf))):
```

```
> eval(theScore,[g2=0,g3=0,b=0,c=0]);
```

$$[\frac{\partial}{\partial \mathit{x1}\tilde{}}\,\mathrm{g1}\tilde{}(\mathit{x1}\tilde{},\,r\tilde{}),\,-(\frac{\partial}{\partial \mathit{x1}\tilde{}}\,\mathrm{g0}\tilde{}(\mathit{x1}\tilde{},\,r\tilde{})),$$
$$-(\frac{\partial}{\partial r\tilde{}}\,\mathrm{g0}\tilde{}(\mathit{x1}\tilde{},\,r\tilde{}))+\frac{1}{2}\,\frac{\mathrm{g0}\tilde{}(\mathit{x1}\tilde{},\,r\tilde{})\,a\tilde{}}{r\tilde{}}-\frac{1}{2}\,\frac{\mathrm{g0}\tilde{}(\mathit{x1}\tilde{},\,r\tilde{})}{r\tilde{}},$$
$$-(\frac{\partial}{\partial r\tilde{}}\,\mathrm{g1}\tilde{}(\mathit{x1}\tilde{},\,r\tilde{}))+\frac{1}{2}\,\frac{\mathrm{g1}\tilde{}(\mathit{x1}\tilde{},\,r\tilde{})\,a\tilde{}}{r\tilde{}}-\frac{1}{2}\,\frac{\mathrm{g1}\tilde{}(\mathit{x1}\tilde{},\,r\tilde{})}{r\tilde{}},\,0,\,0,\,0,\,0]$$

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